Sparse sensing and machine learning strategies for characterizing complex dynamical systems

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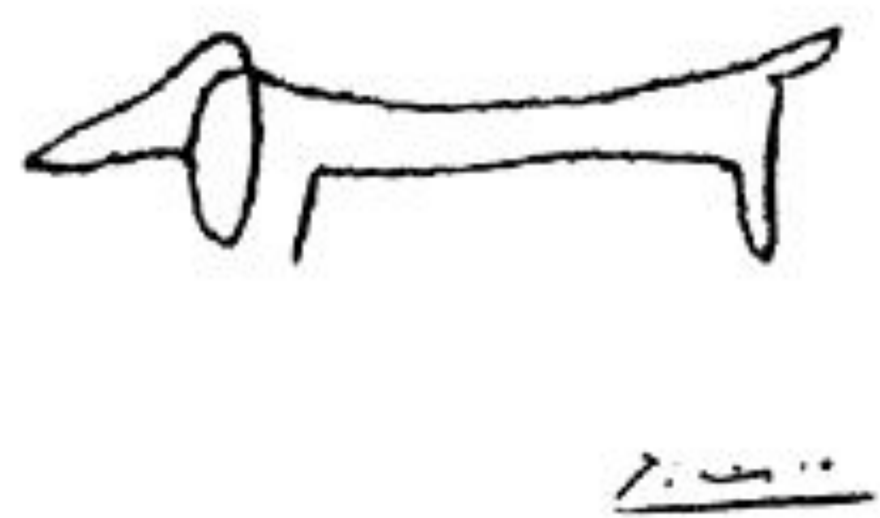
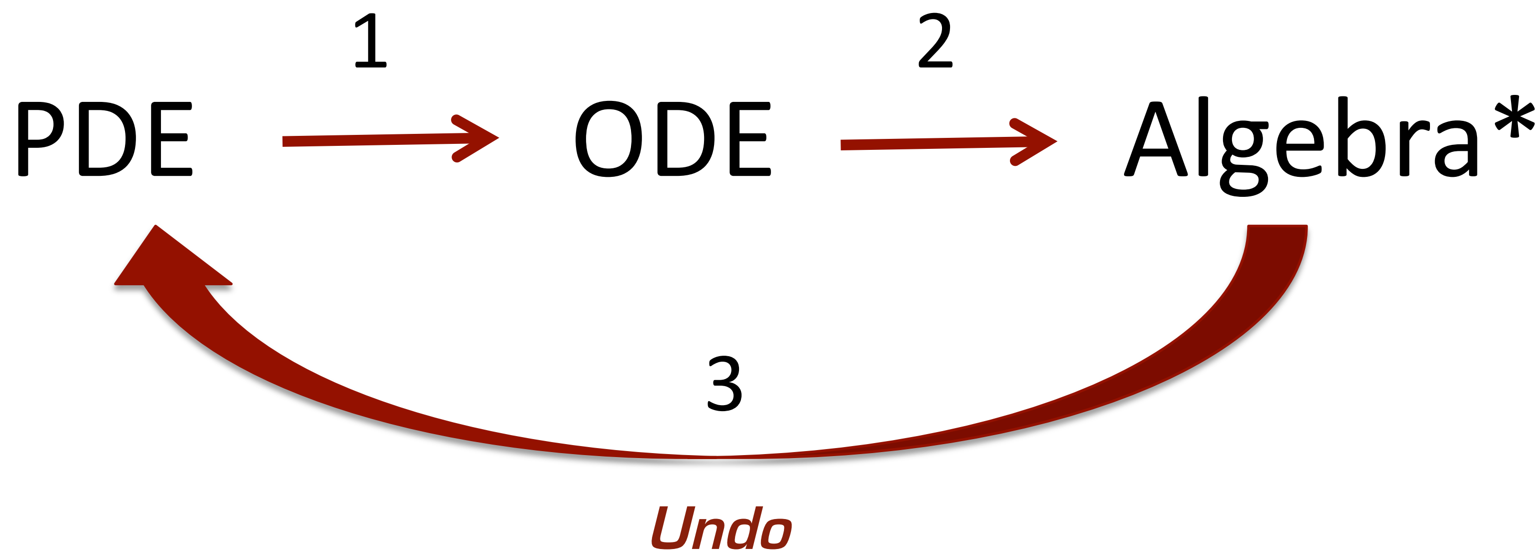


PDEs & Complex Dynamical Systems

Dimensionality Reduction

- Getting the right basis
  - POD modes
- Minimal dynamics
  - dynamical systems of the cheap

How to solve a PDE: 3 easy steps



$$\dot{u} = N(x, u, u_x, u_{xx}, \ldots)$$

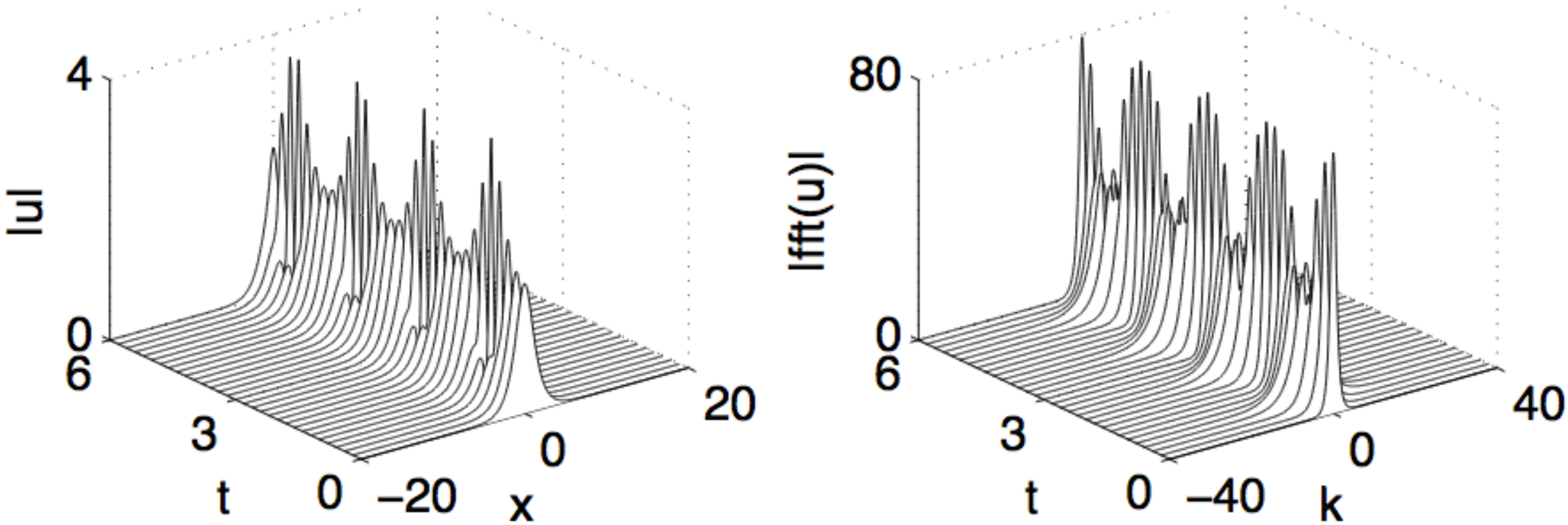
Eigenfunction expansion (separation of variables)

- $u[x,t] = \Sigma\, a_n(t)\phi_n(x)$

* Sad fact: This is all you know how to do!

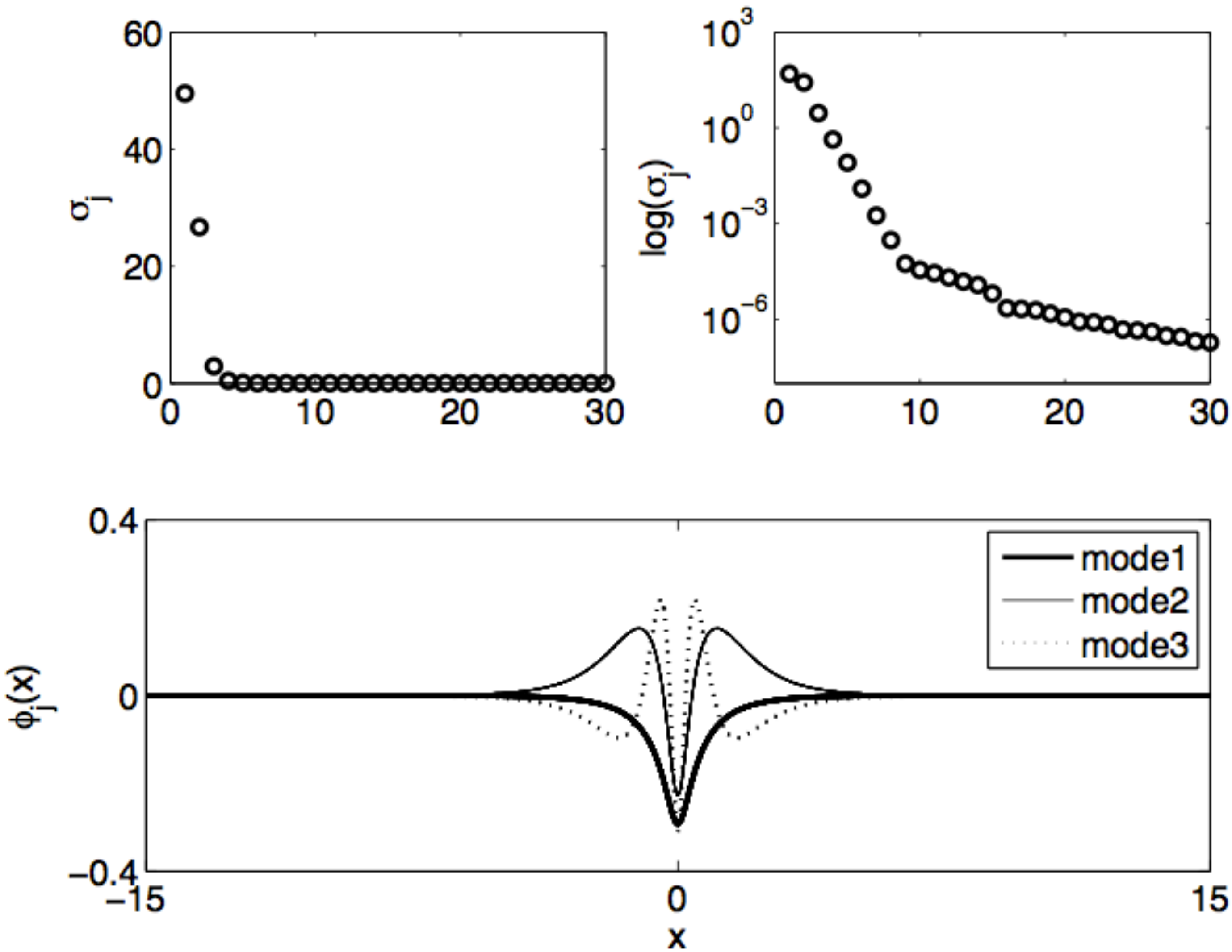
IST on the Cheap

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = 0$$



2 soliton

SVD on Data



60
40
20
0
σj
0
10
20
30
10^3
10^0
10^−3
10^−6
log(σj)
0
10
20
30
0.4
0
−0.4
φj(x)
−15
0
15
x
mode1
mode2
mode3

Galerkin Expansion

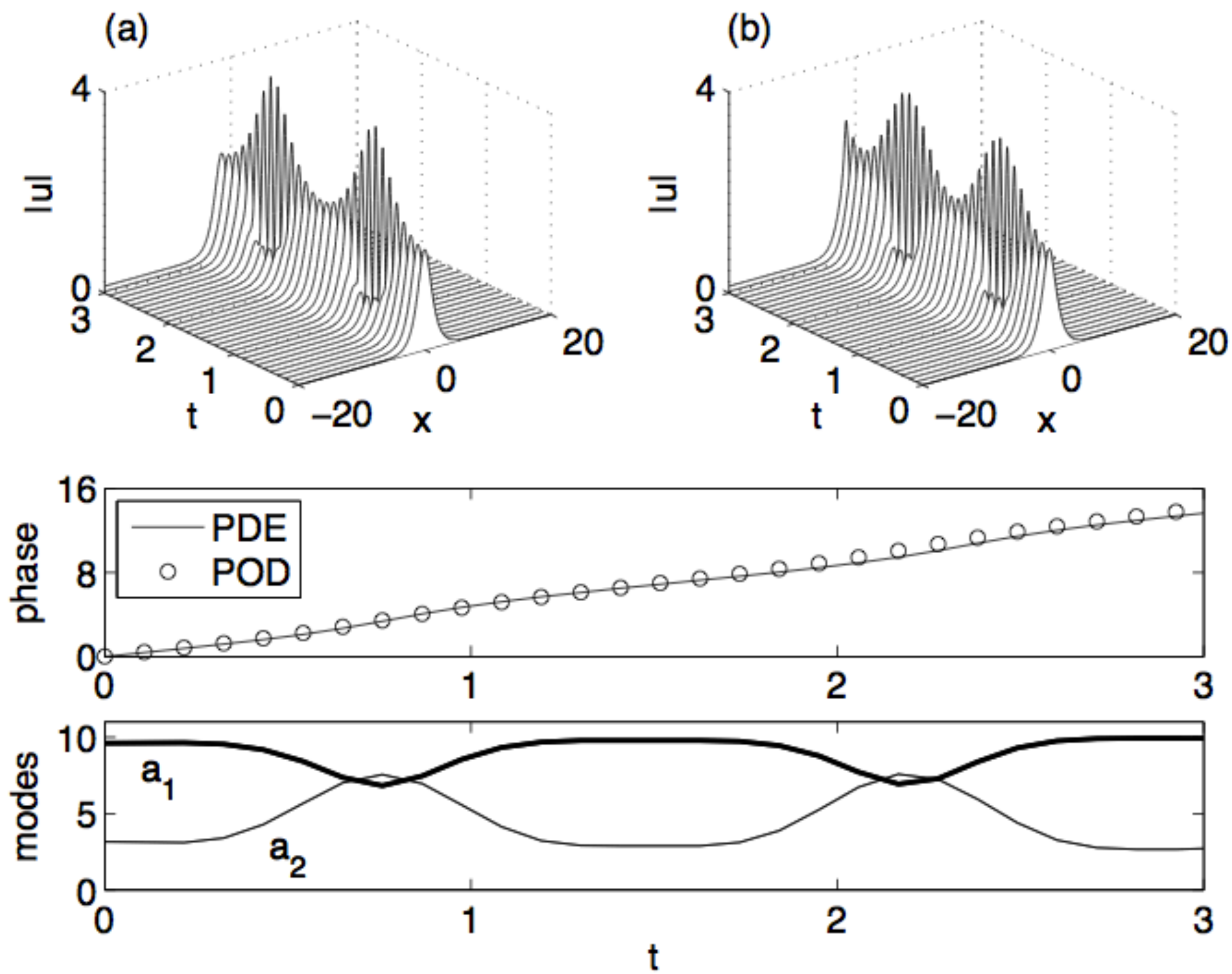
$$u(x,t) = a_1(t)\phi_1(x) + a_2(t)\phi_2(x)$$

$$\begin{aligned}
& i a_{1t} + \alpha_{11} a_1 + \alpha_{12} a_2 + \left(\beta_{111}|a_1|^2 + 2\beta_{211}|a_2|^2\right) a_1 \\
& \quad + \left(\beta_{121}|a_1|^2 + 2\beta_{221}|a_2|^2\right) a_2 + \sigma_{121} a_1^2 a_2^* + \sigma_{211} a_2^2 a_1^* = 0 \\
& i a_{2t} + \alpha_{21} a_1 + \alpha_{22} a_2 + \left(\beta_{112}|a_1|^2 + 2\beta_{212}|a_2|^2\right) a_1 \\
& \quad + \left(\beta_{122}|a_1|^2 + 2\beta_{222}|a_2|^2\right) a_2 + \sigma_{122} a_1^2 a_2^* + \sigma_{212} a_2^2 a_1^* = 0
\end{aligned}$$

$$\begin{aligned}
\alpha_{jk} &= (\phi_{j\,xx}, \phi_k)/2 \\
\beta_{jkl} &= (|\phi_j|^2 \phi_k, \phi_l) \\
\sigma_{jkl} &= (\phi_j^2 \phi_k^*, \phi_l)
\end{aligned}$$

$$\begin{aligned}
a_1(0) &= \frac{(2\text{sech}(x), \phi_1)}{(\phi_1, \phi_1)} \\
a_2(0) &= \frac{(2\text{sech}(x), \phi_2)}{(\phi_2, \phi_2)} .
\end{aligned}$$

Data Driven IST



Sensory Processing: Manduca Sexta

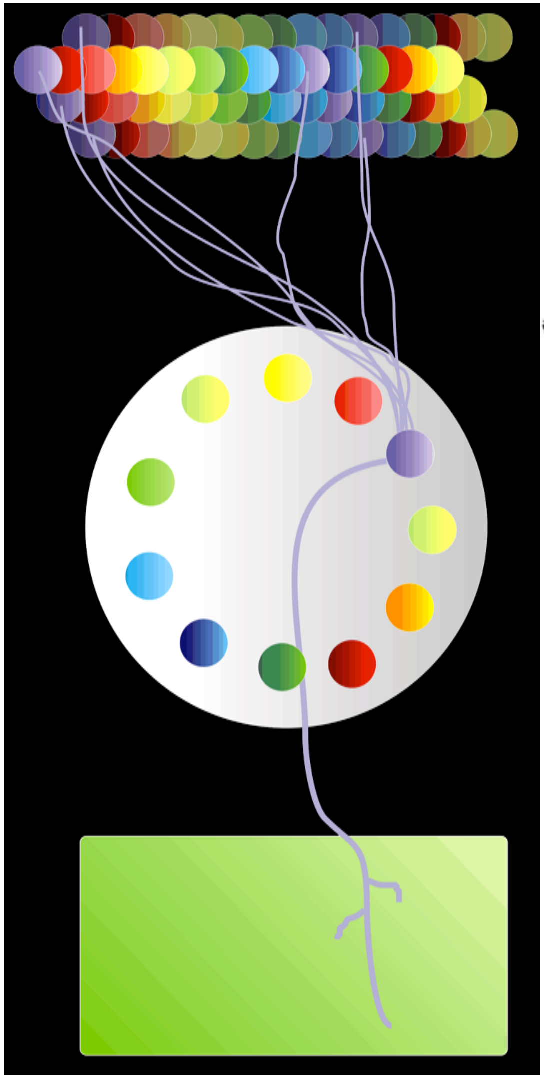
Olfactory receptor cells

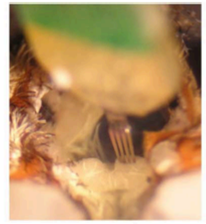
10^6 neurons

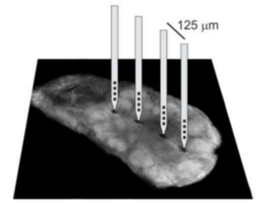
Antennal lobe (AL)

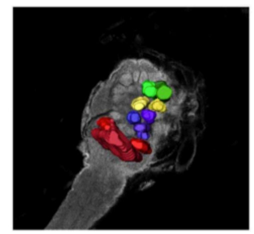
10^3 neurons

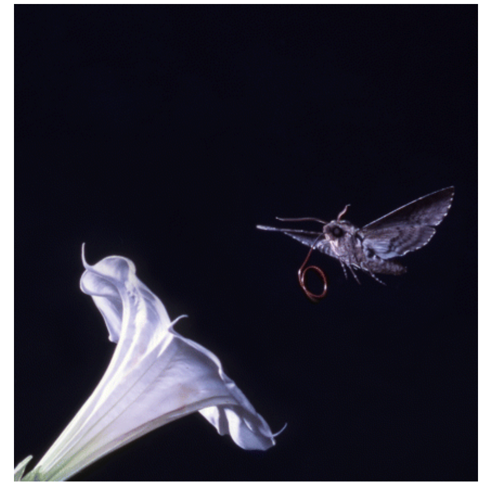
Mushroom body





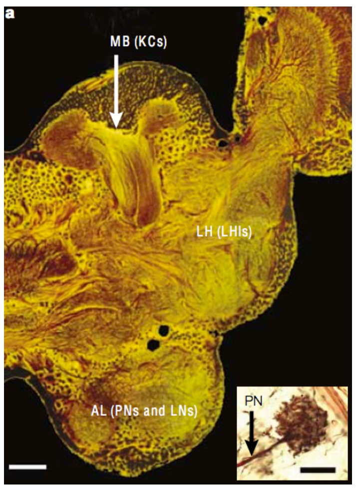


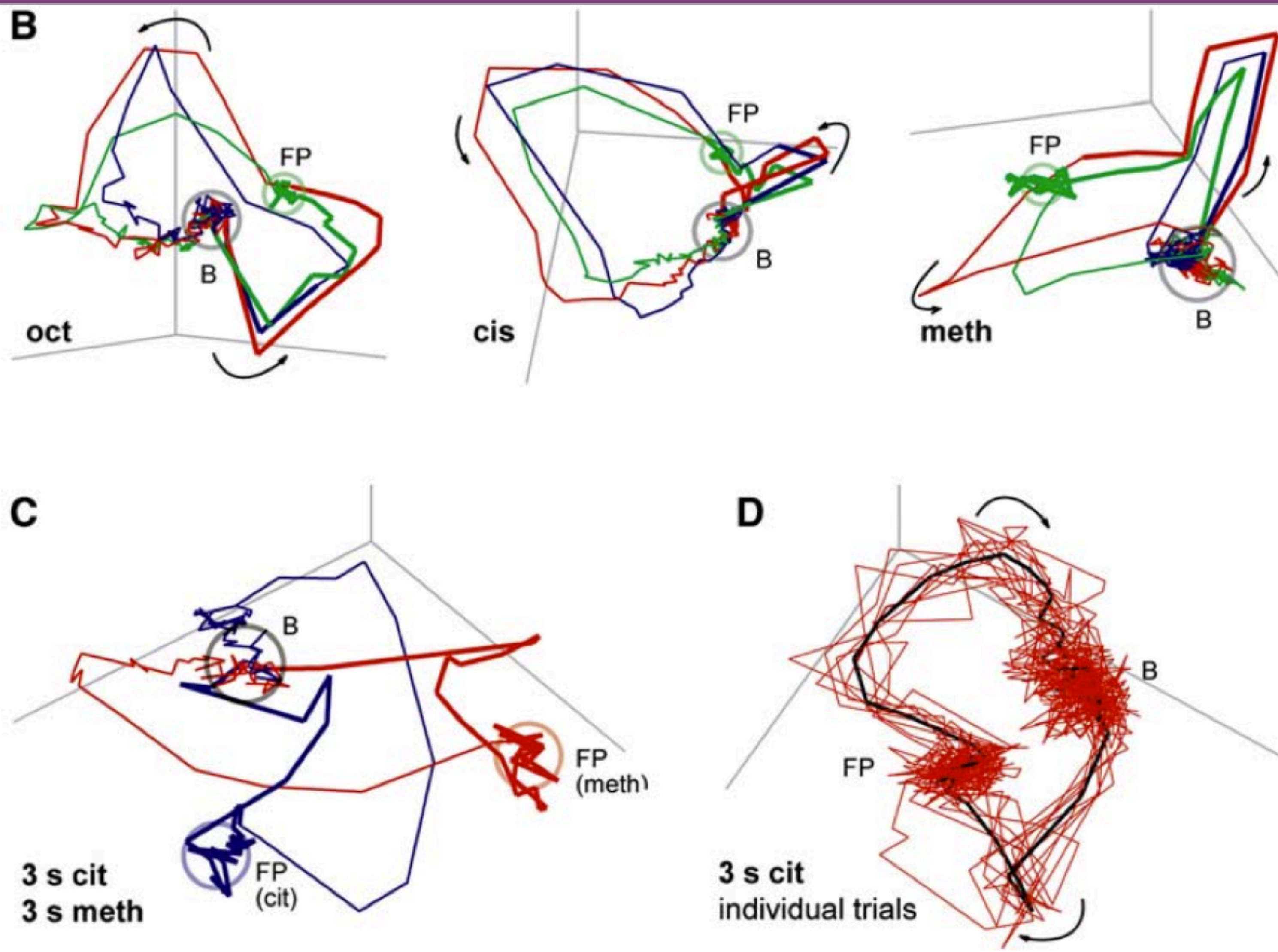




Eli Shlizerman + Jeff Riffell, UW Biology

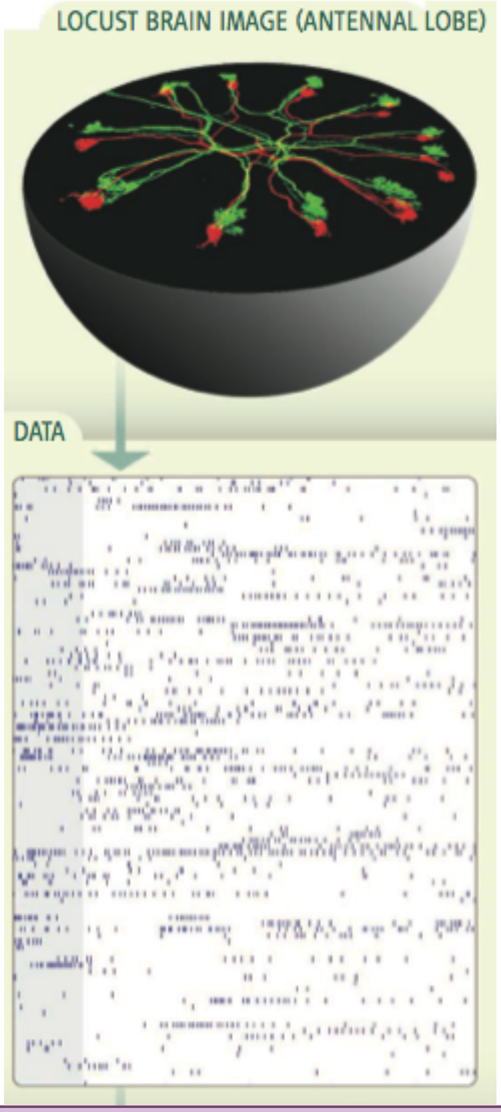
Dynamical Systems and Fixed Points (Locust)

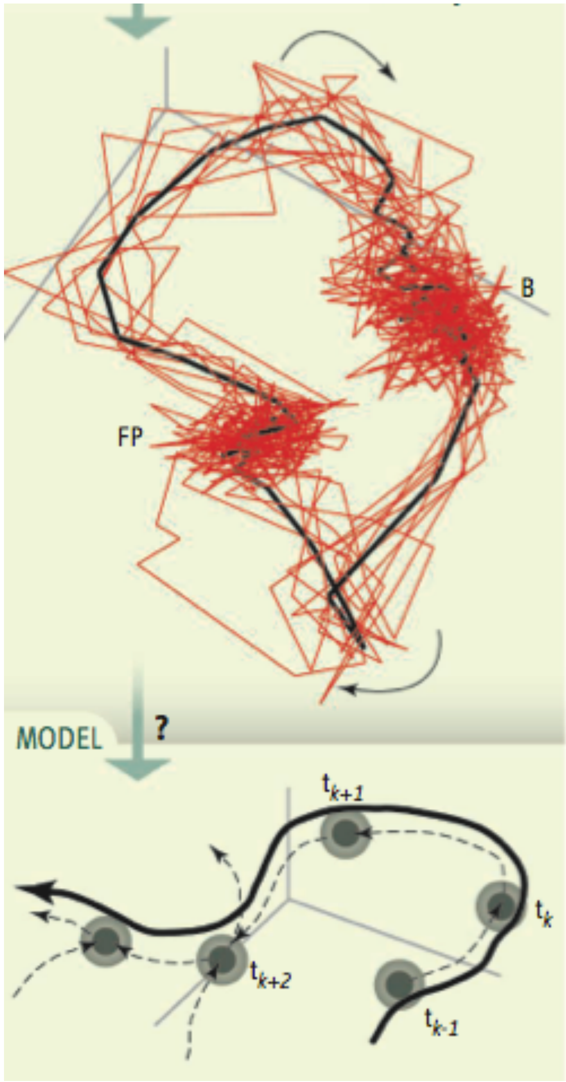




Mazor & Laurent, Neuron 2005

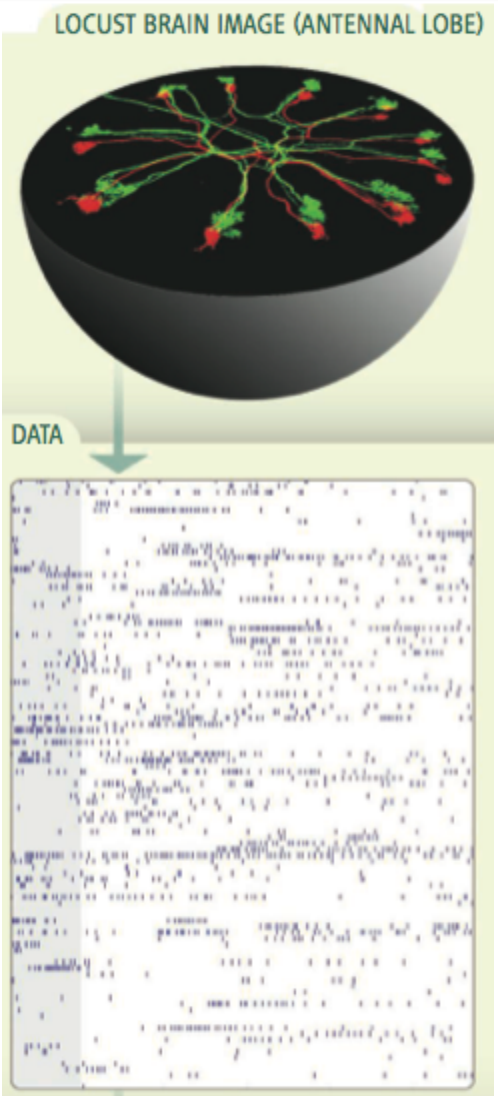
Encoding Dynamics

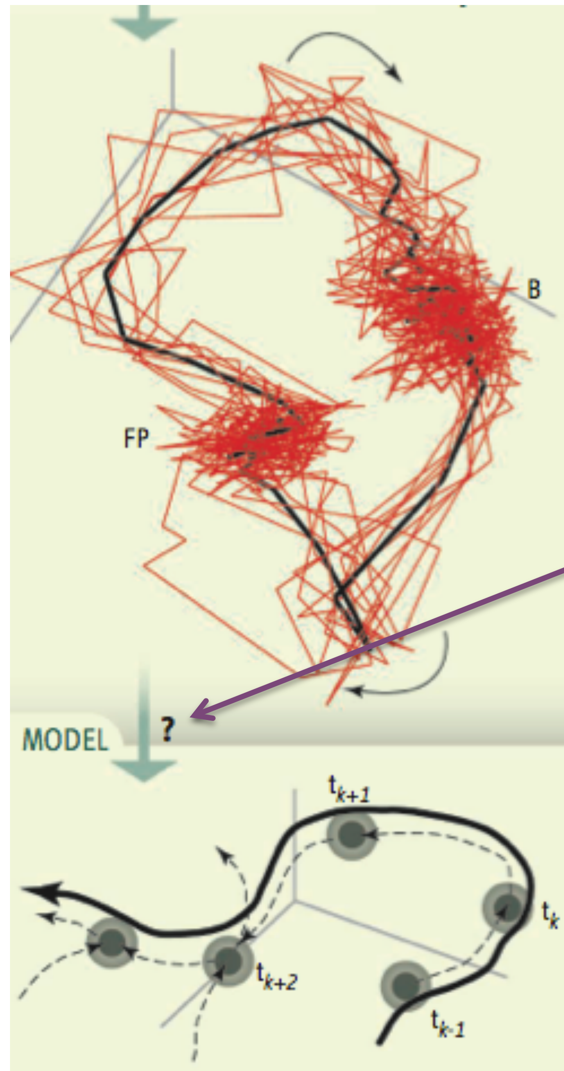




Rabinovich, Huerta, Laurent, Science (2008)

Encoding Dynamics (?)

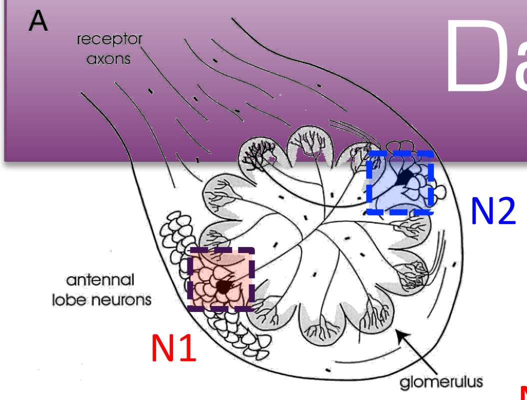


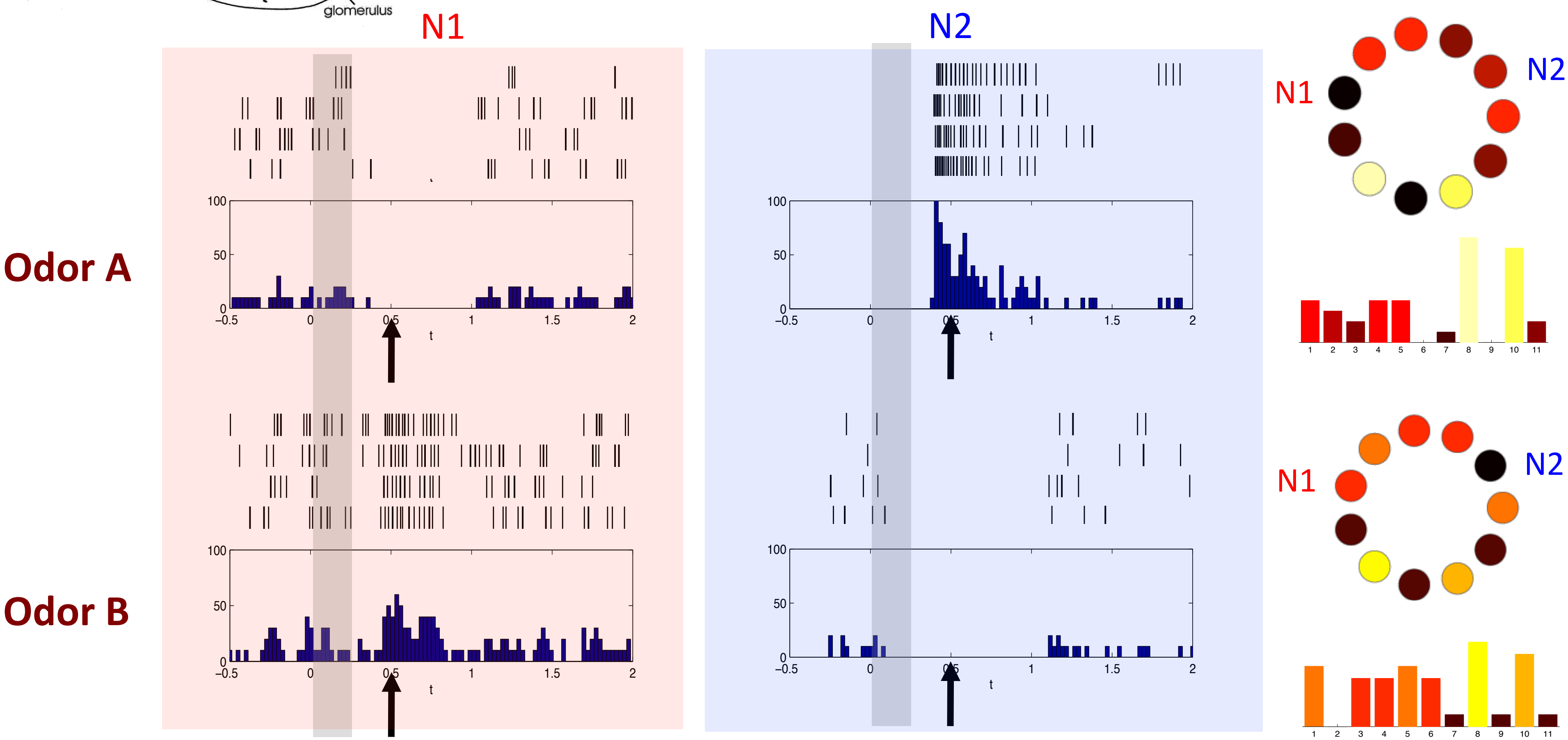


Our Goal

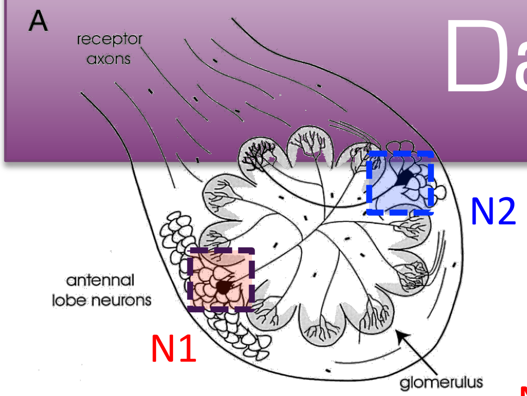
Rabinovich, Huerta, Laurent, Science (2008)

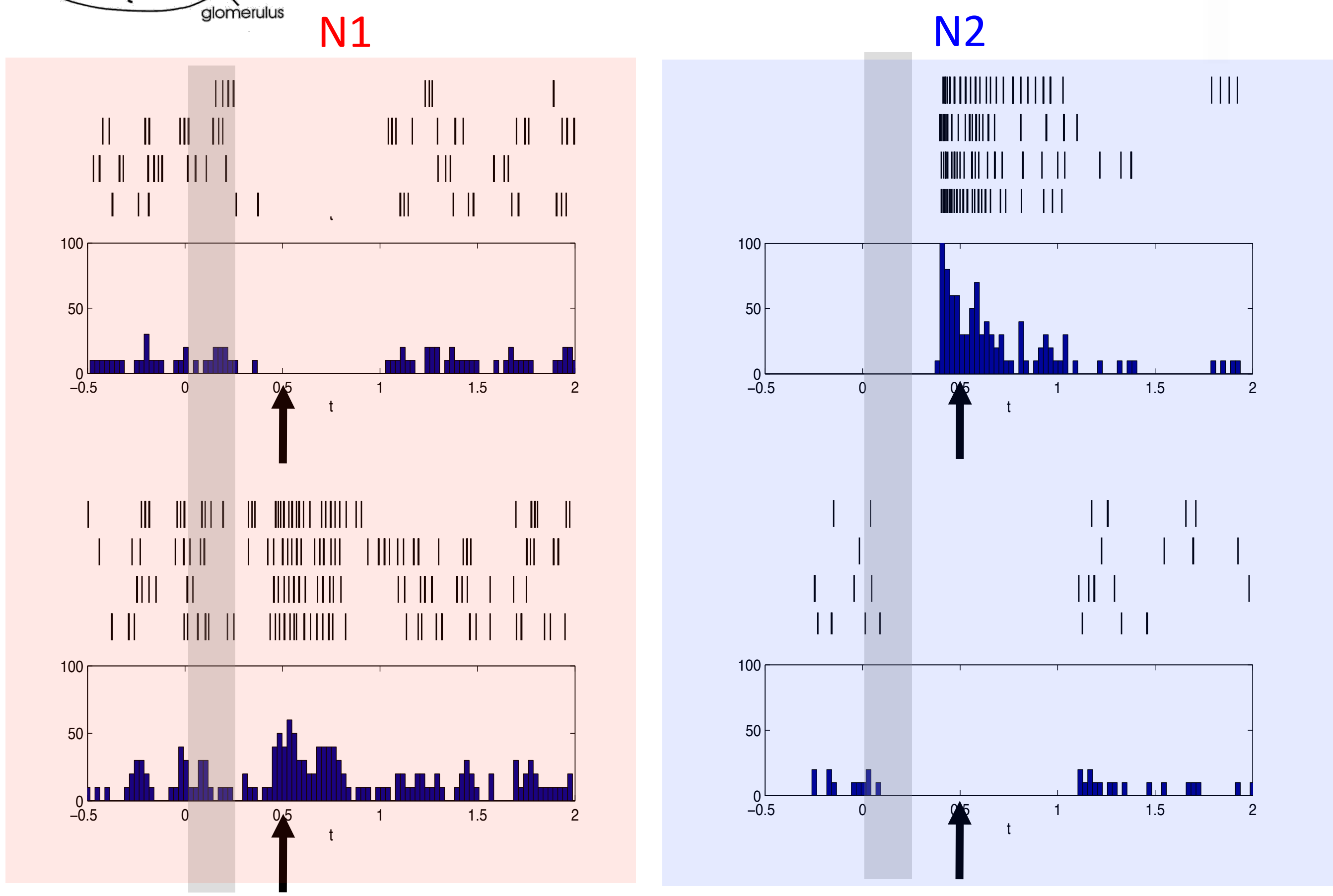
Data Driven Projections





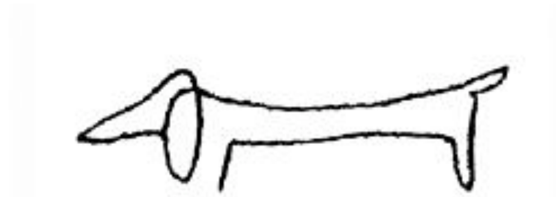
Data Driven Projections

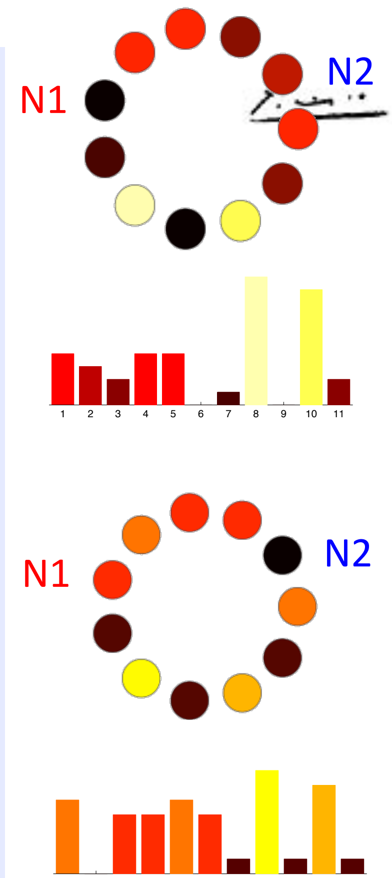




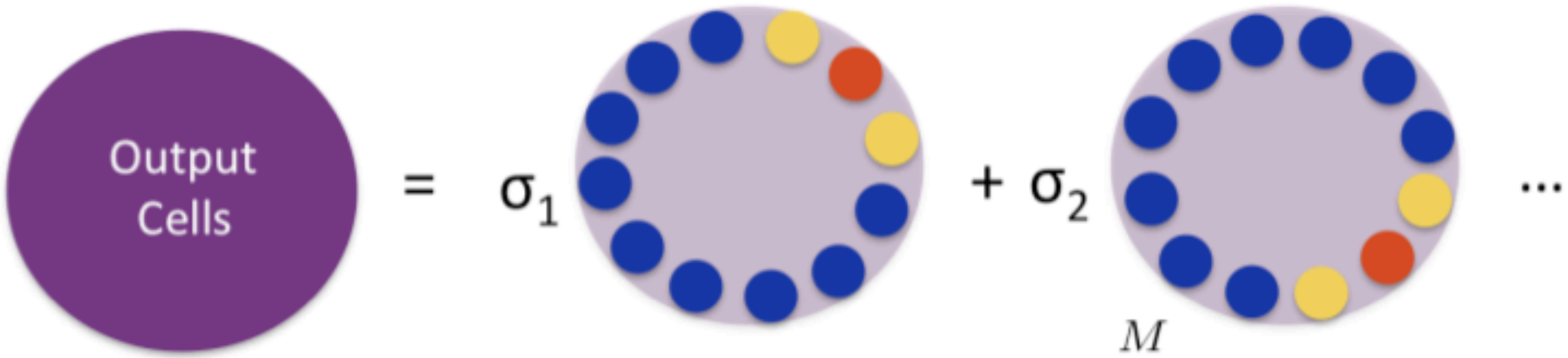
Odor A

Odor B





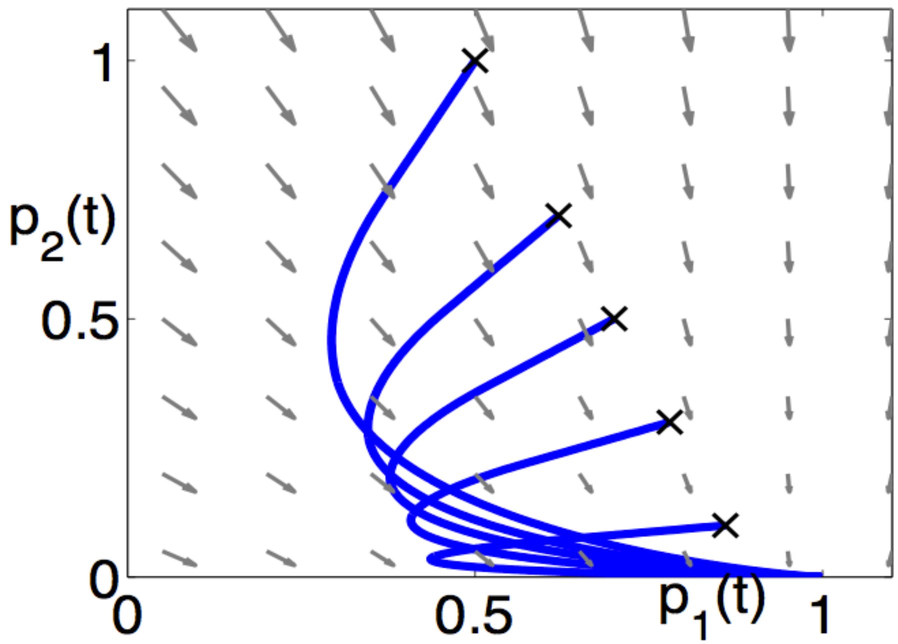
Spatio-Temporal Coding Modes

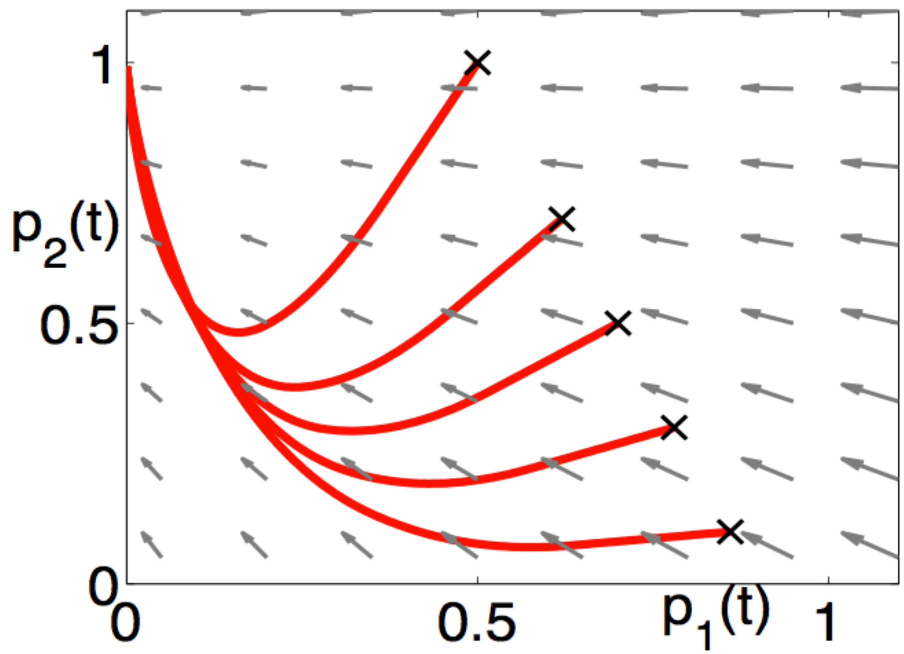


$$\vec{V}^{out}(t) = \sum_{m=1}^{M} \sigma_m a_m(t) \vec{V}^{PC}$$

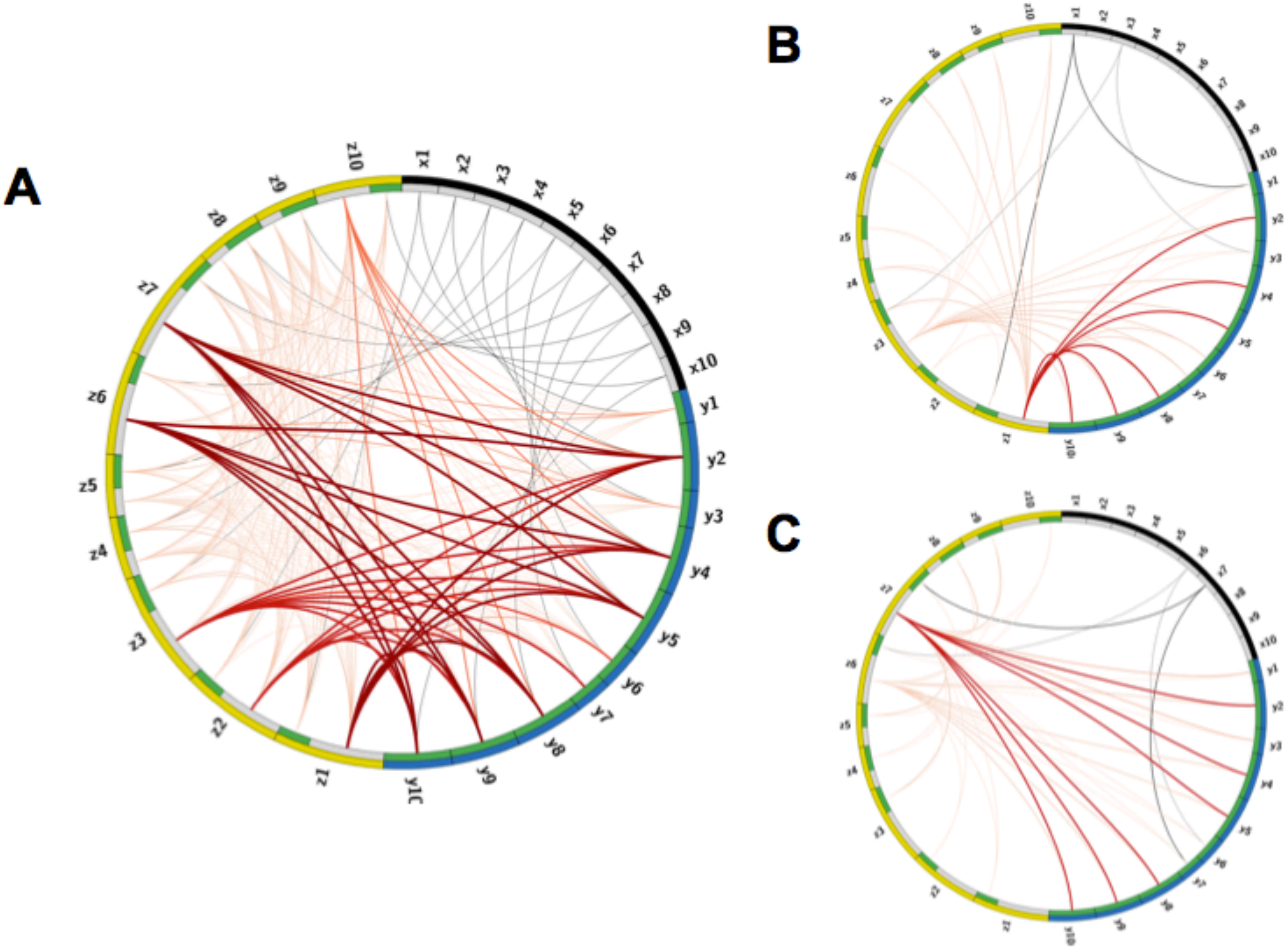
Model of olfaction: ***Spatio-temporal competing modes***

Encoding Dynamics



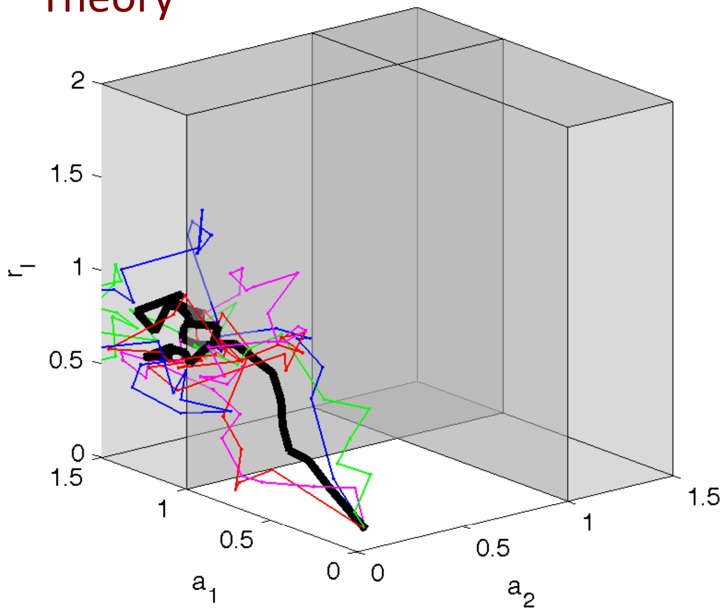


AL Connectome



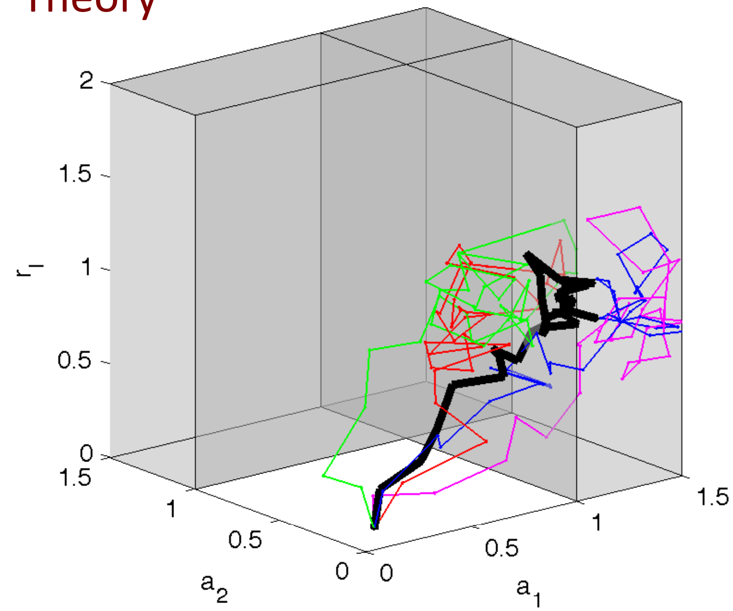
Encoding Competition Dynamics

Theory

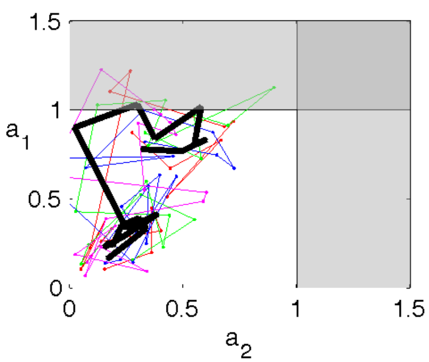


3X3 ODEs

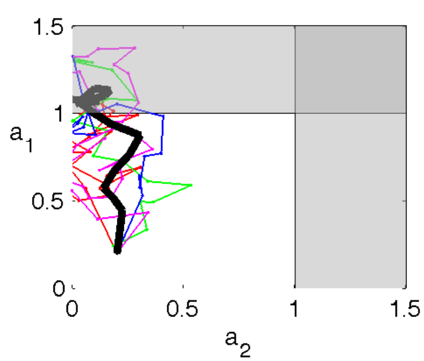
Theory



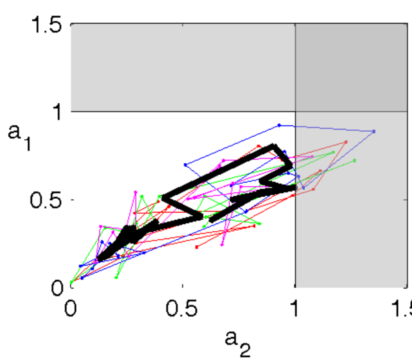
Experiment



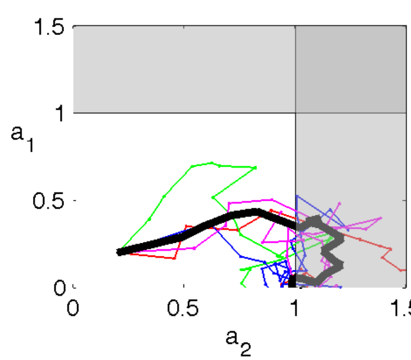
Theory



Experiment

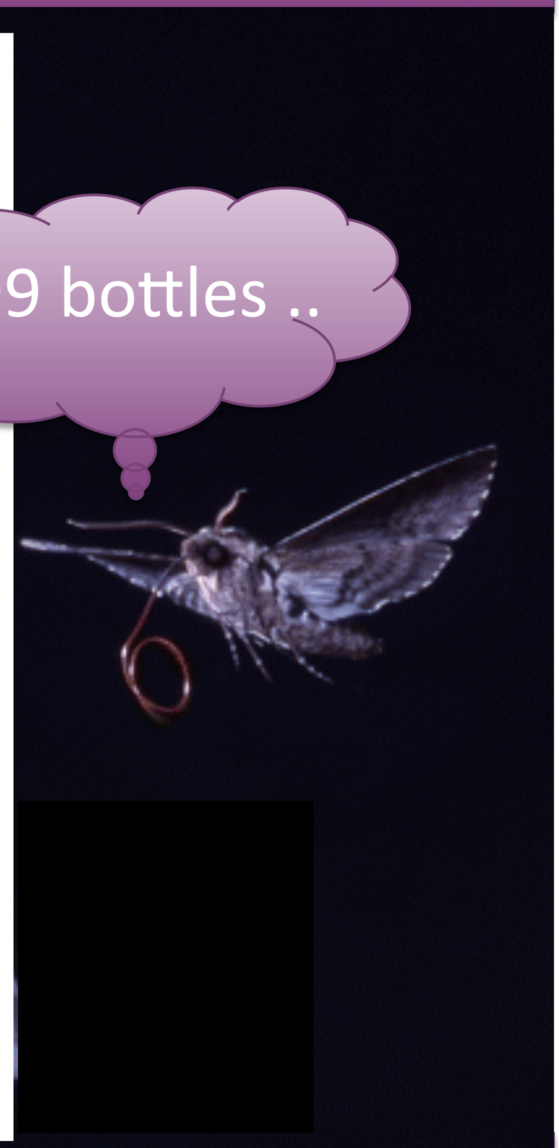


Theory



The Role of Inhibition

Lack of Decision Making



Learning and Enconding: Library Structure

New Code Generation

YUMM!!!!

Nature + Nurture

A Nice Talk by Candes

Blah blah blah ...

compressive sensing ....

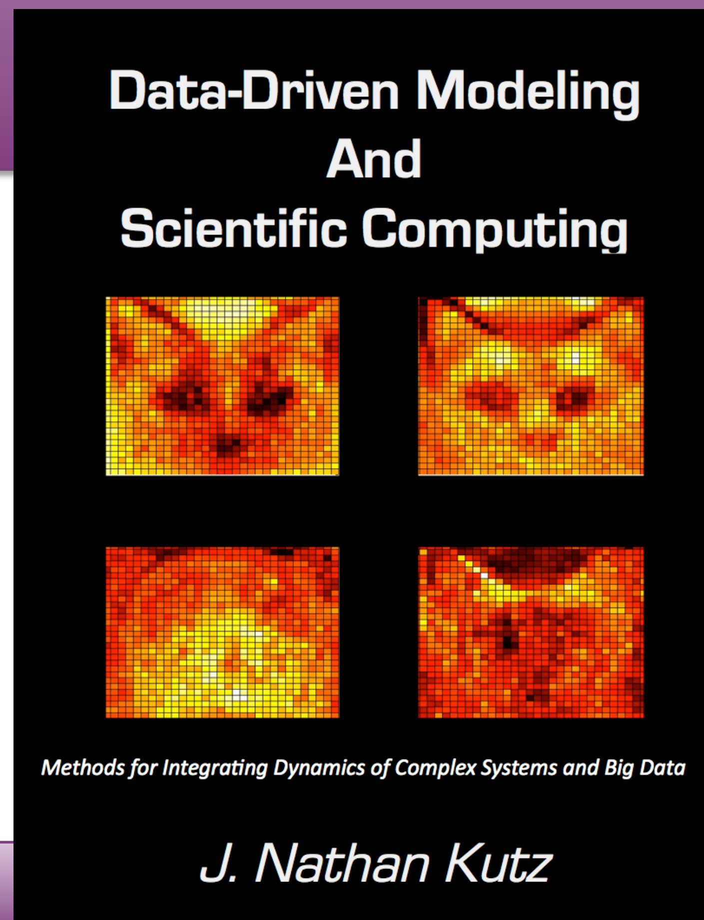
blah blah blah ....

L1 optimization ... sparsity .....

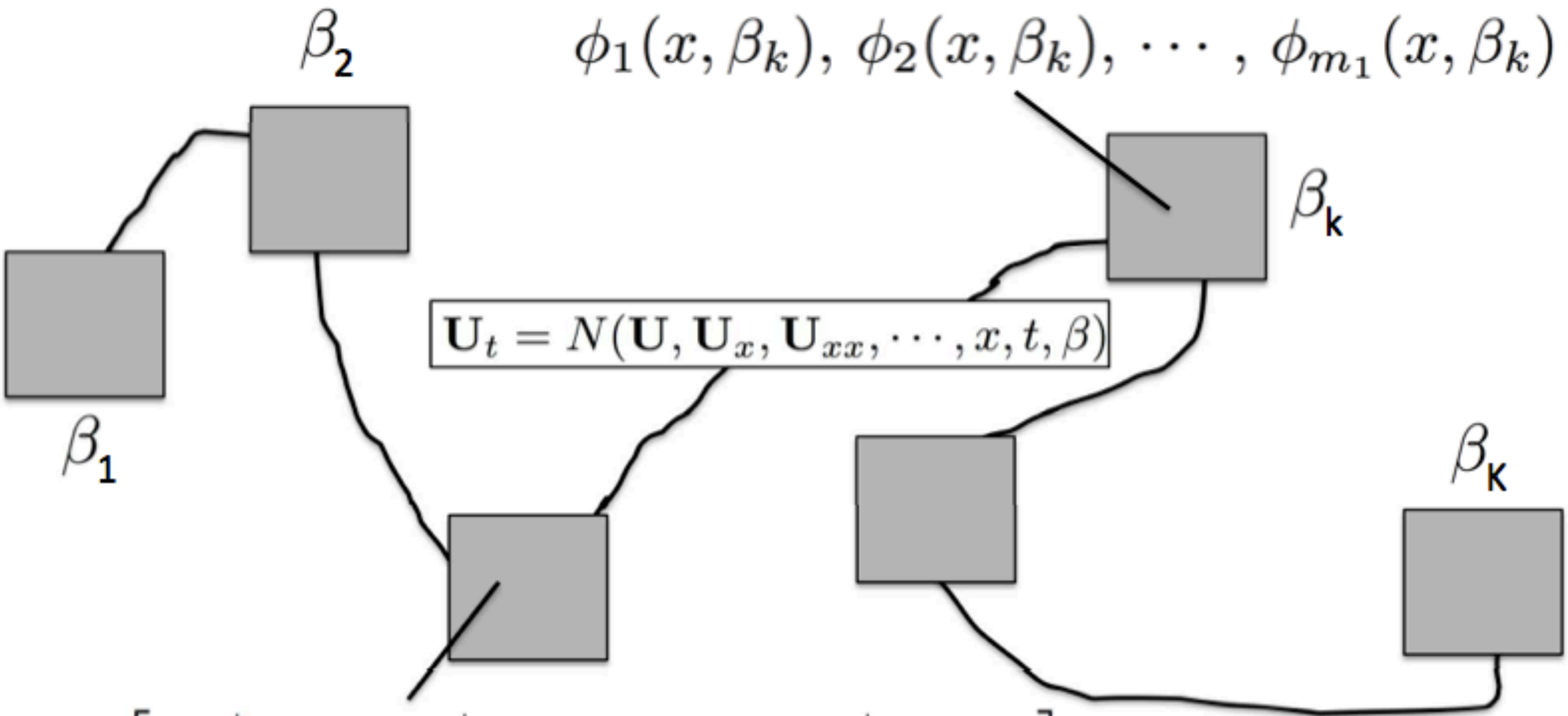
blah blah blah ..... C'est magnifique!

Integrating Machine Learning and Compressive Sensing

Example from this book (Oxford 2013)



General PDE Reduction Concepts



$$\mathbf{A}_{\beta_k} = \begin{bmatrix} | & | & & | \\ \mathbf{U}(t,x) & \mathbf{U}(t+\Delta t, x) & \cdots & \mathbf{U}(t+m_k\Delta t, x) \\ | & | & & | \end{bmatrix}$$

Expand in Low-Dimensional Bases

$$\mathbf{U}(x,t) = \sum_{k=1}^{K} \sum_{m=1}^{m_k} a_{km}(t)\phi_m(x,\beta_k) = \psi_L \mathbf{a}$$

sparse measurement matrix

solution expansion

$$\hat{\mathbf{U}} = \Phi \mathbf{U}$$

$$\hat{\mathbf{U}} = (\Phi \psi_L)\mathbf{a}$$

highly –undetermined system!

Promoting Sparsity

$$\mathbf{U}(x,t) = \sum_{k=1}^{K} \sum_{m=1}^{m_k} a_{km}(t)\phi_m(x,\beta_k) = \psi_L \mathbf{a}$$

$$\hat{\mathbf{U}} = (\Phi\psi_L)\mathbf{a}$$ solution expansion

promote sparsity

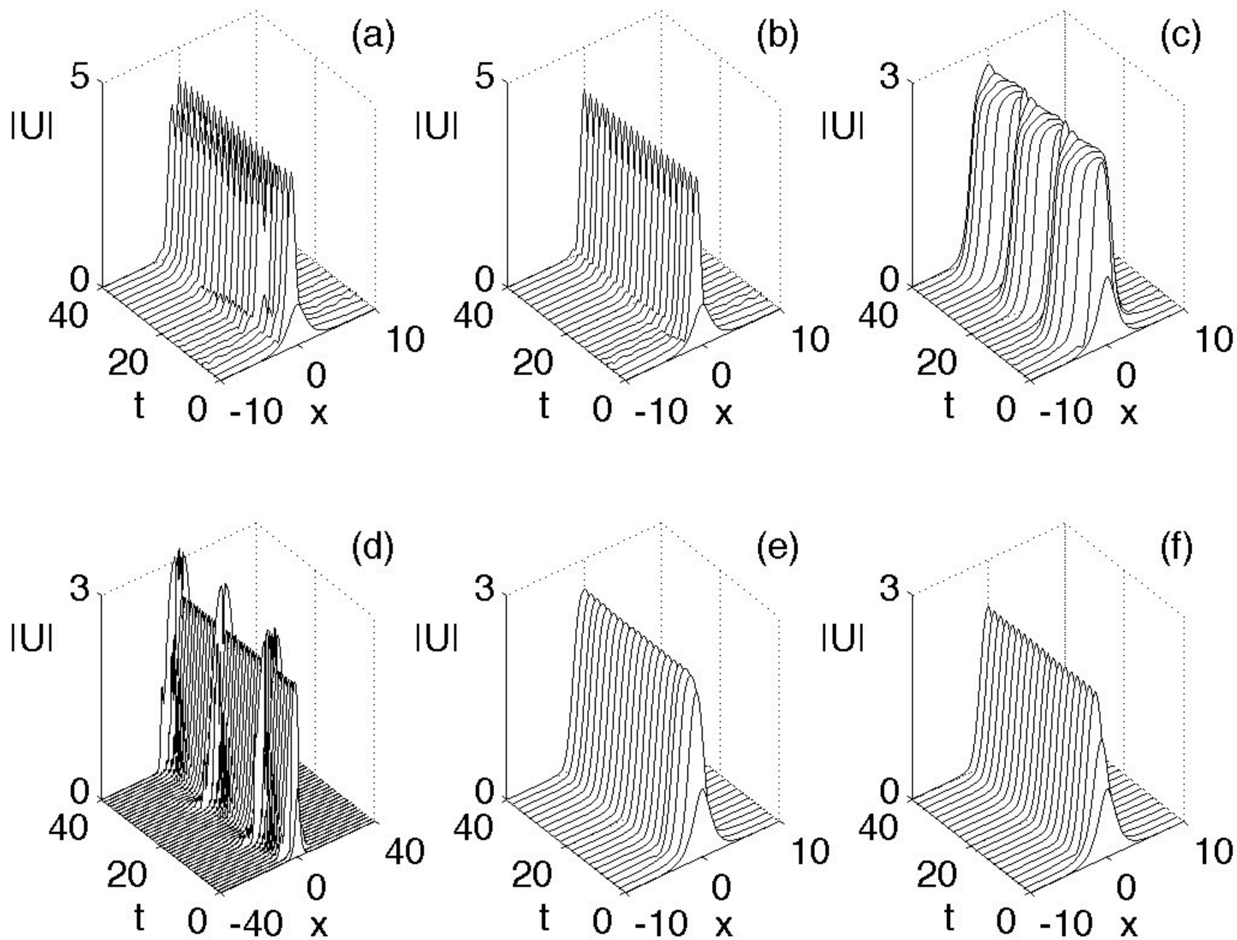
$$\min ||a||_1$$

CQGLE - A Prototypical Example

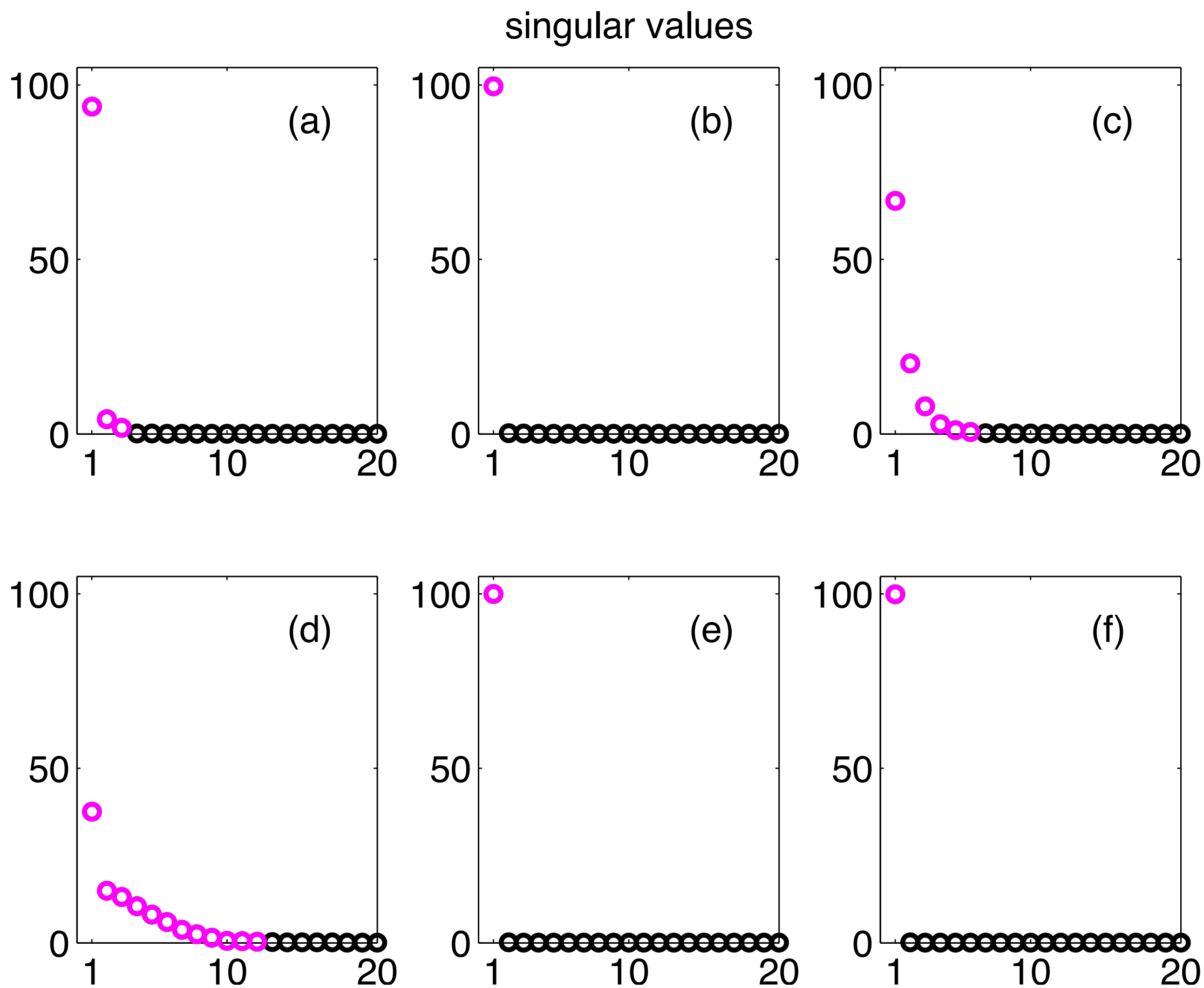
$$iU_t + \left(\frac{1}{2} - i\tau\right) U_{xx} - i\kappa U_{xxxx} + (1 - i\mu)|U|^2 U + (\nu - i\sigma)|U|^4 U - i\gamma U = 0$$

	τ	κ	μ	ν	σ	γ	description
β_1	-0.3	-0.05	1.45	0	-0.1	-0.5	3-hump, localized solution
β_2	-0.3	-0.05	1.4	0	-0.1	-0.5	localized solution with small side lobes
β_3	0.08	0	0.66	-0.1	-0.1	-0.1	breather
β_4	0.125	0	1	-0.6	-0.1	-0.1	exploding (periodic) dissipative soliton
β_5	0.08	-0.05	0.6	-0.1	-0.1	-0.1	fat bump
β_6	0.08	-0.05	0.5	-0.1	-0.1	-0.1	dissipative soliton

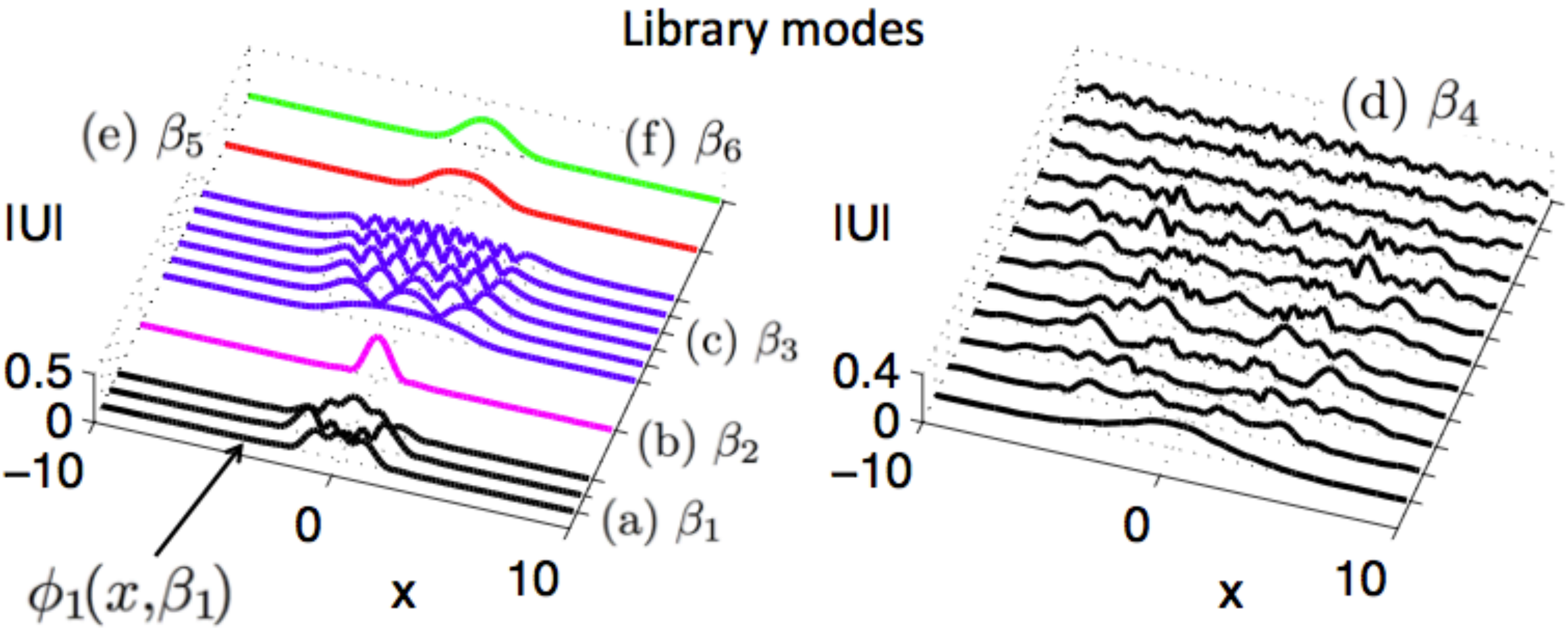
PDE Dynamics



Low-Dimensionality



Library Modes



Measurements and Reconstruction

$$\beta = \begin{cases} \beta_1 & t \in [0, 100) \\ \beta_2 & t \in [100, 200) \\ \beta_3 & t \in [200, 300] \end{cases}$$

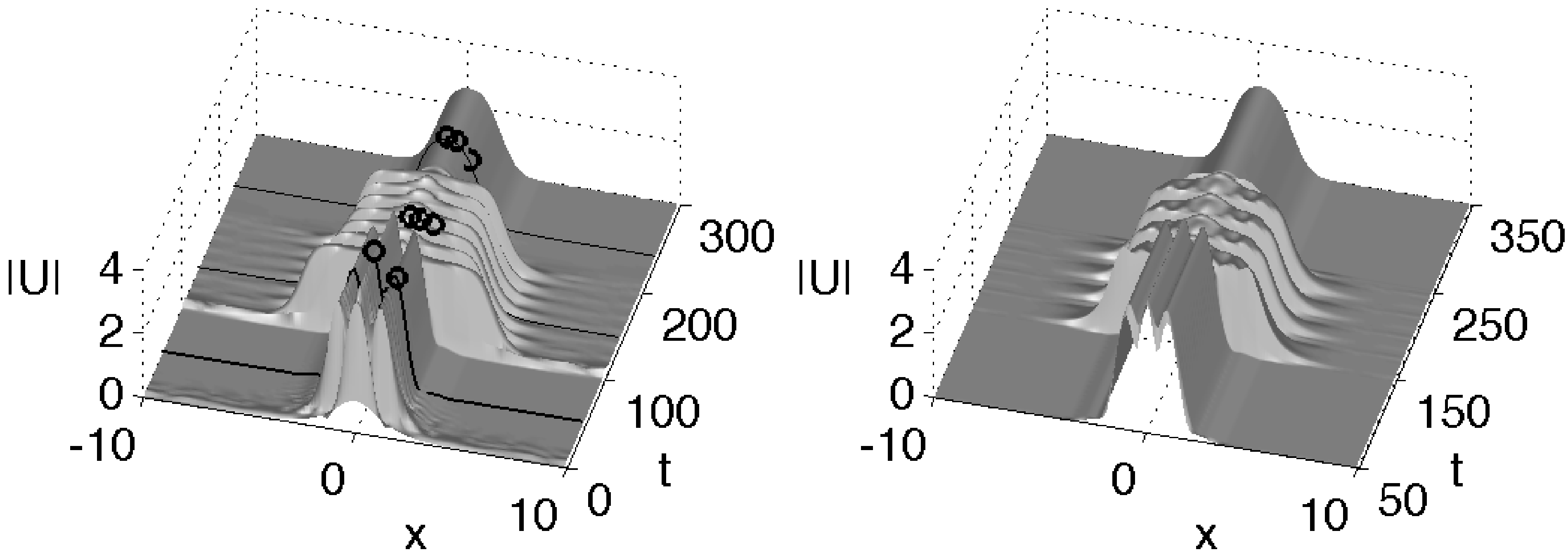
$m = 3$ measurements at $x_0 = 0, x_1 = 0.5, x_2 = 1$

$p = 24$ number of library elements

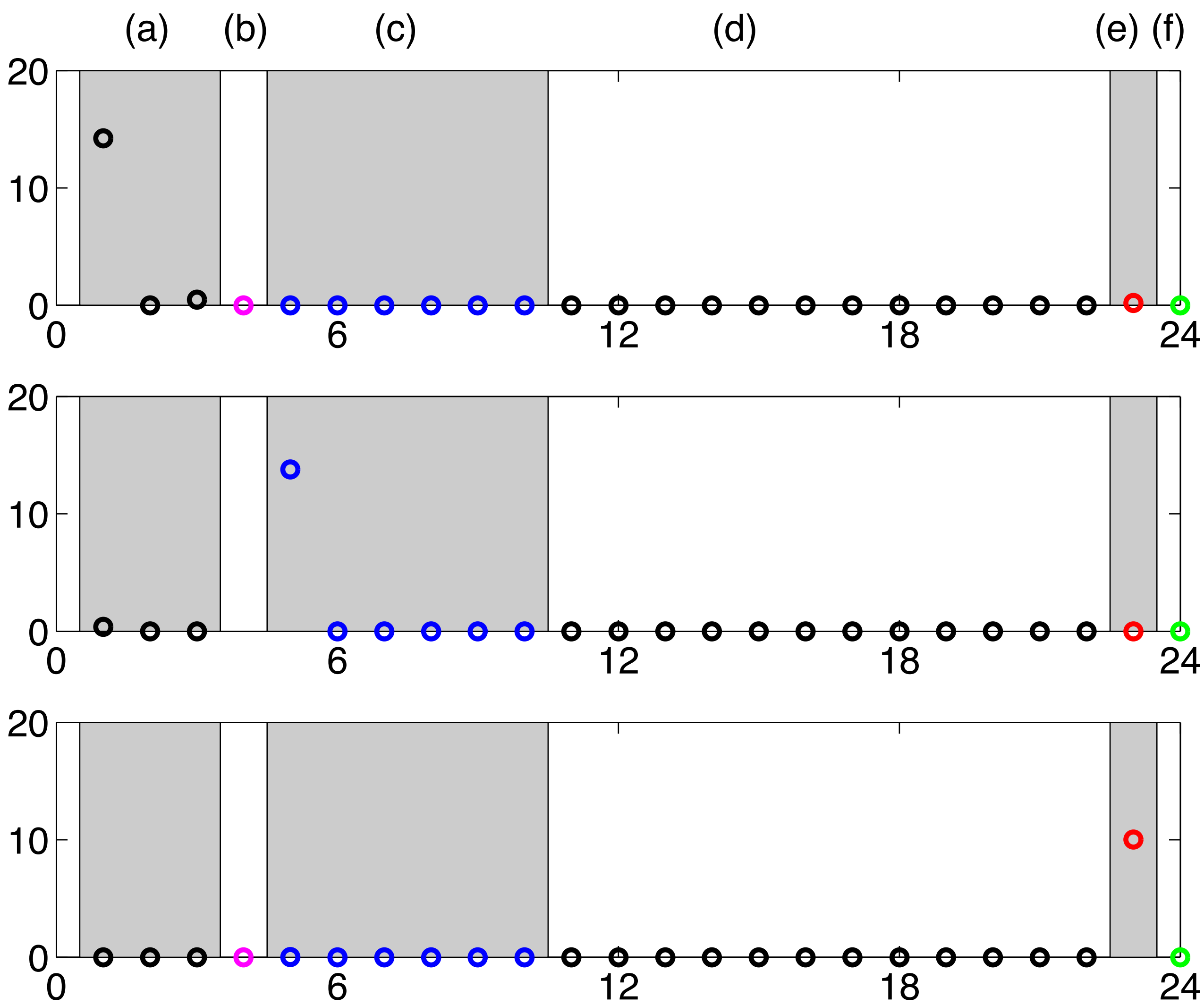
$n = 128$ dimension of original system

$$\Phi = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & & & & & & & 0 \\ 0 & \cdots & & & \cdots & 0 & 1 & 0 & \cdots & & & & 0 \\ 0 & \cdots & & & & & & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

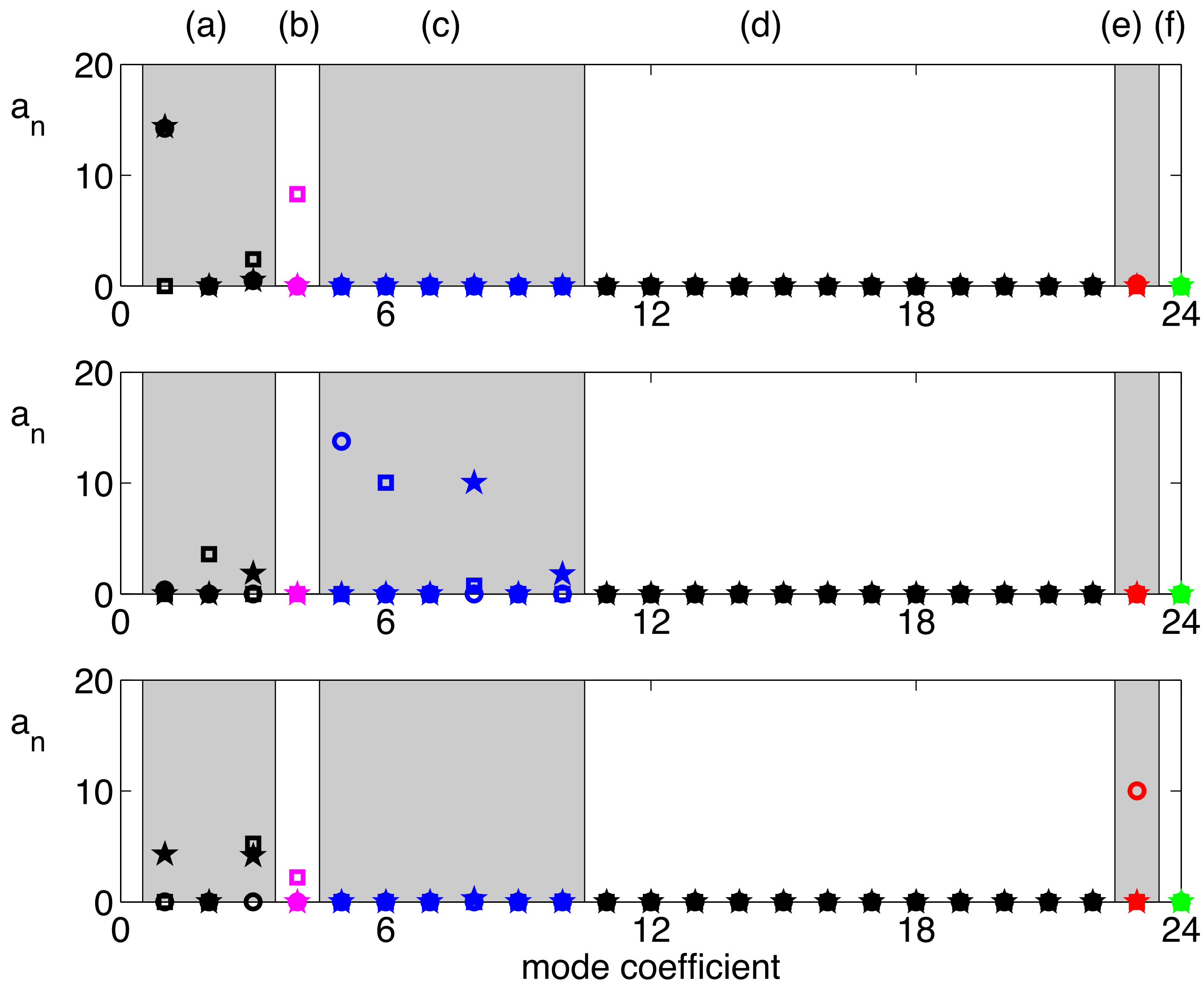
Measurement and Reconstruction



Library Identification



Poor Measurements



Algorithm

- Run extensive simulations of the governing PDE (23.1.1) at values of the bifurcation parameter β that produce signature dynamics in the system.

- For each signature value β_k $(k = 1, 2, \cdots, K)$, sample the low-rank dynamics (slow manifold) through a snapshot based method. Alternatively, other dimensionality reduction techniques can be used to produce the correct low-rank encodings such as dynamic mode decomposition or equation free modeling. Retain only the principal components (POD modes) that capture a prescribed high percentage of the data.

- Construct a library ψ_L of the p POD vectors sampled at the various β_k.

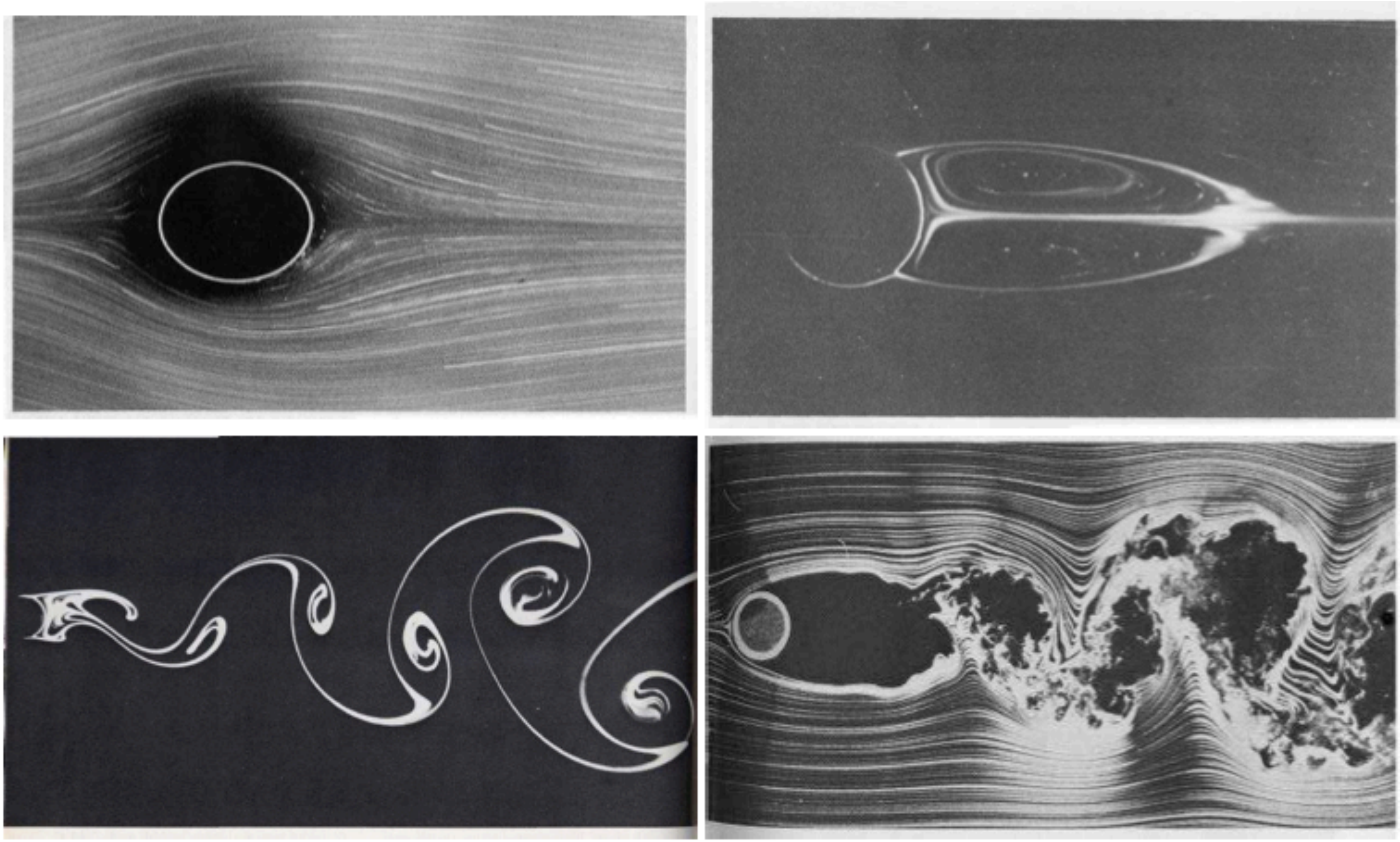
- Building the library is a one-time cost. Once done, use sparse measurements $\tilde{\mathbf{U}}$ and a compressive sensing scheme (convex L^1 optimization) to reconstruct the best-fit (in an L^1 sense) sparse evaluation of the full state of the system $\mathbf{U}$ at any given time t by evaluating $\mathbf{a}$.

- Either the sparse sampling can be used continuously to produce a sparse representation of the state of the system in terms of the library modes ψ_L, or occasional sparse sampling can be used to generate the correct low-rank POD dynamics of $\mathbf{a}$.

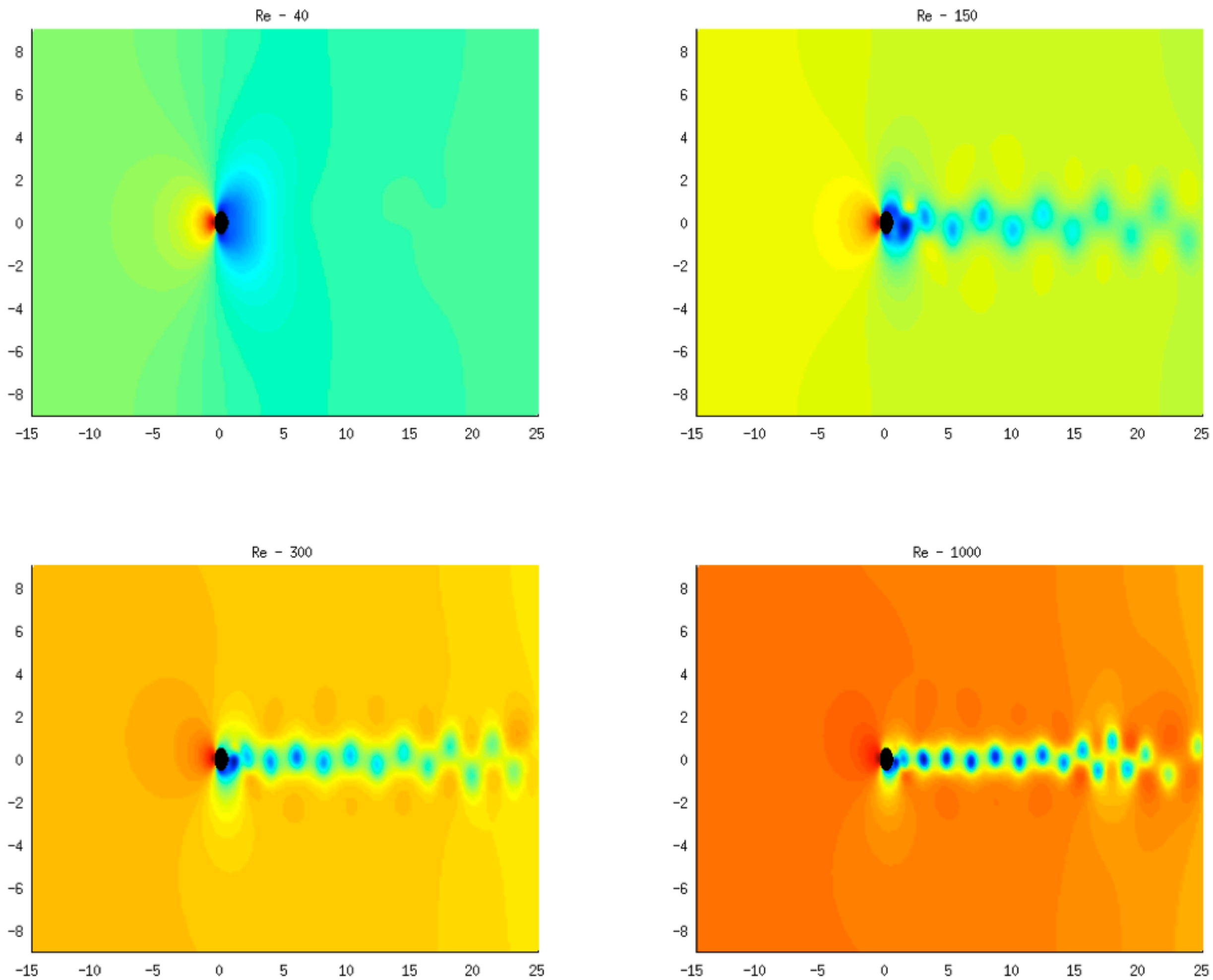
Flow Around a Cylinder

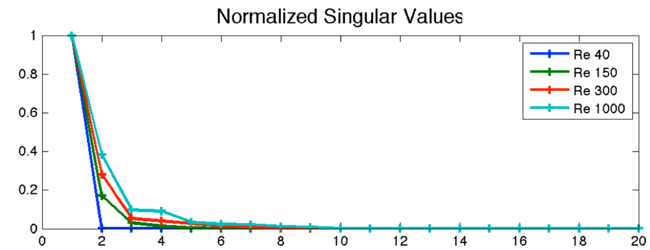
Ido Bright + Guang Lin
UW AMATH PNNL

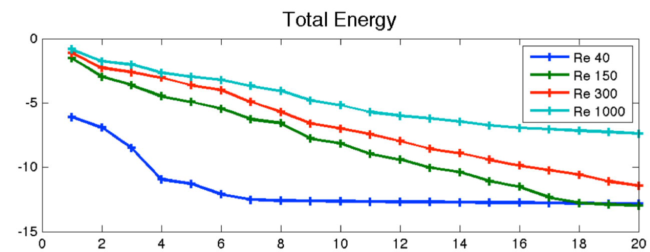
Compressive Sensing for PDE Dynamics



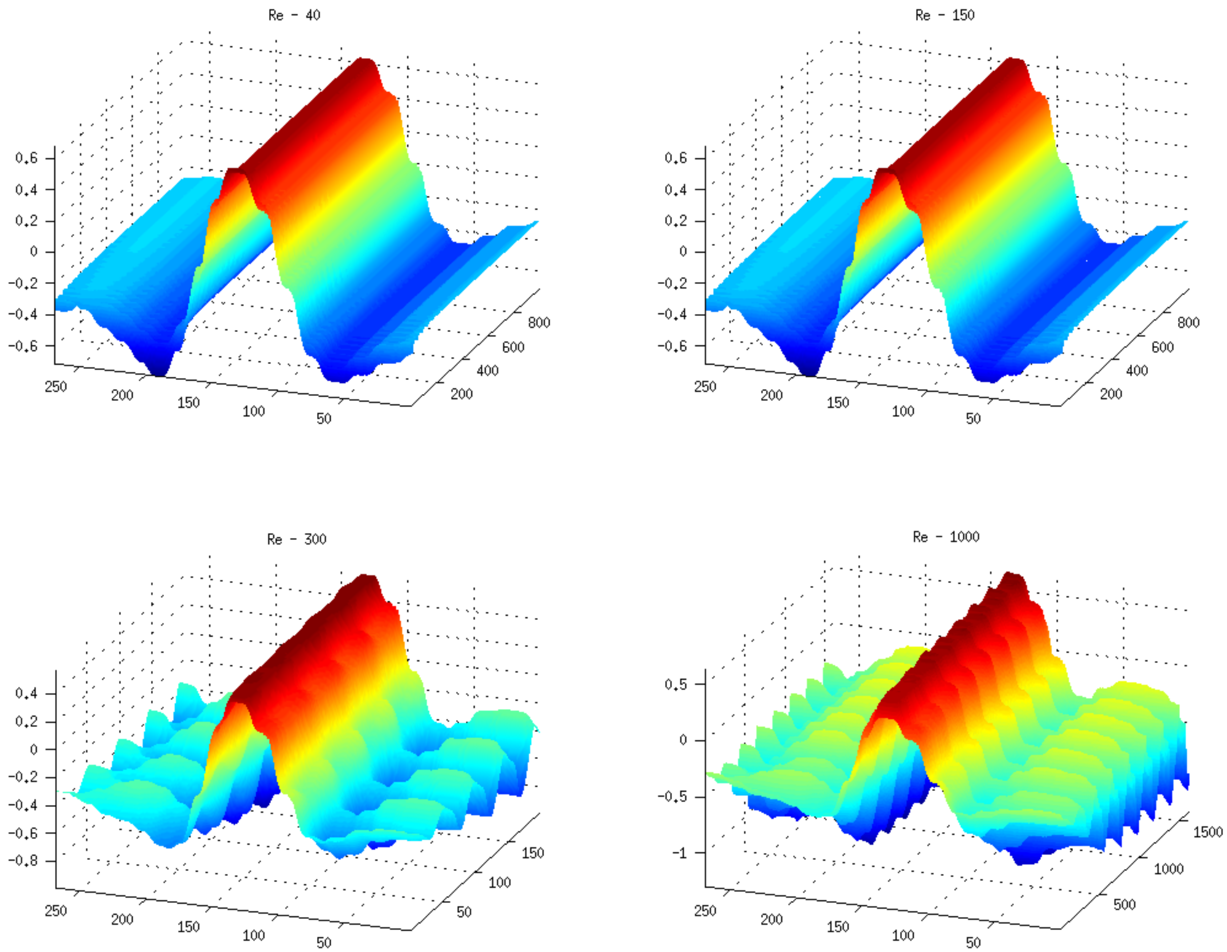
Pressure Field



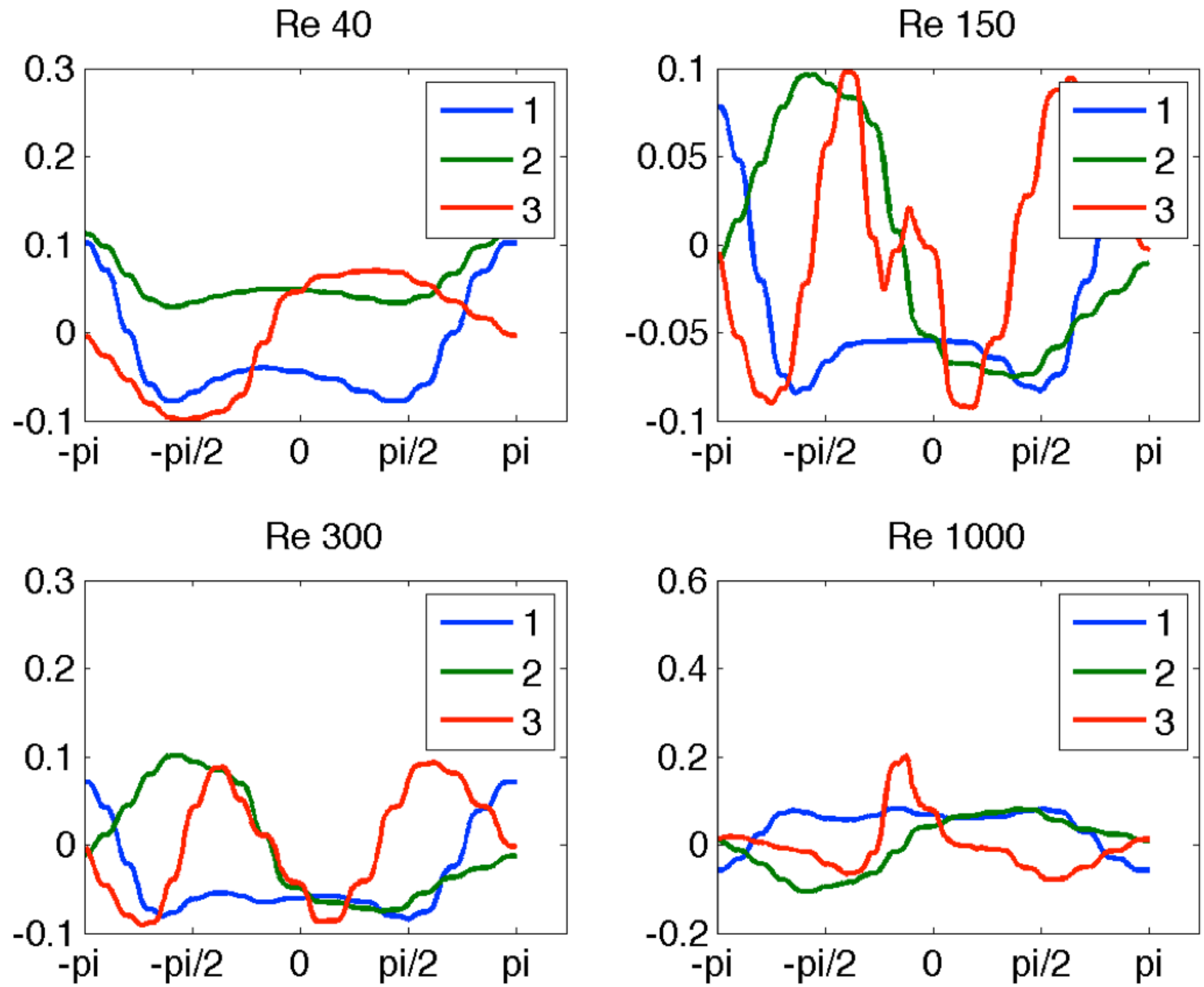




Pressure Dynamics on Cylinder

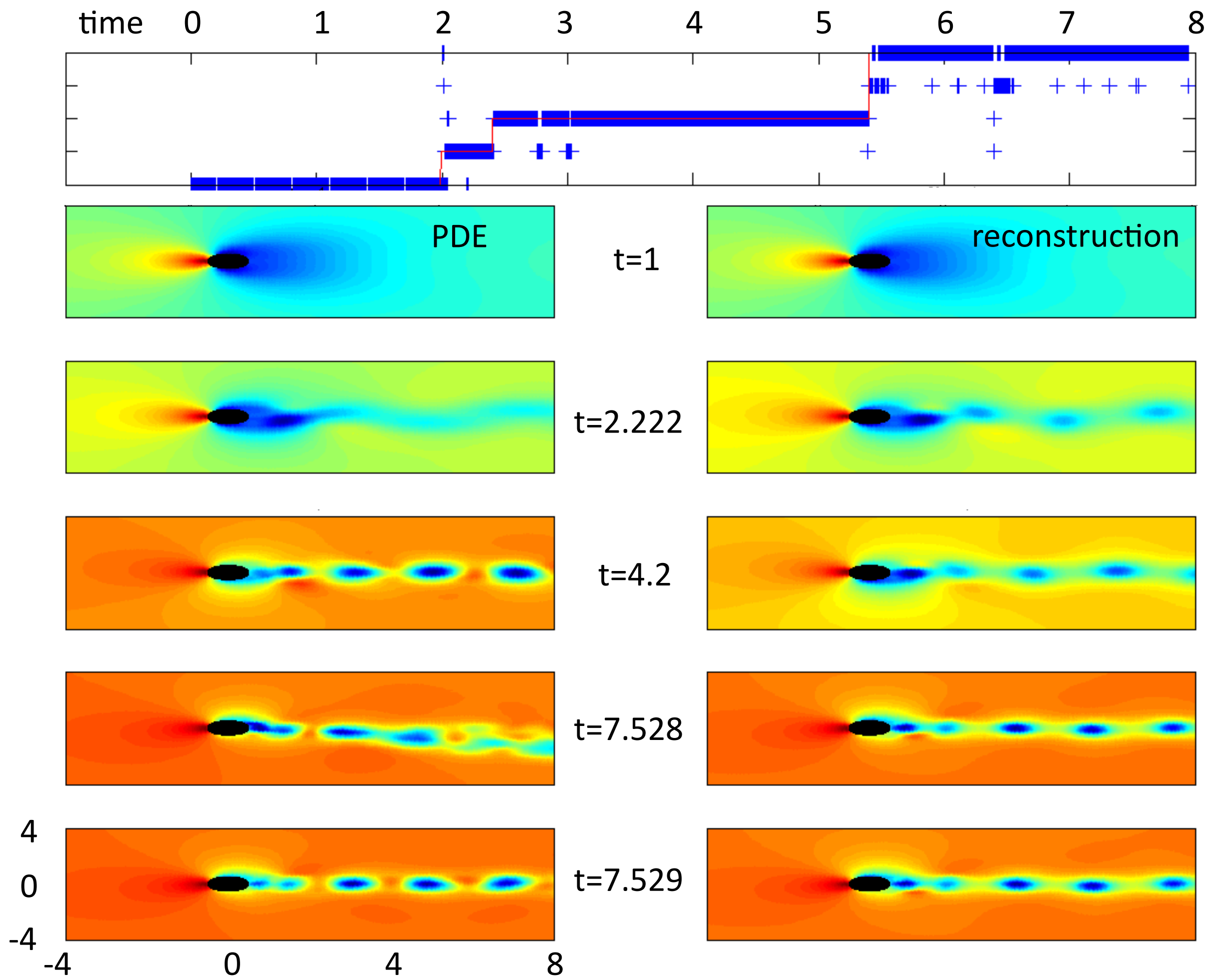


POD Modes



Re 40
0.3
0.2
0.1
0
-0.1
-pi
-pi/2
0
pi/2
pi
1
2
3
Re 150
0.1
0.05
0
-0.05
-0.1
-pi
-pi/2
0
pi/2
pi
1
2
3
Re 300
0.3
0.2
0.1
0
-0.1
-pi
-pi/2
0
pi/2
pi
1
2
3
Re 1000
0.6
0.4
0.2
0
-0.2
-pi
-pi/2
0
pi/2
pi
1
2
3

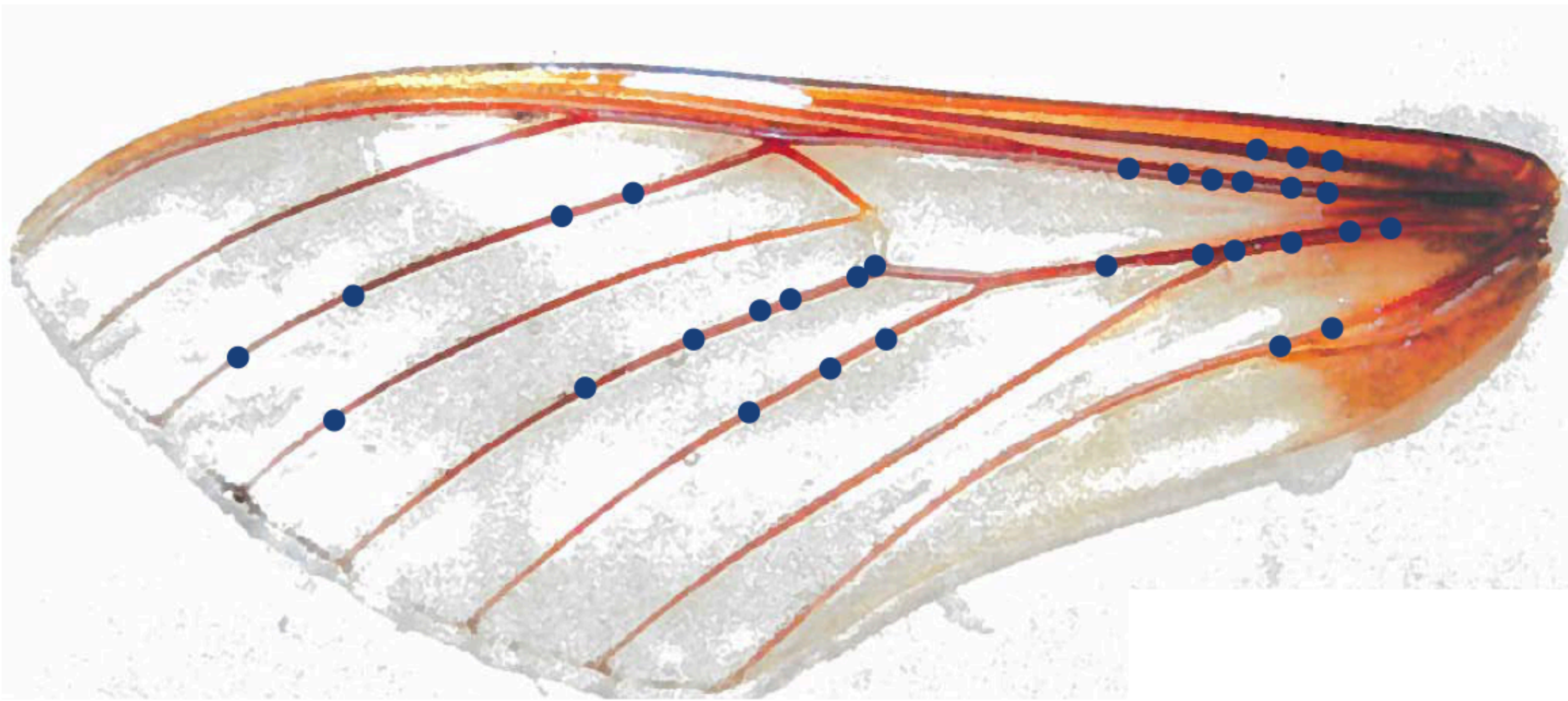
Compressive Sensing Reconstruction



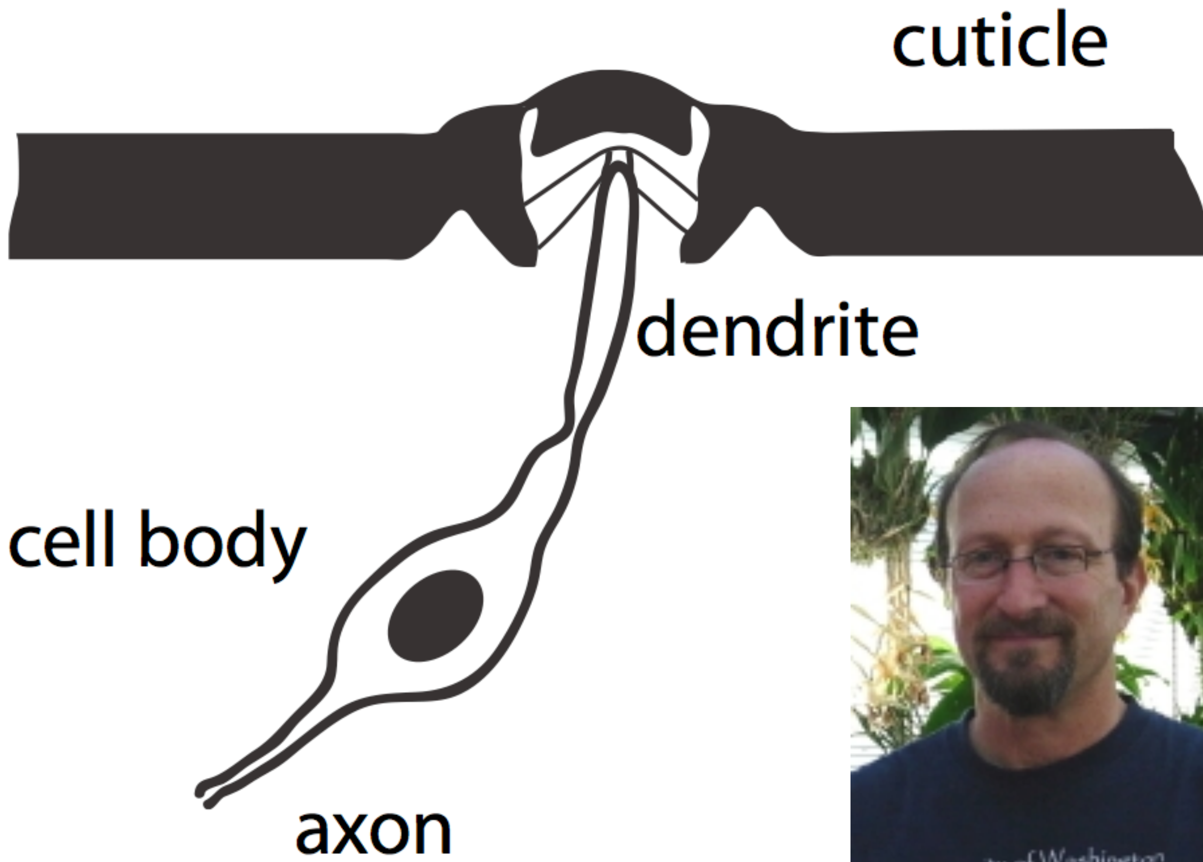
Neuro-Sensory Application?

Steven Brunton
UW AMATH

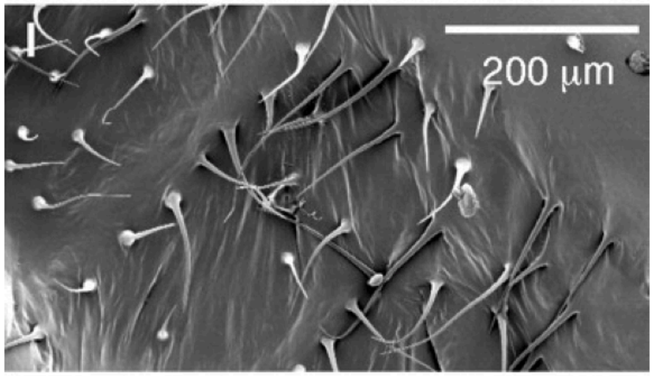
Encoding Flight Dynamics



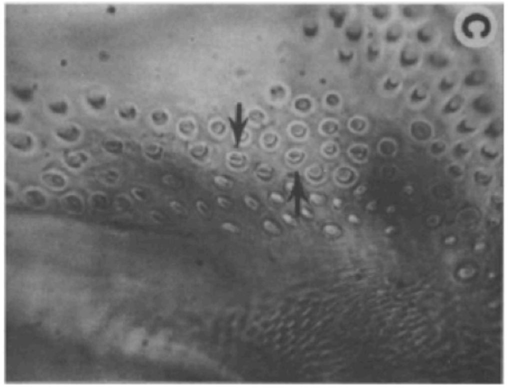
Capaniform Sensilla



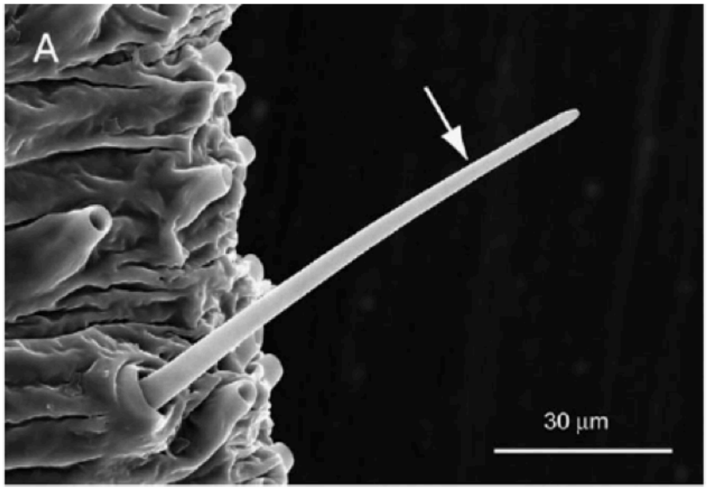
Insect Wing Sensors



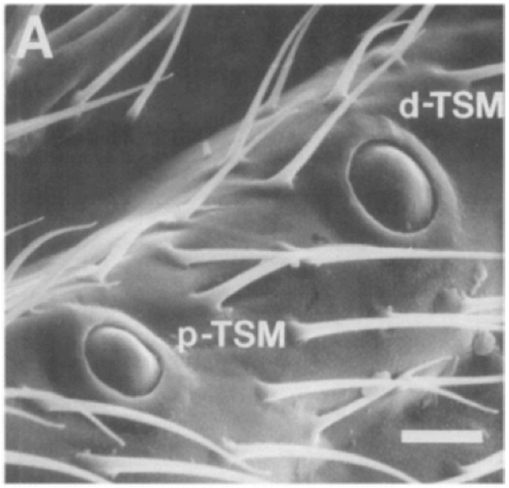
Locust (Page and Matheson, 2004)



Moth (Orona and Agee, 1987)



Moth (Ai et al., 2010)



Fly (Dickinson and Palka, 1987)

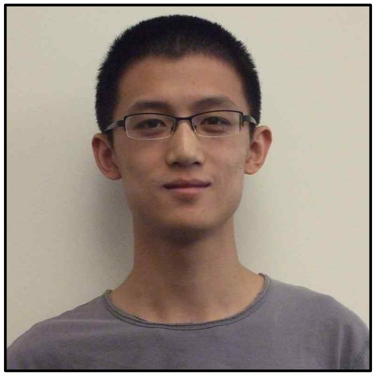
Filiform sensilla

Campaniform sensilla

Question

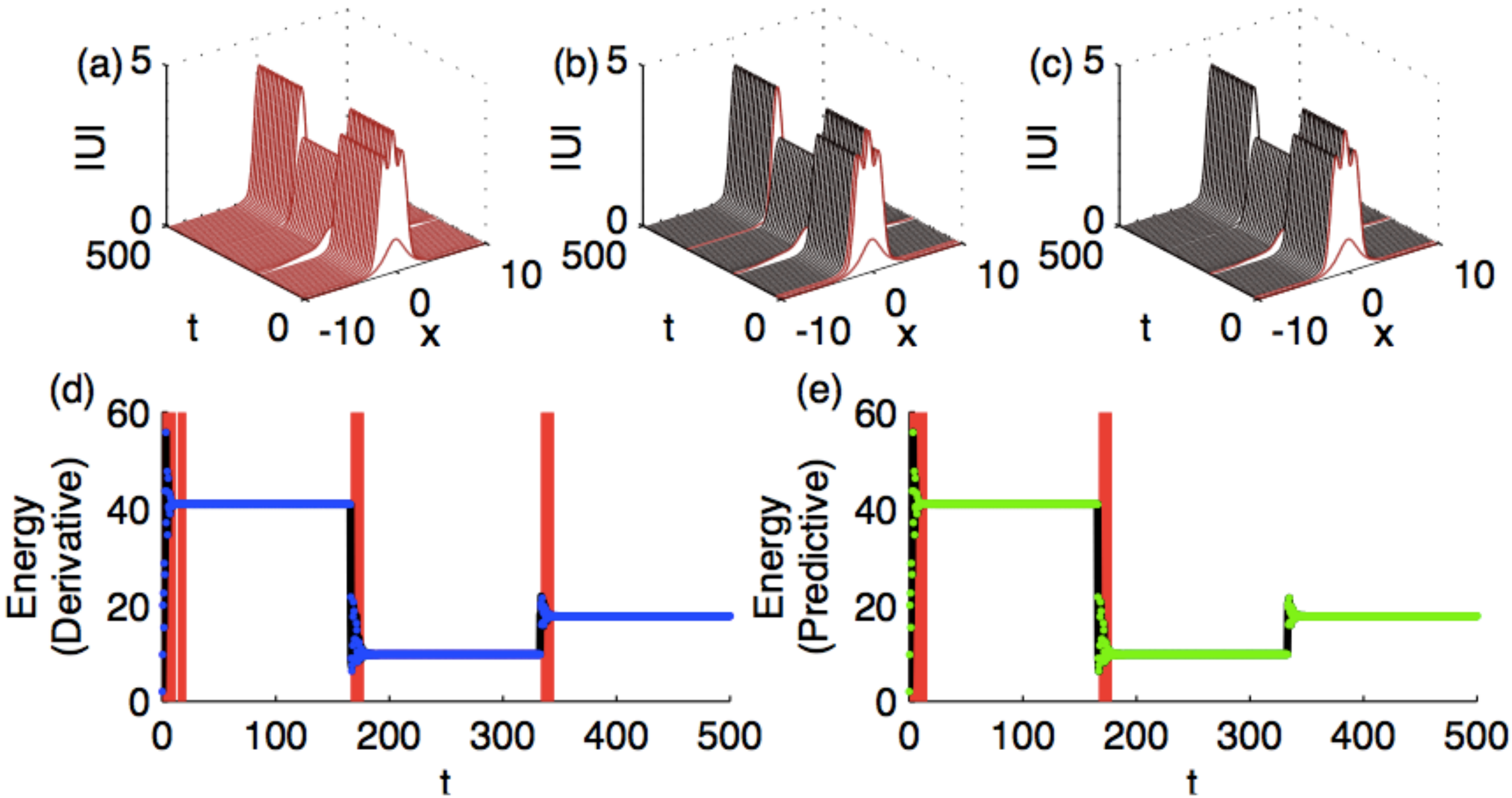
Can the campaniform sensilla on insect wings encode body dynamics or wing loading?

Application to Numerical Algorithms



Xing Fu + Matthew Williams + Peter Schmid
UW AMATH Princeton Ecole Polytechnic

Adaptive Time-Stepping Algorithms



Combine POD + DMD

Conclusions

Dynamical Systems

+ Machine Learning (robust PCA: L+M+S)

+ Compressive Sensing (time + space)

+ Control

= Transformative Paradigm