On Time-Frequency Sparsity and Uncertainty

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* partially based on joint works with François Auger, Pierre Borgnat and Éric Chassande-Mottin

Heisenberg (classical)

Moments

Second-order (variance-type) measures for the individual time and frequency spreadings of a signal $x(t) \in L^2(\mathbb{R})$ with spectrum $X(\omega)$:

$$\Delta_t^2(x) := \frac{1}{E_x} \int t^2 \, |x(t)|^2 \, dt \quad ; \quad \Delta_\omega^2(X) := \frac{1}{E_x} \int \omega^2 \, |X(\omega)|^2 \, \frac{d\omega}{2\pi}$$

Theorem (Weyl, '27; Gabor, '46, ...)

$$\Delta_t(x) \, \Delta_\omega(X) \geq \frac{1}{2},$$

with lower bound attained for Gabor "logons", i.e., Gaussian waveforms $x_*(t) = C\, e^{\alpha t^2}$, $\alpha < 0$

Heisenberg (time-frequency)

Joint moment

Second-order (variance-type) measure for the time-frequency spreading of a signal $x(t)$ with energy distribution $C_x(t, \omega; \varphi)$:

$$\Delta_{t\omega}(C_x) := \frac{1}{E_x} \iint \left(\frac{t^2}{T^2} + T^2 \omega^2 \right) C_x(t, \omega; \varphi) \, dt \, \frac{d\omega}{2\pi}$$

Theorem (Janssen, '91)

$$\begin{aligned} \textit{Wigner} \quad &\Rightarrow \quad \Delta_{t\omega}(W_x) \geq 1 \\ \textit{Spectrogram} \quad &\Rightarrow \quad \Delta_{t\omega}(S_x^h) \geq 2, \end{aligned}$$

with lower bounds attained for logons and matched Gaussian windows, i.e., $h(t) = x(t) = x_*(t)$

time-frequency "covariance"

Definition (Cohen, '95)

$$c(x) := \int t\,|x(t)|^2\,\frac{d}{dt}\arg x(t)\,dt$$

Interpretation

Covariance quantifies the coupling between time and instantaneous frequency

$$c(x) = \langle t\,\omega_x(t)\rangle$$

Intuition

Covariance is zero in case of no coupling

$$c(x) = \langle t\,\omega_x(t)\rangle = \langle t\rangle\,\langle\omega_x(t)\rangle = \langle t\rangle\,\langle\omega\rangle = 0$$

Schrödinger

Theorem (Schrödinger, '30)

$$\Delta_t(x)\,\Delta_\omega(X) \geq \frac{1}{2}\sqrt{1 + c^2(x)},$$

with lower bound attained for Gaussian waveforms of the form $x_*(t) = e^{\alpha\, t^2 + \beta\, t + \gamma}$, *with* $Re\{\alpha\} < 0$.

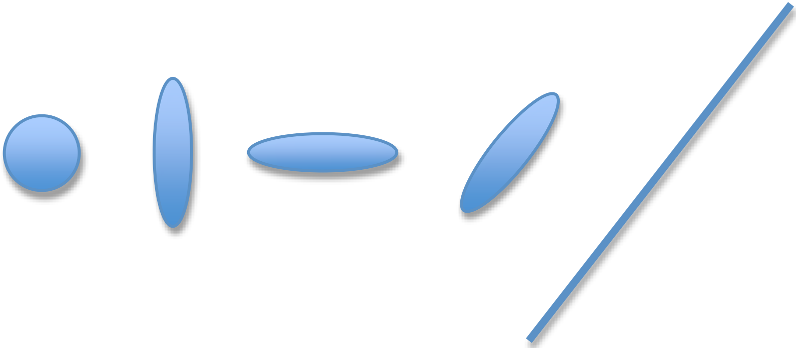
Terminology

Waveforms x_* attaining the lower bound correspond to "squeezed states" in quantum mechanics and "linear chirps" in signal theory

Interpretation

Possibility of localization in the plane beyond pointwise energy concentration

from “logons” to “chirps”



effective localization

Wigner and beyond

$$\lim_{Re\{\alpha\}\to 0_-} W_{x_*}(t,\omega) = \delta\left(\omega - (\beta + 2\, Im\{\alpha\} t)\right)$$

1. Generalization: localization on more "arbitrary" curves of the plane by modifying the symmetry rules underlying the Wigner distribution (Gonçalvès and F., '96)
2. *Caveat*: global localization holds for monocomponent signals only

Ways out

1. "sparsity"
2. "reassignment"
3. constrained EMD (Pustelnik *et al.*, '12 + poster)

rationale

Fourier

Duality between distribution and correlation:

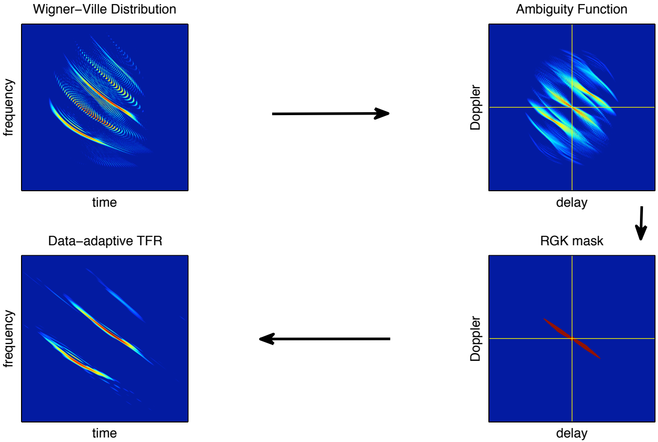
$$C_x(t,\omega;\varphi) = \mathcal{F}_{\xi\to t}\mathcal{F}_{\tau\to\omega}\left\{\varphi(\xi,\tau)\, A_x(\xi,\tau)\right\},$$

with $A_x(\xi,\tau) = \mathcal{F}_{t\to\xi}\mathcal{F}_{\omega\to\tau}\left\{W_x(t,\omega)\right\}$ a TF correlation function ("Ambiguity Function")

Consequences

1. signal terms located around the origin of the AF plane
2. cross-terms located outside
3. "low-pass" filtering (e.g., $A_h^*(\xi,\tau)$ for the spectrogram)
   $\Rightarrow$ trade-off between localization and cross-terms reduction

example


Wigner–Ville Distribution
frequency
time
Ambiguity Function
Doppler
delay
RGK mask
Doppler
delay
Data–adaptive TFR
frequency
time

sparsity perspective

Discrete time

Signal of dimension N $\Rightarrow$ TFD of dimension N^2 when computed over N frequency bins

Few AM-FM components

$K \ll N \Rightarrow$ at most $KN \ll N^2$ non-zero values in the TF plane

Sparsity

Minimizing ℓ_0-norm not feasible, but near-optimal solution by minimizing ℓ_1-norm (in the spirit of "basis pursuit" or "compressed sensing")

sparse solution

From ℓ_2 to ℓ_1 (F. & Borgnat, '08)

1. Select a domain Ω neighbouring the origin of the AF plane
2. Find the sparsest time-frequency distribution $\rho(t,\omega)$ by solving the program

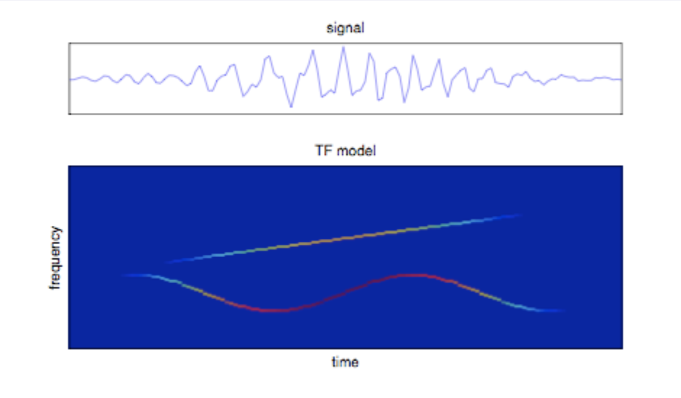
$$\min_{\rho} \|\rho\|_1 \text{ s.t. } \mathcal{F}\{\rho\} - A_x = 0|_{(\xi,\tau)\in\Omega}$$

3. The exact equality over Ω can be relaxed according to

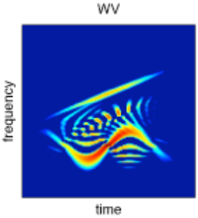
$$\min_{\rho} \|\rho\|_1 \text{ s.t. } \|\mathcal{F}\{\rho\} - A_x\|_2 \leq \epsilon|_{(\xi,\tau)\in\Omega}$$

toy example

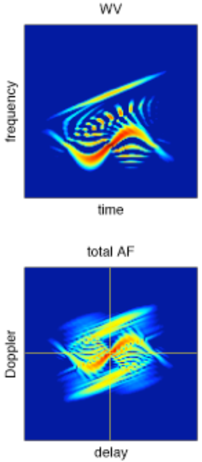
2-component AM-FM signal, 128 data points



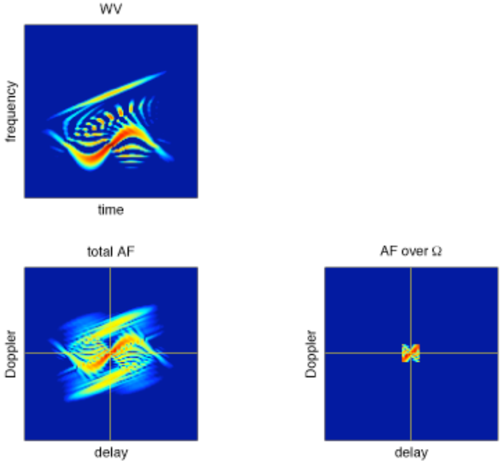
Wigner-Ville



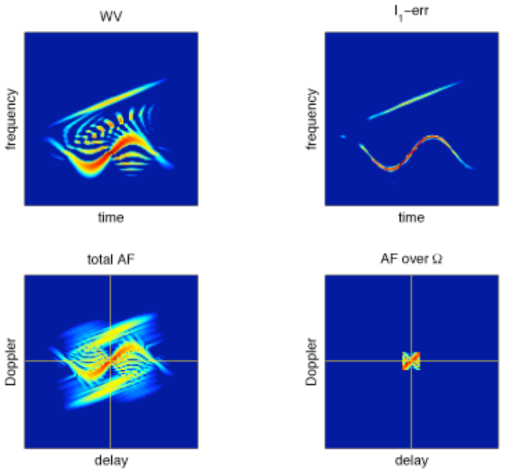
ambiguity function



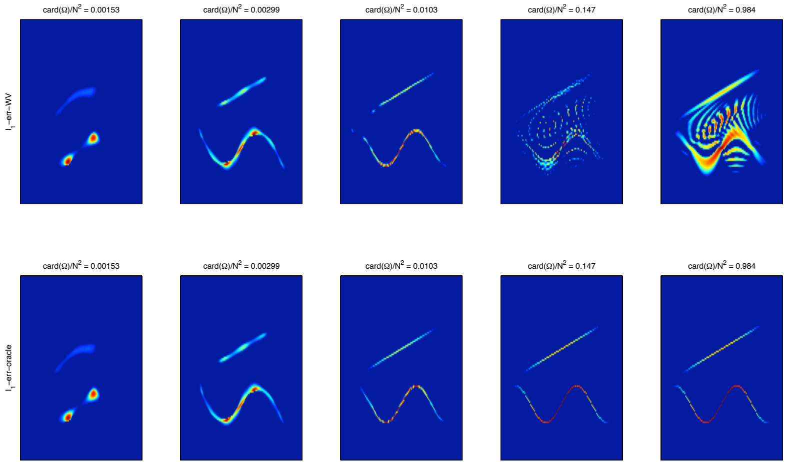
selection


WV
frequency
time
total AF
Doppler
delay
AF over Ω
Doppler
delay

sparse solution



domain selection



domain selection

Interpretation

- $|\Omega|$ too small $\Rightarrow$ not enough auto-terms $\Rightarrow$ TFD not enough "sharp": penalty with entropy (H)
- $|\Omega|$ too large $\Rightarrow$ inclusion of cross-terms $\Rightarrow$ TFD "sharp" but discontinuous: penalty with total variation (TV)

Result

$$|\Omega|_* = \arg\min_{\Omega}(H + TV)$$

$\sim$ "Heisenberg cell", i.e., minimum quantity of information necessary for coding (in both magnitude and phase) auto-terms

rationale

Smoothing relationship

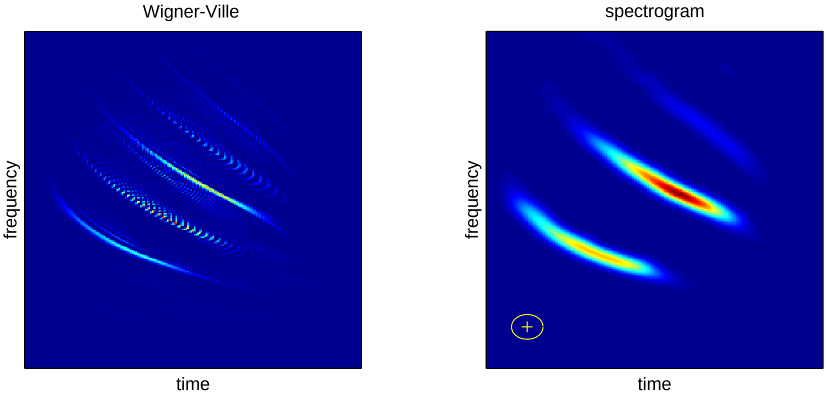
$$C_x(t,\omega;\varphi) = \iint W_x(s,\xi)\,\Phi(s-t,\xi-\omega)\,dt\,\frac{d\omega}{2\pi}$$

Key idea (Kodera *et al.*, '76, Auger & F., '95)

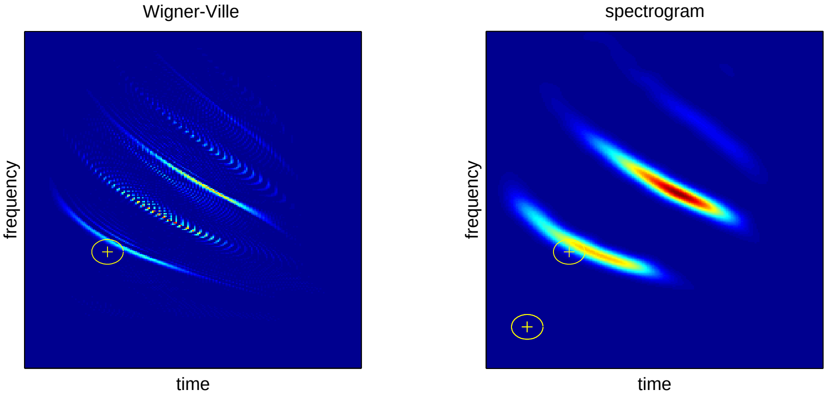
1. replace the geometrical center of the smoothing TF domain (defined by $\Phi(t,\omega)$) by the center of mass of the Wigner distribution over this domain
2. reassign the value of the smoothed distribution to this local center of mass:

$$\hat{C}_x(t,\omega;\varphi) = \iint C_x(\tau,\xi;\varphi)\delta\left(t-\hat{t}_x(\tau,\xi),\omega-\hat{\omega}_x(\tau,\xi)\right)\,d\tau\,\frac{d\xi}{2\pi}$$

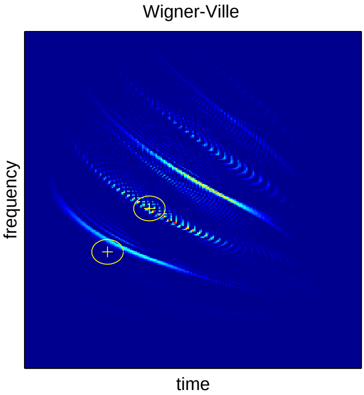
spectrogram = smoothed Wigner

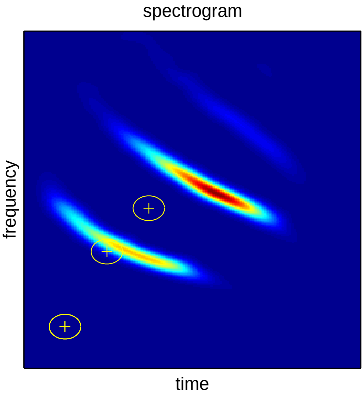


spreading of auto-terms

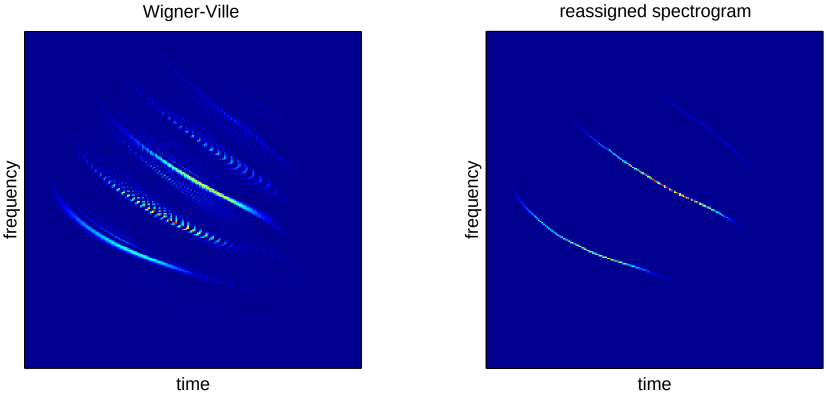


cancelling of cross-terms





reassignment



reassignment and localization

Two examples

For a Gaussian window $h(t) = \pi^{-1/4}\, e^{-t^2/2}$, reassigned spectrograms are asymptotically perfectly localized for

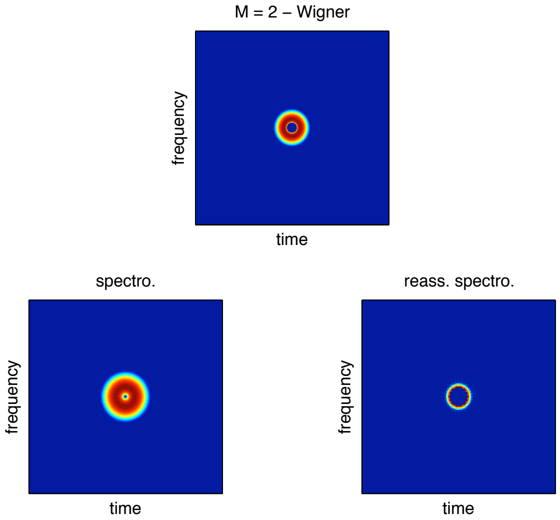
1. linear chirps $c_T(t)$ of infinite duration:

$$\lim_{T\to\infty} \hat{S}^h_{c_T}(t,\omega) = \delta(\omega - at)$$

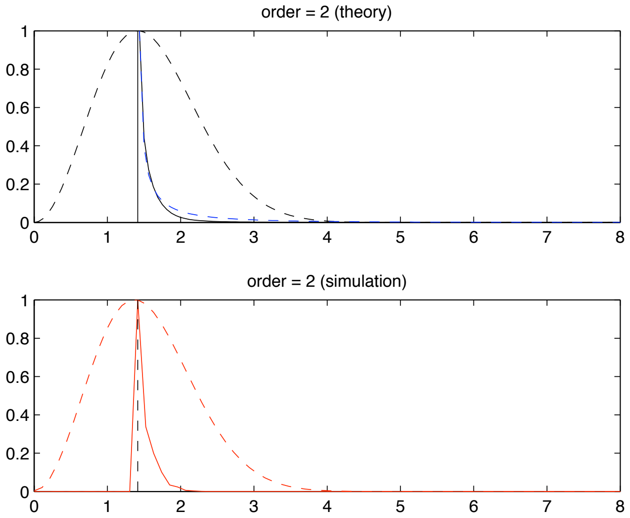
2. Hermite functions $h_M(t)$ of infinite order (F., *JFAA*'12):

$$\hat{S}^h_{h_M}(t,\omega) \overset{M\to\infty}{\longrightarrow} \delta(t^2 + \omega^2 - 2(M-1))$$

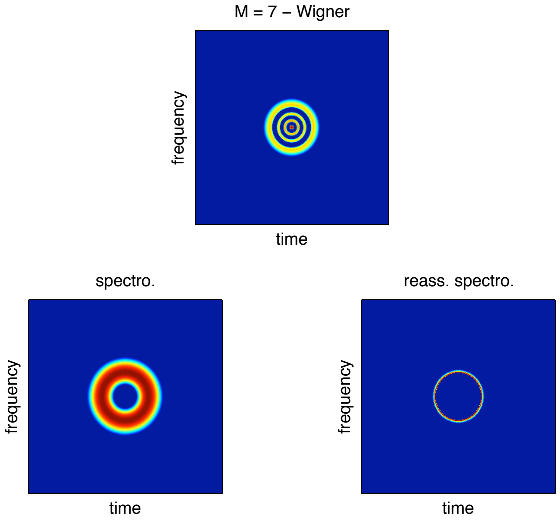
Hermite functions



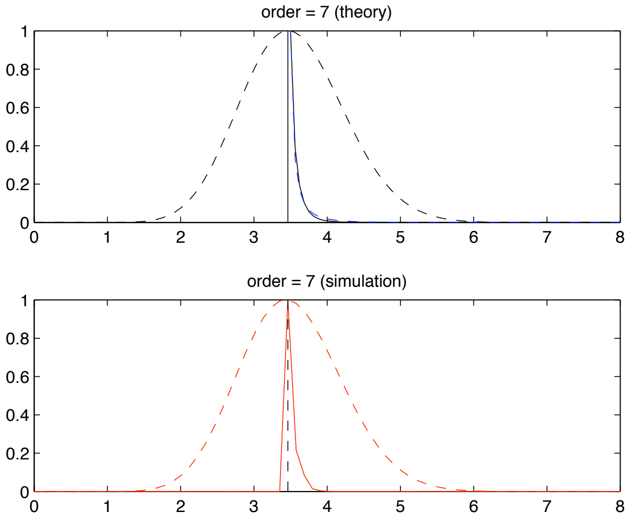
Hermite functions



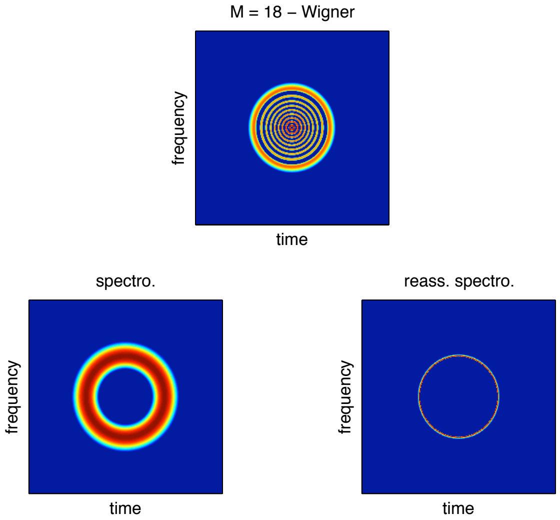
Hermite functions



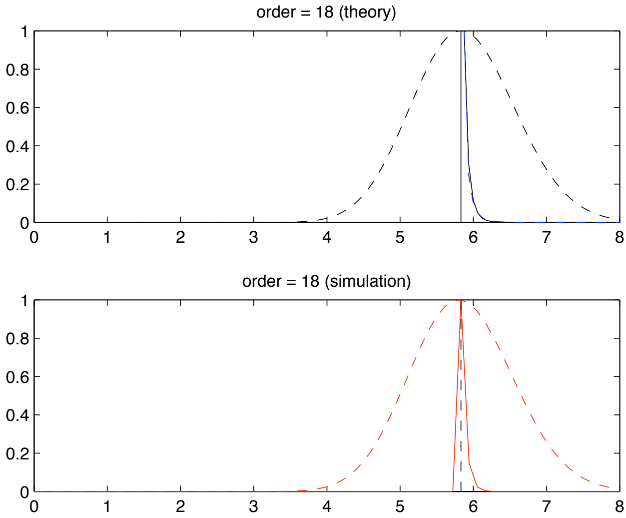
Hermite functions



Hermite functions



Hermite functions



reassignment and uncertainty

Result

The minimum uncertainty Gabor logon $h(t) = \pi^{-1/4}\, e^{-t^2/2}$ is such that

$$W_h(t,\omega) = 2\, e^{-(t^2+\omega^2)} \Rightarrow \Delta_{t\omega}(W_h) = 1$$

$$S_h^h(t,\omega) = e^{-\frac{1}{2}(t^2+\omega^2)} \Rightarrow \Delta_{t\omega}(S_h^h) = 2$$

$$\hat{S}_h^h(t,\omega) = 4\, e^{-2(t^2+\omega^2)} \Rightarrow \Delta_{t\omega}(\hat{S}_h^h) = \frac{1}{2}$$

Interpretation

$\frac{1}{2} < 1 \Rightarrow$ Heisenberg defeated?

reassignment and uncertainty

Resolving the logons paradox

1. General remark: as for Fourier, sharp localization not to be confused with resolution (i.e., ability to separate closely spaced components)
2. Complete picture: reassignment = squeezed distribution + vector field $\mathbf{r}_x(t,\omega) = (\hat{t}_x(t,\omega) - t, \hat{\omega}_x(t,\omega) - \omega)^t$ such that

$$\mathbf{r}_x(t,\omega) = \frac{1}{2}\nabla \log S_x^h(t,\omega)$$

3. Interpretation: basins of attraction
4. Open question: geometry of such basins and relationship with Heisenberg?

factorization

Bargmann

With a "circular" Gaussian window, the STFT can be expressed as $F_x^h(z) = \mathcal{F}_x^h(z)\, e^{-|z|^2/4}$, with $z = \omega + \jmath t$ and $\mathcal{F}_x^h(z)$ an entire function of order at most 2

Weierstrass-Hadamard

$$\mathcal{F}_x^h(z) = e^{Q(z)} \prod_n (1 - \tilde{z}_n) \exp\left(\tilde{z}_n + \tilde{z}_n^2/2\right),$$

where $Q(z)$ is a quadratic polynomial and $\tilde{z}_n = z/z_n$

Interpretation

Characterization by zeroes and, by duality, by local maxima

more on extrema

Motivation

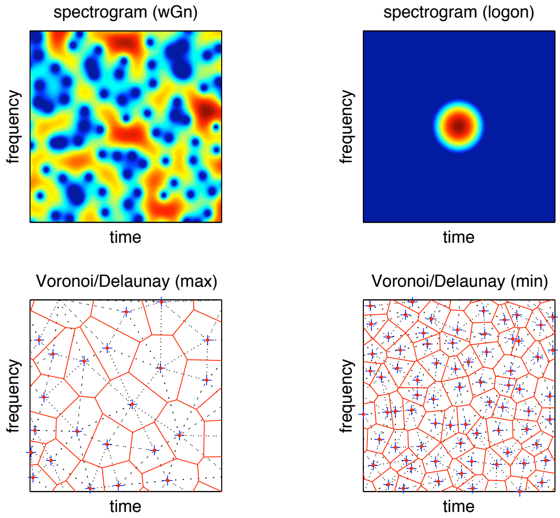
Better understanding of reassigned distributions:

1. Simplified modeling
2. Ultimate localization properties

Proposed approach

1. Data: white Gaussian noise
2. Time-frequency distribution: Gaussian spectrogram
3. Characterization: identification of local extrema (zeroes and maxima) + Voronoi tessellation and Delaunay triangulation
4. Analysis: distribution of cell areas, lengths between extrema and heights of local maxima

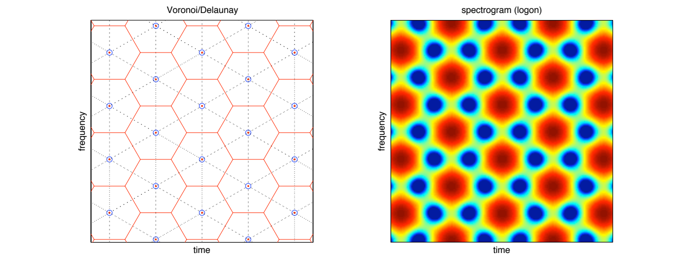
an example


spectrogram (wGn)
spectrogram (logon)
Voronoi/Delaunay (max)
Voronoi/Delaunay (min)
frequency
time

time-frequency patches

Mean arrangement

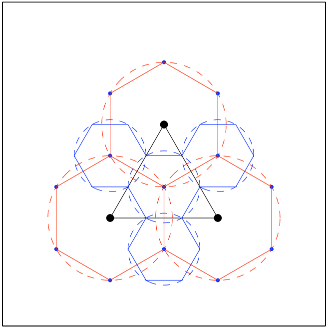
1. Average connectivity $\approx 6 \Rightarrow$ tiling with hexagonal cells



2. Maximum packing of circular patches

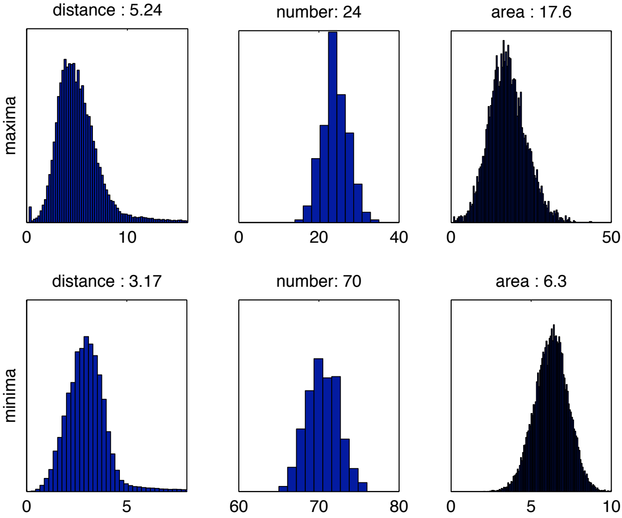
predictions

Hexagonal tiling geometry

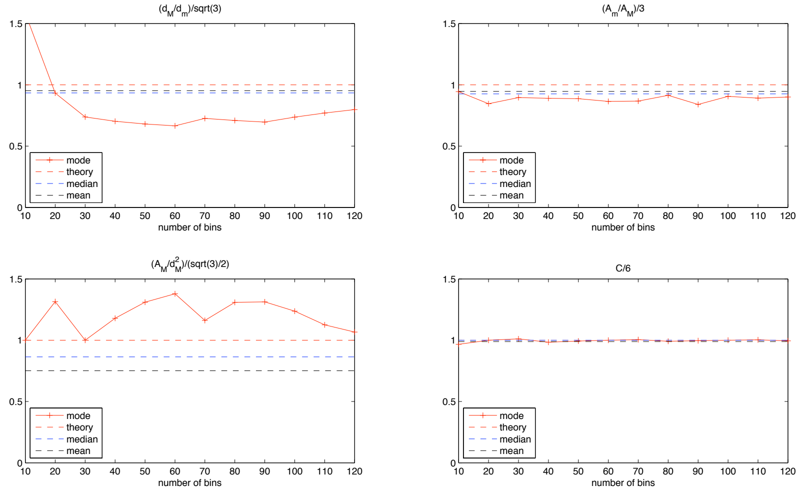


$$D_M/d_m = \sqrt{3};\ N_M/N_m = 1/3;\ A_M/A_m = 3$$

simulation results



comparison with model



analysis

Distributions of lengths and areas

1. Ratios max/min: dispersion but reasonable agreement (in the mean) between experimental results and theoretical predictions
2. Ranges of values: if we call "effective domain" of the minimum uncertainty logon the circular domain which encompasses 95% of its energy, its radius and area are equal to ~ 2.6 and 21.8, to be compared to the values $d_M/\sqrt{3} \sim 3$ and $2\pi/(3\sqrt{3})A_M \sim 21.8$ attached to the hexagonal tiling

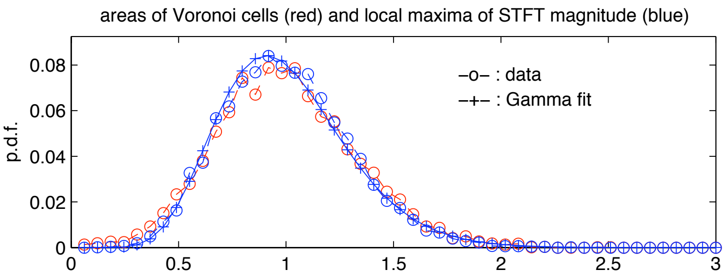
Interpretation

Tiling cells $\sim$ minimum uncertainty logons

local maxima and Voronoi cells areas (1)

Distributions

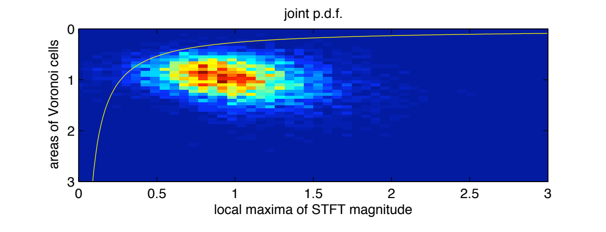
1. Heights: well-described by a Gamma distribution
2. Areas: *idem* and similar to the heights distribution for a proper renormalization



local maxima and Voronoi cells areas (2)

Proposition

The value $|F|_*$ of a local maximum of the STFT magnitude and the area A of the associated Voronoi cell satisfy the uncertainty-type inequality $A.|F|_* \geq 3\sqrt{6}$



Sketch of proof: $2 \leq \iint_A (t^2+\omega^2)\,|F_x^h(t,\omega)|^2 dt\,\frac{d\omega}{2\pi} \leq \left(\frac{|F|_*\,R^2(A)}{2}\right)^2$

beyond the mean model

Random Gabor expansion

Previous results suggest a possible modeling of Gabor spectrograms as

$$S_x^h(t,\omega) = \left| \sum_m \sum_n c_{mn} \, F_h^h(t - t_m, \omega - \omega_n) \right|^2$$

with

1. locations (t_m, ω_n) distributed on some suitable randomized version of a triangular grid
2. magnitudes of the weights c_{mn} Gamma-distributed
3. locations and weights partly correlated

Concluding remarks

1. Time-frequency energy distributions
   - Sparse representations
   - Ultimate localization constrained by uncertainty
2. Spectrogram geometry
   - New insights on reassignment
   - Complete characterization by local maxima in the Gabor case?
3. Analysis/synthesis
   - Modeling sparse time-frequency distributions?
   - New approaches to data driven representations?

some references

Available at perso.ens-lyon.fr/patrick.flandrin/publis.html and/or upon request at flandrin@ens-lyon.fr, with some Matlab codes at http://perso.ens-lyon.fr/patrick.flandrin/software2.html

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