An Adaptive Iterative Filter Method for Signal Decompositions and Instantaneous Frequency Analysis

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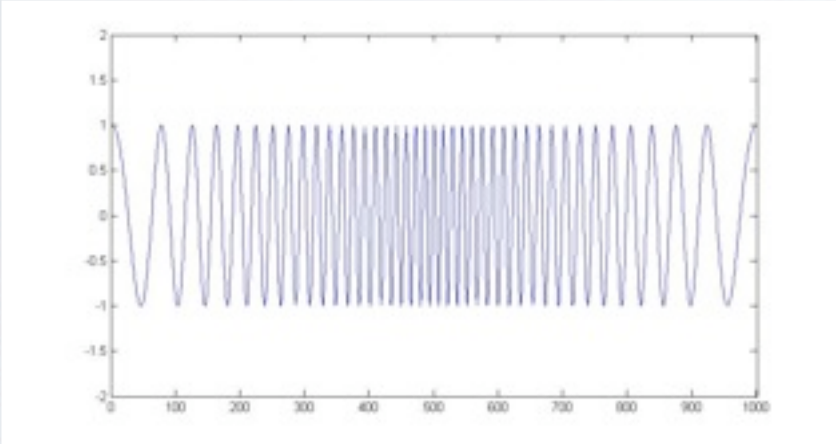
Outline

- Instantaneous Frequency
- Classical EMD
- Iterative Filters
- Filter Selects
- Examples
- Conclusions and Future Work

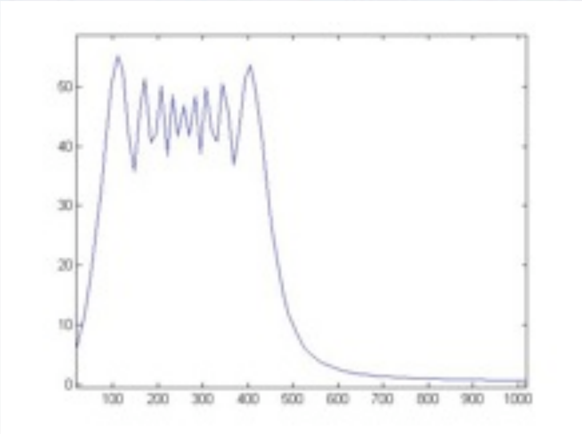
Introduction and Motivation

- Classical signal analysis methods, such as FFT or wavelets based algorithms, are good for linear and stationary signals, but may not be effective to handle nonlinear and non-stationary signals.

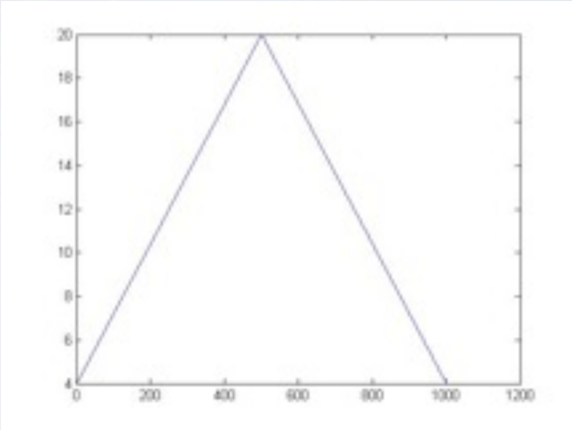


$X(t) = \cos(4\pi\lambda(t)t), t \in [0, 0.5]$, $x(t) = x(1-t), t \in [0.5, 1]$,

$\lambda(t) = 4 + 32 * t$



Fourier spectrum

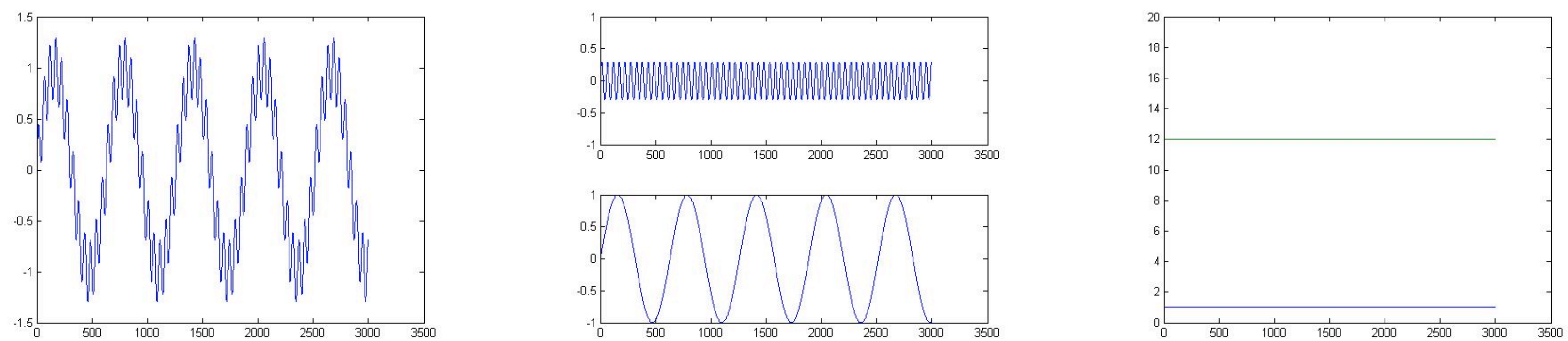


Time dependent frequency

- Goals: develop adaptive signal decomposition methods to treat nonlinear and non-stationary signals more effectively.
- Inspired by the Empirical Mode Decomposition (EMD) method, the Hilbert-Huang Transform (HHT), pioneered by Huang, etc. ('98).

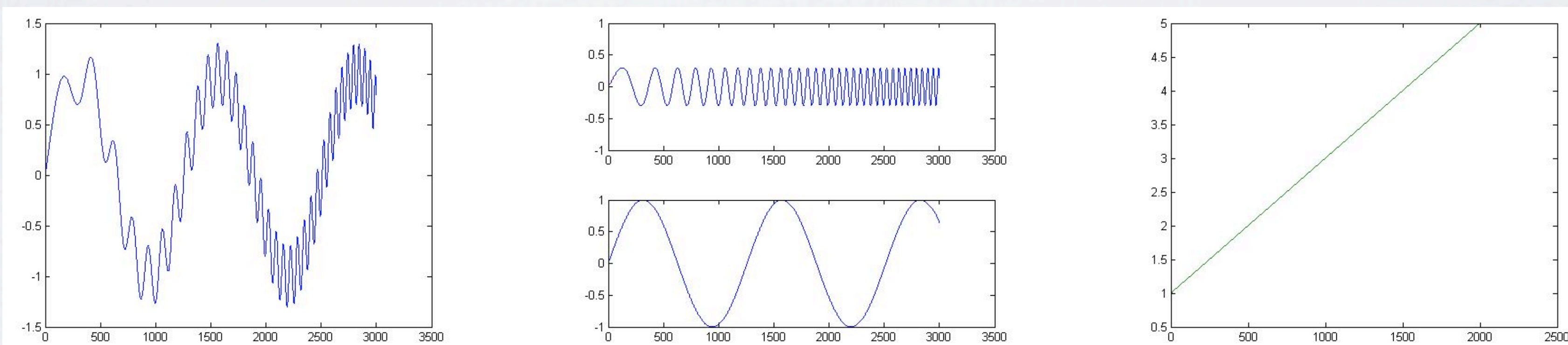
Introduction and Motivation

We want to achieve:



x_1 components frequency

$x_1 = \sin t + 0.3 \sin 12t$



x_2 components frequency

$x_2 = \sin 0.5t + 0.3 \sin (0.2t^2 + t)$

Our strategy: localize treatments and analysis.

Instantaneous Frequency (IF)

- A commonly used one is through Hilbert transform:

$$Y(t) = H(X)(t) = \frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{X(\tau)}{t-\tau} d\tau$$

Analytical signal: $Z(t) = X(t) + iY(t) = a(t)e^{i\theta(t)}$

$$\theta(t) = \arctan\left(\frac{Y(t)}{X(t)}\right), \quad a(t) = \sqrt{X^2(t) + Y^2(t)}.$$

Instantaneous Frequency: $\omega(t) = \frac{d\theta(t)}{dt}$

- Example:

$$X(t) = \sin(\omega t), \quad Y(t) = H(X)(t) = \cos(\omega t), \quad \omega(t) = \omega.$$

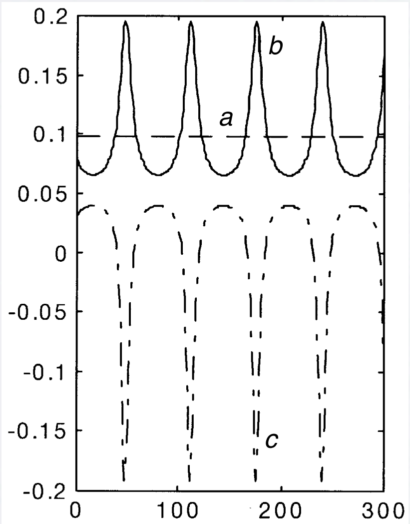
Instantaneous Frequency (IF)

- The IF defined are controversial: may lead to inconsistency or negative frequencies, which are meaningless.

- An example: $X(t) = \cos(t) + b, \quad Y(t) = \sin(t)$

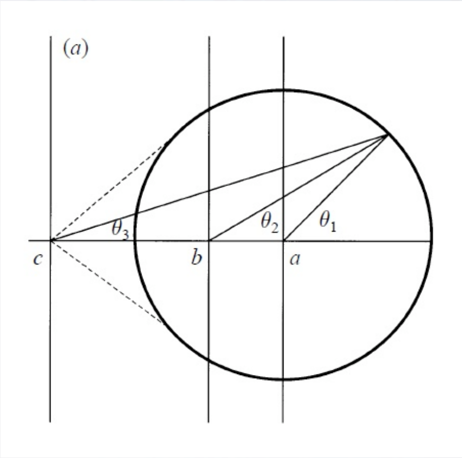
(1) If $b = 0$, $\omega(t) = 1$, this is perfect.

(2) If $b \neq 0$, $\omega(t)$ can obtain a continuum reading.



- Solid line: $b > 0$,
- Dash-dot line: $b < 0$.

picture from Huang etc. ('98).



Instantaneous Frequency (IF)

- Many different ways to introduce Instantaneous Frequency of a signal. The approach we take: find a moving average curve *v(t)*, such that *((x(t)-v(t)), (x(t)-v(t))')* form a regular rotation.

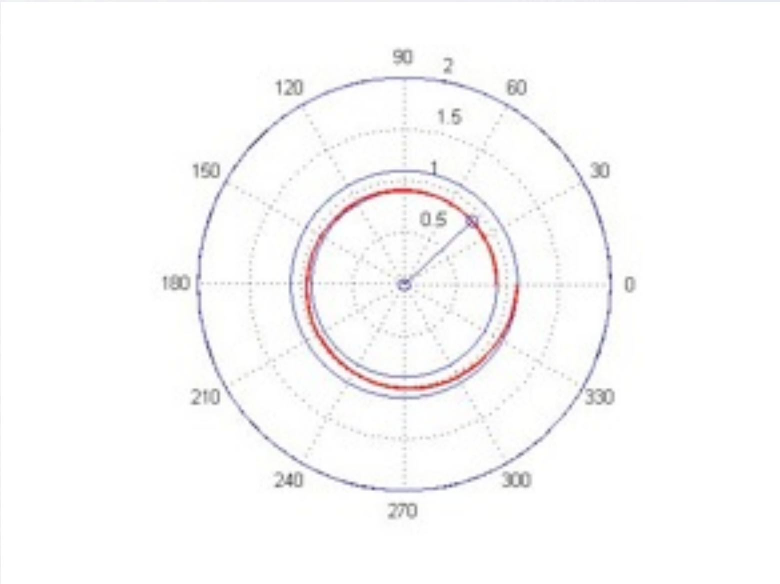
- By rotation speed: Given signal $X(t)$

$$X_u(t) = \frac{X(t)}{a(t)} \qquad a(t) \text{ the envelope of } X(t)$$

$$Y(t) = \frac{dX(t)}{dt} \qquad Y_u(t) = \frac{Y(t)}{b(t)} \qquad b(t) \text{ is the envelope of } Y(t)$$

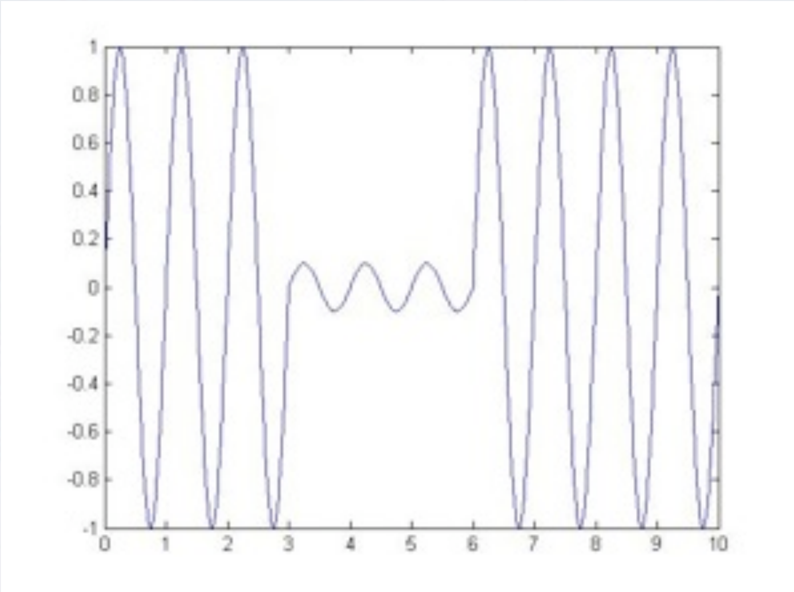
rotation angle: $\omega(t) = arctan\frac{Y_u(t)}{X_u(t)}$

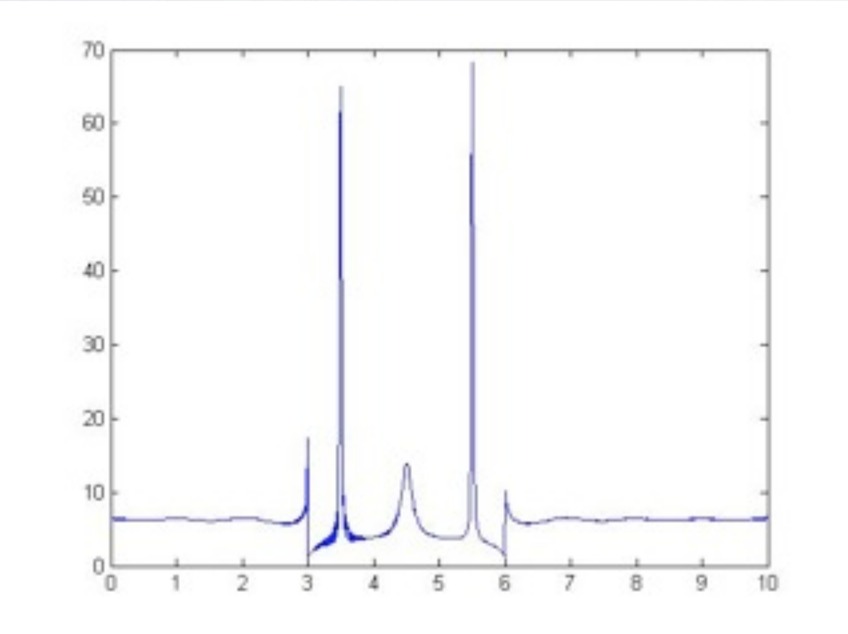
Instantaneous Frequency: $\omega(t) = \frac{d\theta(t)}{dt}$



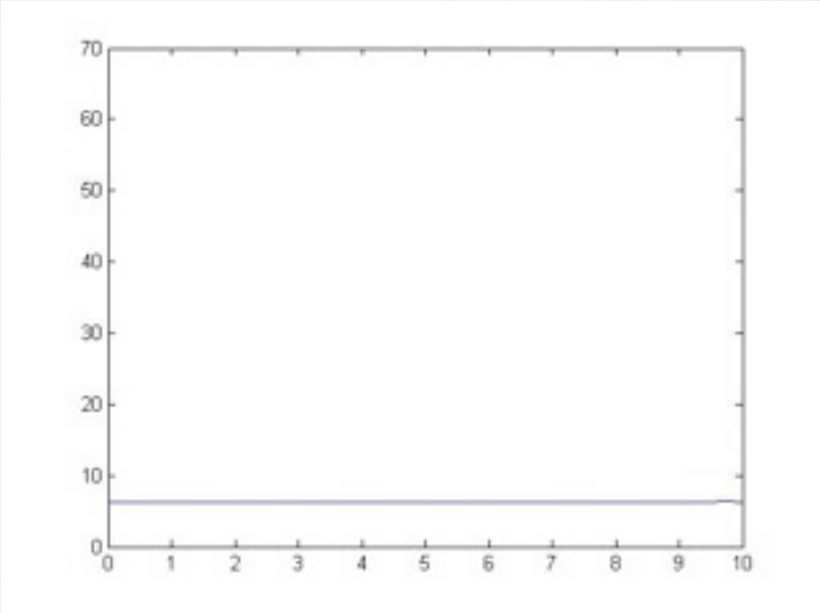
We use this definition

Instantaneous Frequency (IF)





IF from Hilbert transform



IF from rotation angle

Empirical Mode Decomposition (EMD)

- EMD decomposes the signal into summation of intrinsic mode functions (IMF's), an adaptive nonlinear sparse representation,

$$X(t) = I_1(t) + I_2(t) + \cdots + I_m(t) + W(t)$$

$I_i(t)$ is an IMF, $W(t)$ is the trend function.

- Each IMF has well behaved instantaneous frequency.

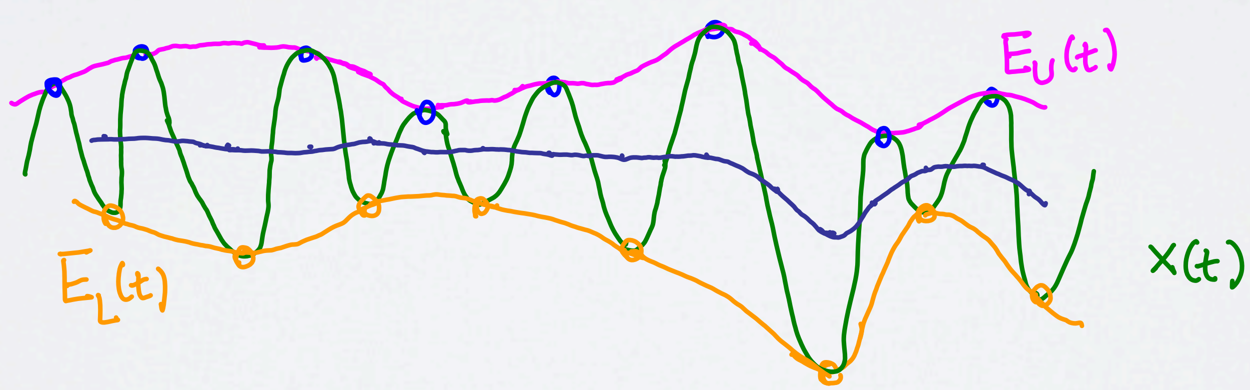
- Each IMF has certain properties that make it look like a mono-component wave.

  - A zero crossing between a minimum and maximum points.
  - Average of upper and lower envelopes (defined by the extrema) is near zero.

- The decomposition is achieved by a sifting algorithm.

Sifting Algorithm for EMD

- Find all the local max and all the local min of $X(t)$.
- Obtain the "upper envelope" $E_U(t)$ by connecting the local max through a cubic spline.
- Obtain the "lower envelope" $E_L(t)$ in a similar way.
- Define $S(X(t)) = X(t) - \frac{1}{2}(E_U(t) + E_L(t))$.



Iterating S to obtain the IMF's, $I_1(t) = \lim_{n\to\infty} S^n(X(t))$.

$$I_j(t) = \lim_{n\to\infty} S^n(X(t) - I_1(t) - \cdots - I_{j-1}(t))$$

Challenges of EMD

- One of the main challenges of EMD was lacking of mathematical framework, significant progress has been made in recent years.

- The decomposition is nonlinear: $x = \sum_{k=1}^{N} x_k + f$ $y = \sum_{k=1}^{M} y_k + g$

$z = x + y, z = \sum_{k=1}^{K} z_k + h$ But $z_k \neq x_k + y_k$

- Many alternative algorithms have been proposed and many successful applications of EMD have been studies. Some of them are remarkable.

- Groups (mathematics) working on the subjects: Huang, Daubechies, Hou, Riemenschneider, Yang, Echeverria, Crowe, Flandrin, Rilling, Gonalves, Zhou, Wang, Pines, Peng, Salvino, Wu, Xu, Osher, ...

Iterative Filters

Main Idea: Replace the mean of upper and lower envelopes in the classical EMD algorithm by the average obtain by low pass filters (data dependent),

Moving local average: in the continuous case

$$K_X(X(t)) = \int X(t+s)w(s,X)ds, \quad \int w(s,X)ds = 1.$$

In the discrete case

$$K_X(X(n)) = \sum_j w_j(X)X(n+j), \quad \sum_j w_j(X) = 1.$$

Define

$$S_X(X(t)) = X(t) - K_X(X(t)),$$

And the IMF's are obtained by

$$I_1(t) = \lim_n S_X^n(X(t)), \quad I_k(t) = \lim_n S_X^n(X(t) - \sum_{j=1}^{k-1} I_j(t)).$$

Adaptive Iterative Filters

Two layers of iterations:

Inner iterations: $I_1(t) = \lim_n S_X^n(X(t)), \quad I_k(t) = \lim_n S_X^n(X(t) - \sum_{j=1}^{k-1} I_j(t)).$

$$S_X(X(t)) = X(t) - \int_{-l(t)}^{l(t)} X(s+t)w(s)ds = X(t) - \int_{-L}^{L} X(g(t,s)+t)w(s)ds$$

$g(t,s) : [-l(x), l(x)] \mapsto [-L, L]$, for example: $g(t,s) = l(t)h(s)$.

we take $g(t,s) = l(t)s$. linear map.

Outer iterations: $X(t) = \sum_{i=1}^m I_i(t) + W(t)$

Hope to have decreased numbers of local maxima (minima) in I_i

With certain adaptive low pass filters, the convergence of the iterative filters can be proved.

Adaptive Iterative Filters

Theorem 1

If $X(t)$ strictly increases, and $c_1(t) + l'(t)c_2(t) > 0$, then $S_X(X(t))$ strictly increases.

If $X(t)$ strictly decreases, and $c_1(t) + l'(t)c_2(t) < 0$, then $S_X(X(t))$ strictly decreases.

$$c_1(t) = K_X(X'(t) - X'(l(t)h(s) + t)) = \int_{-L}^{L} [X'(t) - X'(l(t)h(s) + t)]w(s)ds$$

$$c_2(t) = K_X(X'(t) - X'(l(t)h(s) + t)h(s)) = \int_{-L}^{L} [X'(x) - X'(l(t)h(s) + t)]h(s)w(s)ds$$

No new extremal point is generated during the iterations

Filter length decreases on the left of the maxima of $X'(t)$ and grows on the right of the maxima.

Adaptive Iterative Filters

Define: $K_n(X_n(t)) = \int_{-L}^{L} X_n(g(t,s)+t)w_n(s)ds$

$$\epsilon_n = \frac{\|K_{n+1}(X_{n+1}(t))\|_\infty}{\|K_n(X_n(t))\|_\infty} \qquad \delta_n = \frac{\|K_{n+1}(|X_{n+1}(t)|)\|_\infty}{\|K_n(|X_n(t)|)\|_\infty}$$

Theorem II

If $\prod_{i=1}^{n} \epsilon_n \to 0$ as $n \to \infty$, then $X_n(t)$ converges.

If $\prod_{i=1}^{n} \delta_i \to c > 0$ as $n \to \infty$, then $X_\infty(t) \neq 0$.

Or equivalently, some conditions on filter selection, for example:

$$\gamma K_n(X_n(t)) \leq \int_{-(l_n(t)+l_{n+1}(t))}^{l_n(t)+l_{n+1}(t)} X_n(t+s)(w_{n+1} * w_n)(s)ds$$

where γ is close to zero and $K_n(X_n(t)) > 0$.

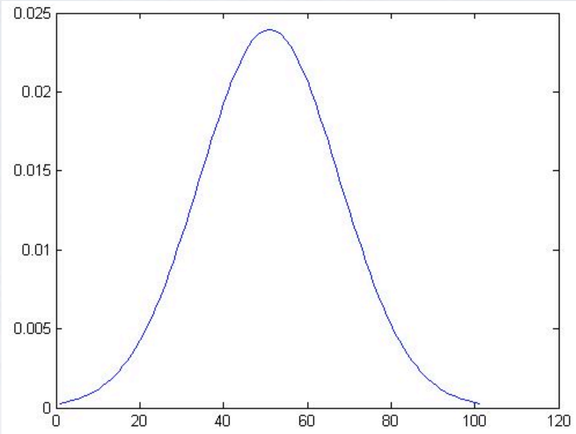
Two filters at the same location in consecutive iterations shouldn't be too much different.

PDE-based Filters

Gaussian Filter:

$$u_t = ku_{xx}$$

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4kt}\right)$$

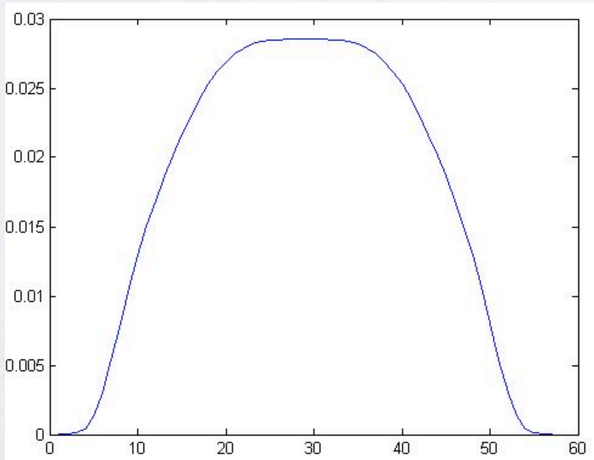


Fokker-Planck equations:

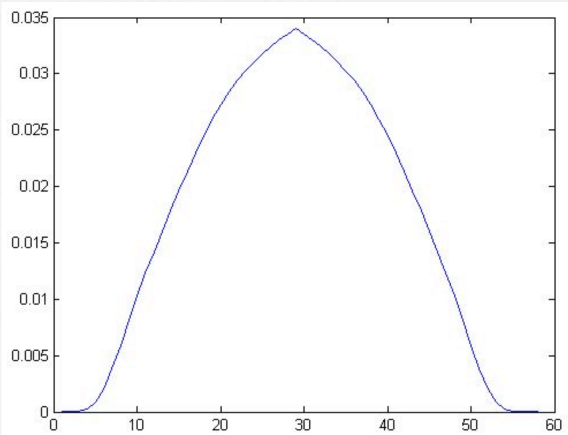
$$u_t = -\alpha(fu)_x + \beta(g^2 u)_{xx}$$

$g(a) = g(b) = 0,\ g(x) > 0$ for $x \in (a,b)$
$f(a) > 0,\ f(b) < 0$

filters (the steady state solution) have finite support

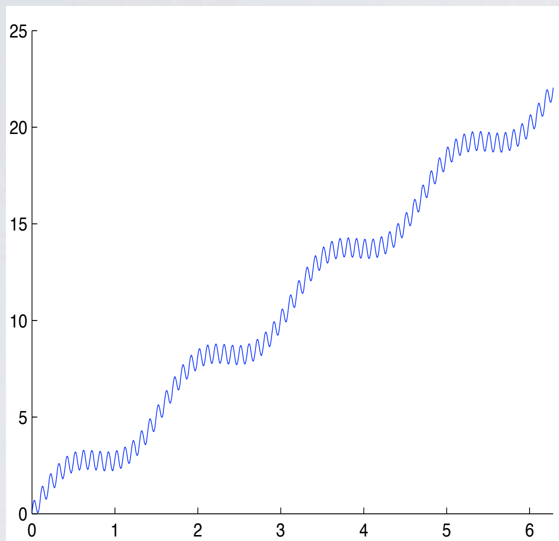


Filter 1

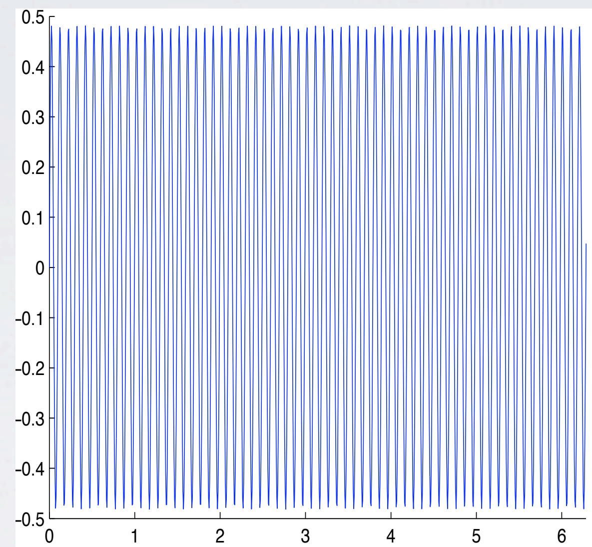


Filter 2

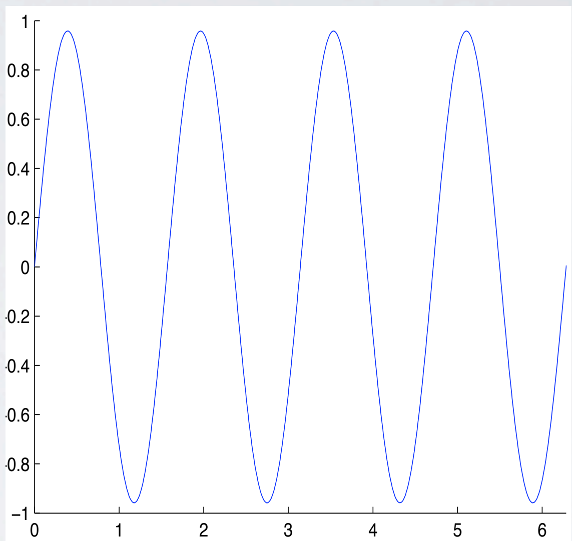
A Simple Example



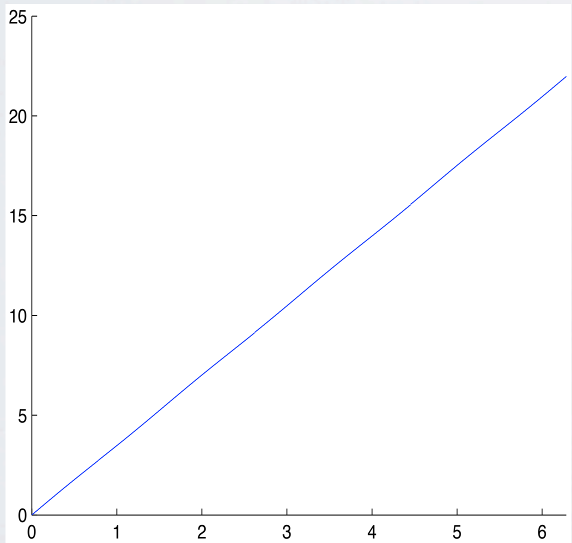
Original signal
$X(t) = \frac{7}{2}t + \sin(4t) + \frac{1}{2}\sin(63t)$



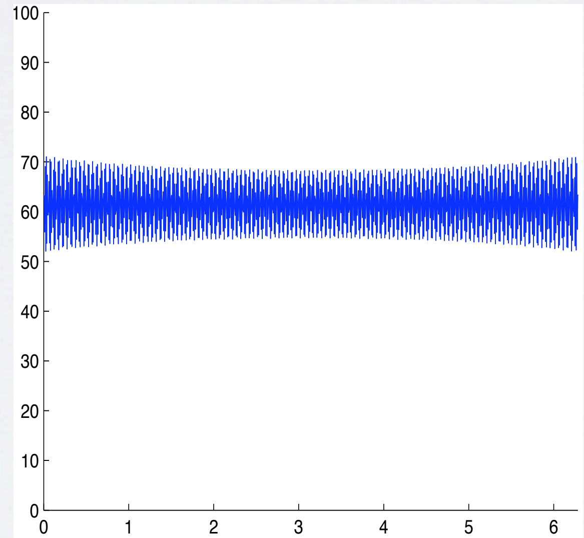
First Component



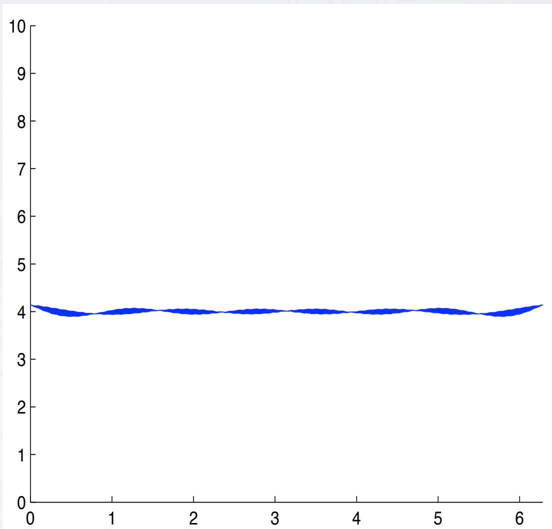
Second Component



The Trend



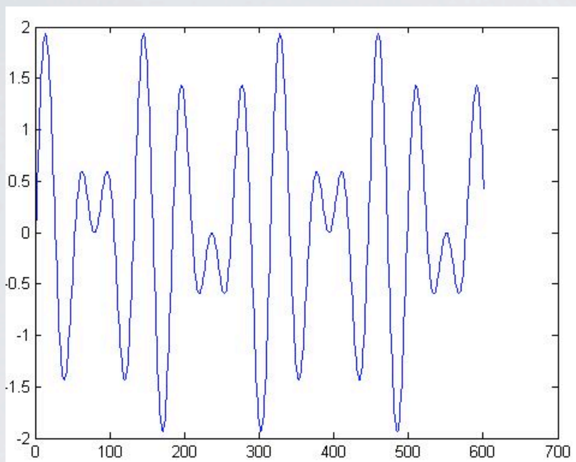
IF for the first IMF



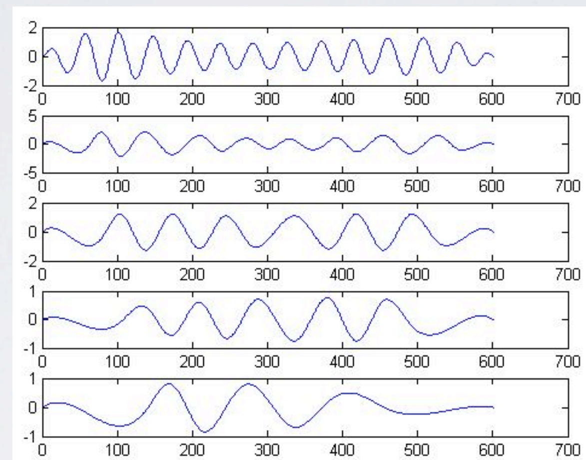
IF for the second IMF

Iterative Filters separate the oscillations from the trend and the IMF's have well separated frequency range, indicate that they are orthogonal.

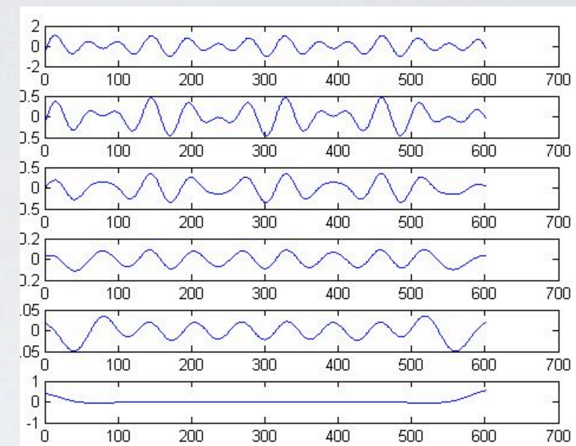
Another Example



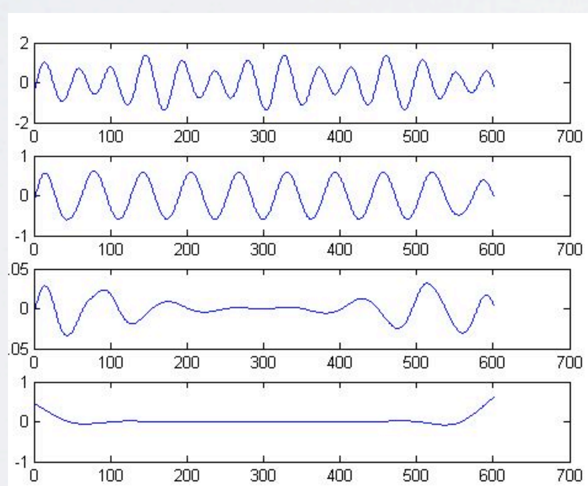
$x = \sin t + \sin 1.4t$



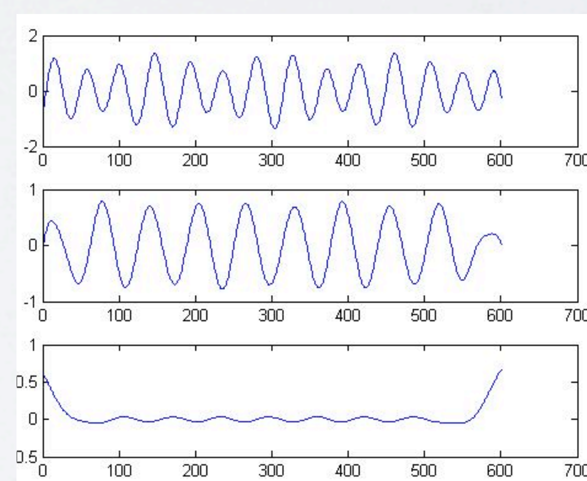
EMD: Sifting



IF: Gaussian



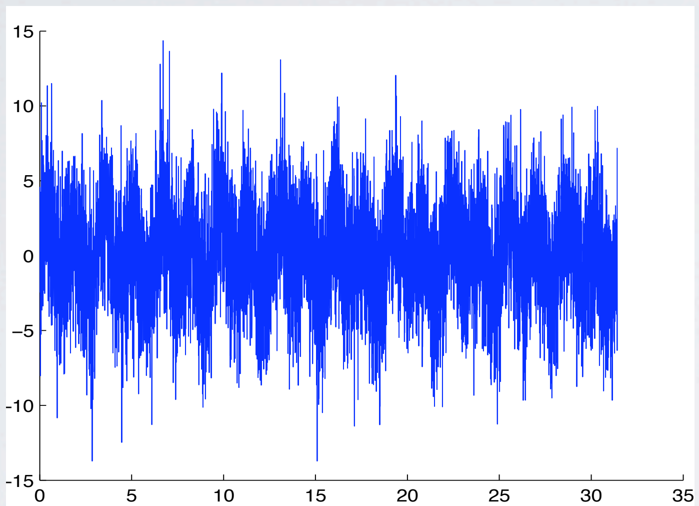
IF: Double Average



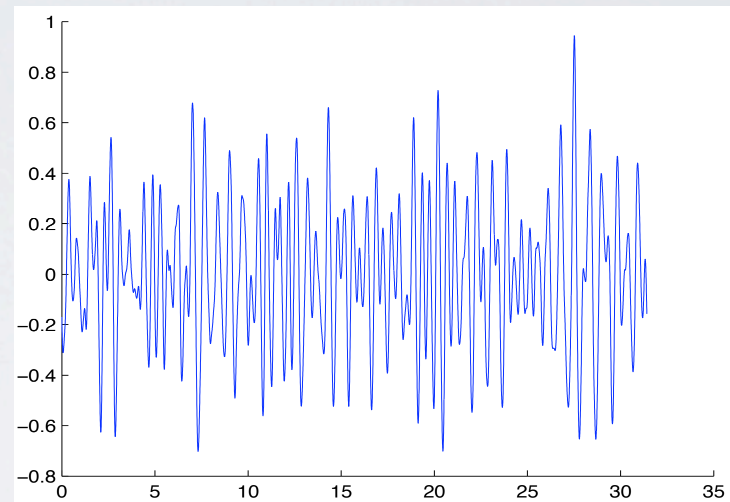
IF: Filter 2

An Example with Noise

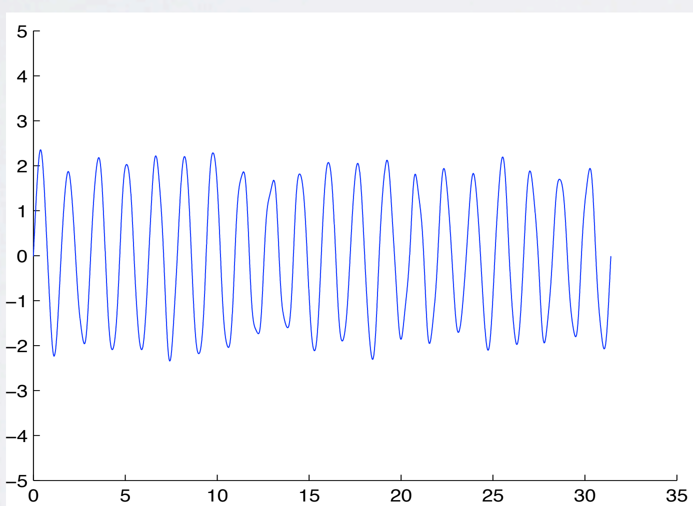
Original



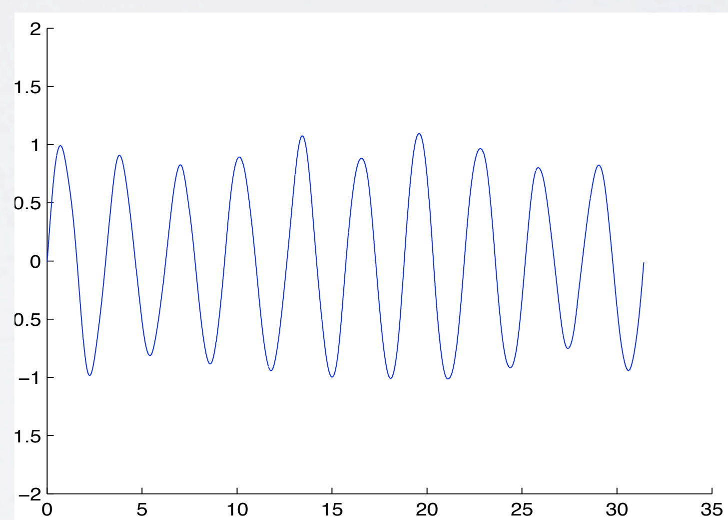
9th IMF



10th IMF

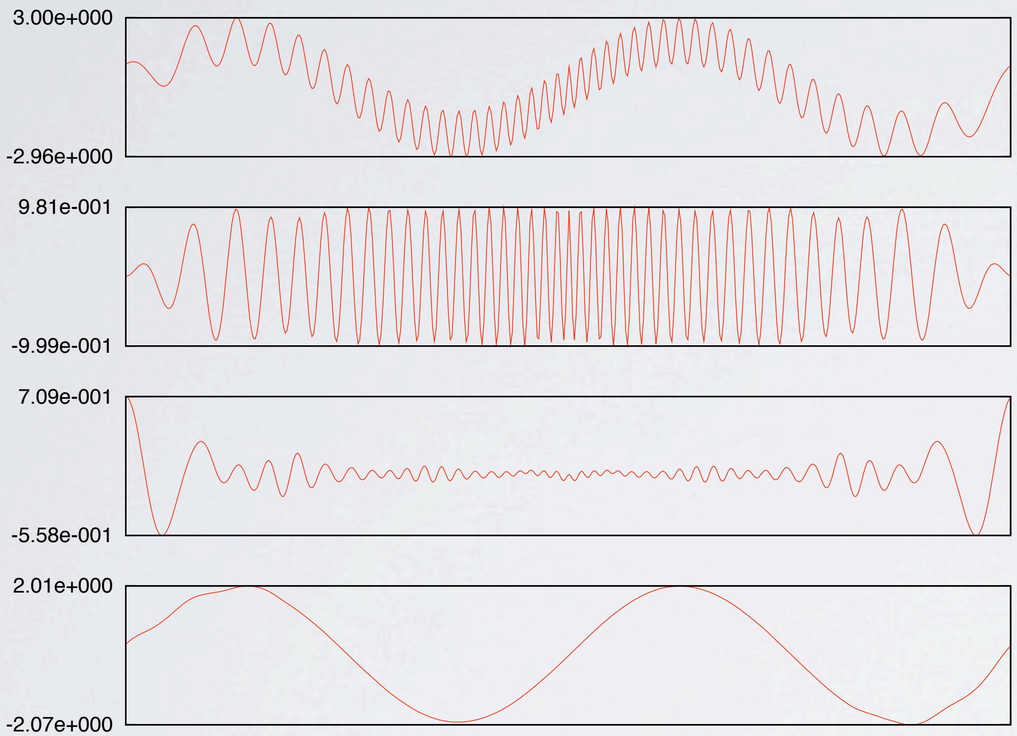


11th IMF

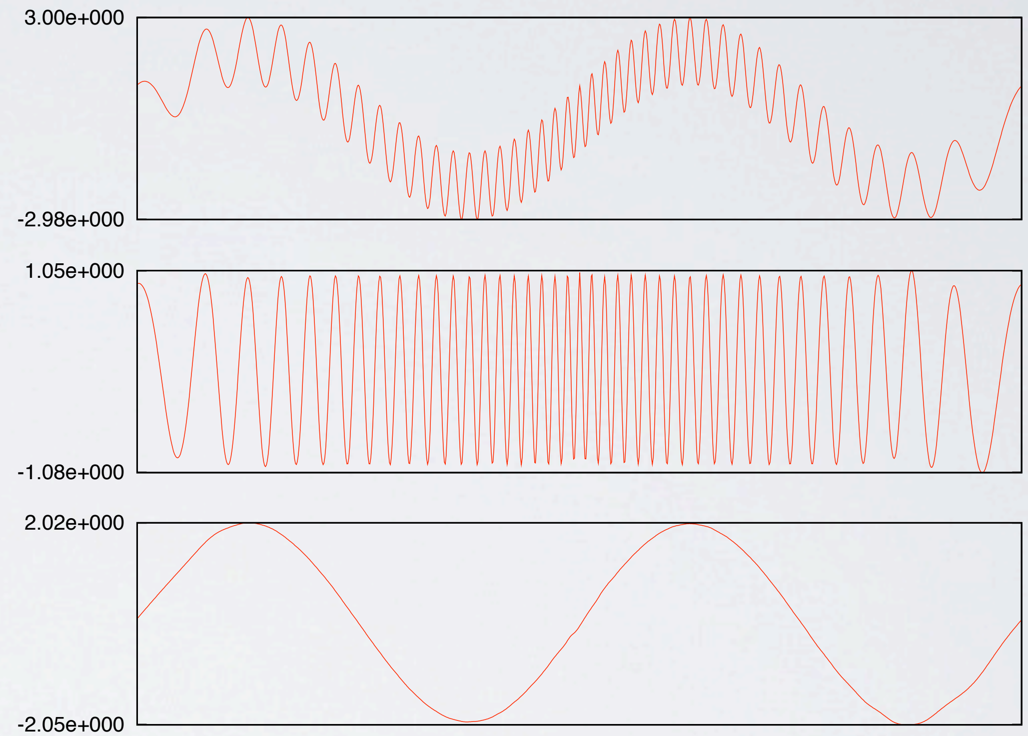


Iterative Filters yield 11 IMF's. The first 9 are essentially noise related. The last two correspond to the sinusoidal components in the original signal, which is actually given as $X(t) = \sin(2t) + 2\sin(4t) + \eta(t)$. The signal-to-noise ratio is $-5.6dB$.

Adaptive Filters



Original (top) and three
IMF's by uniform masks

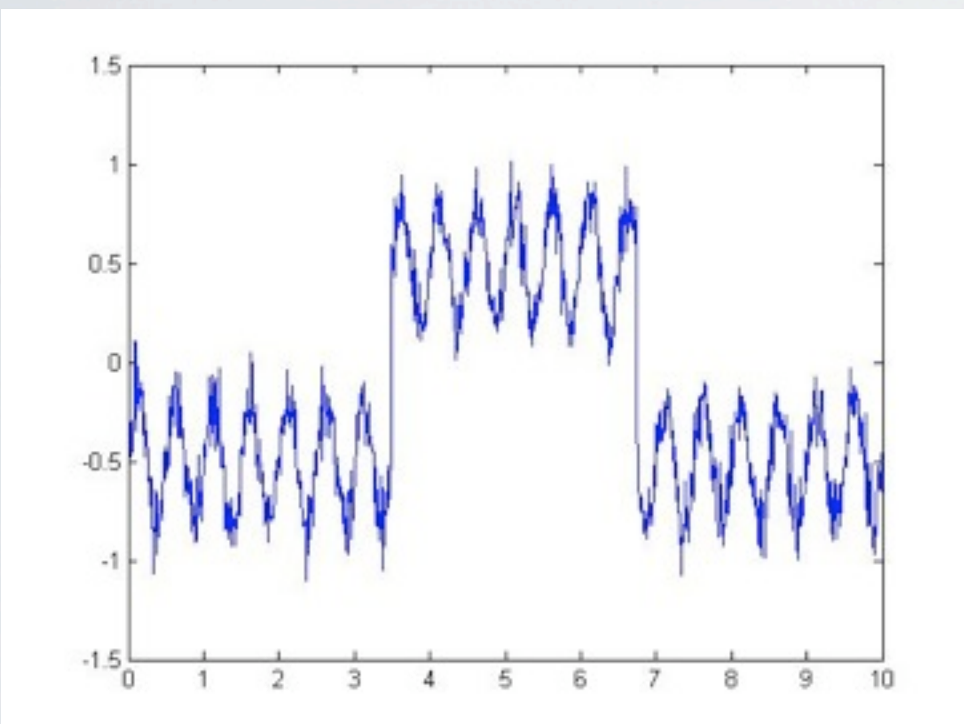


Original (top) and two IMF's
by adaptive filters

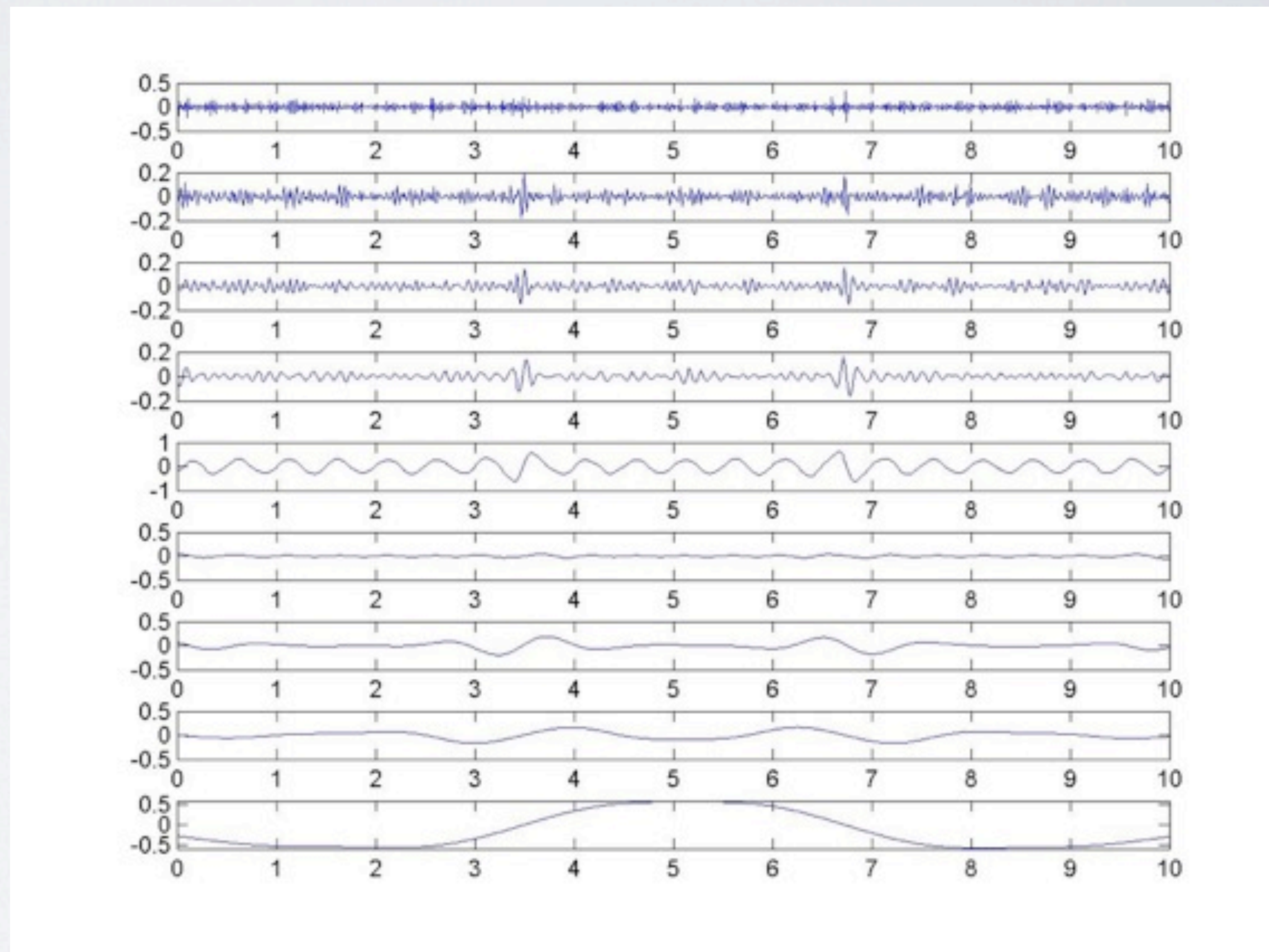
The masks (the length of the filters) are selected adaptively based on the Theorem I and II (density of the local extrema). The adaptive algorithm obtain an almost perfect decomposition for this nonlinear, non-stationary signal.

One-Side (ENO) Adaptive Filters

Sudden Changes in signal need one-side adaption

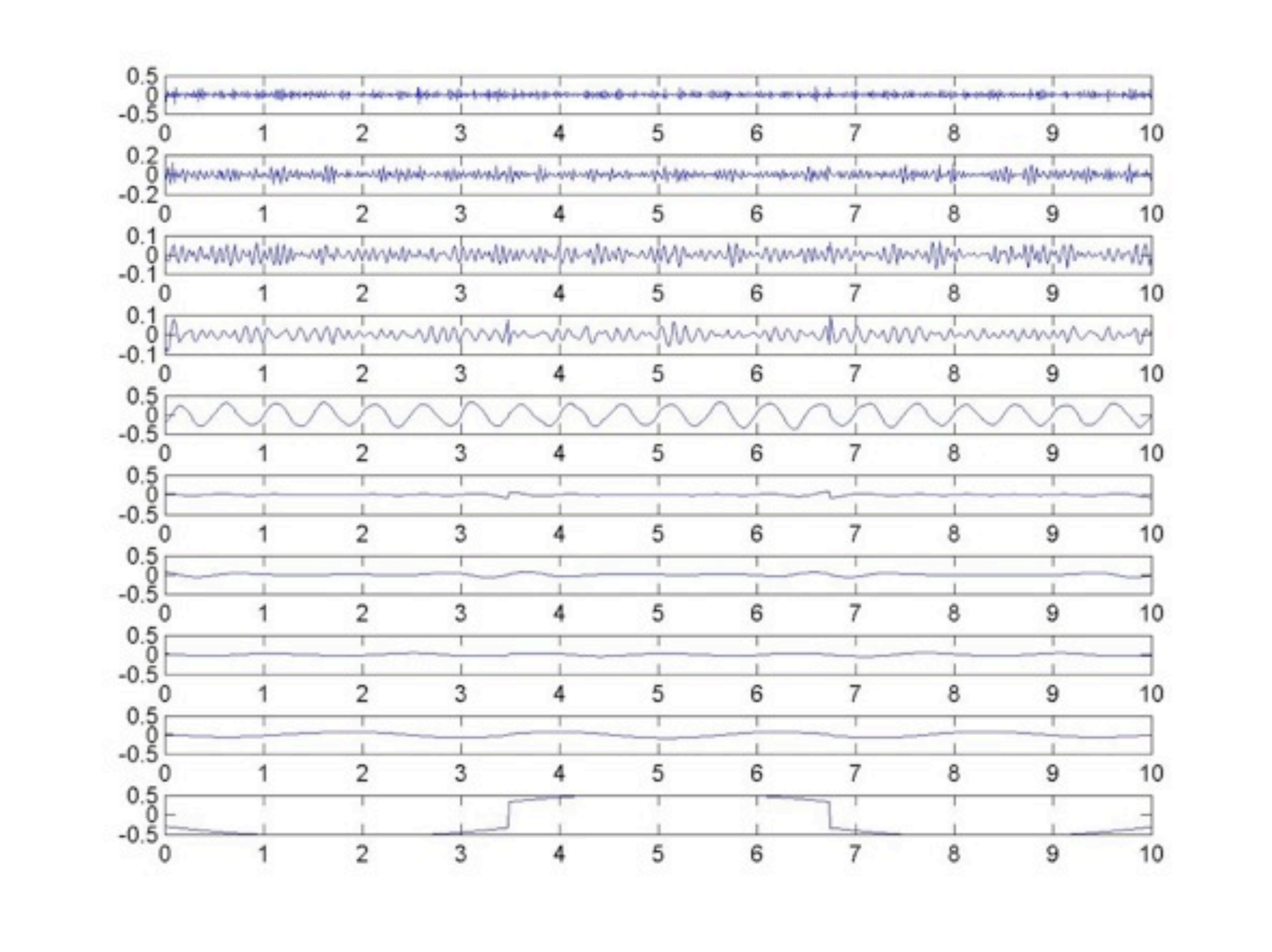


$$x_1(t) = f(t) + 0.3\sin(4\pi t) + n(t),$$



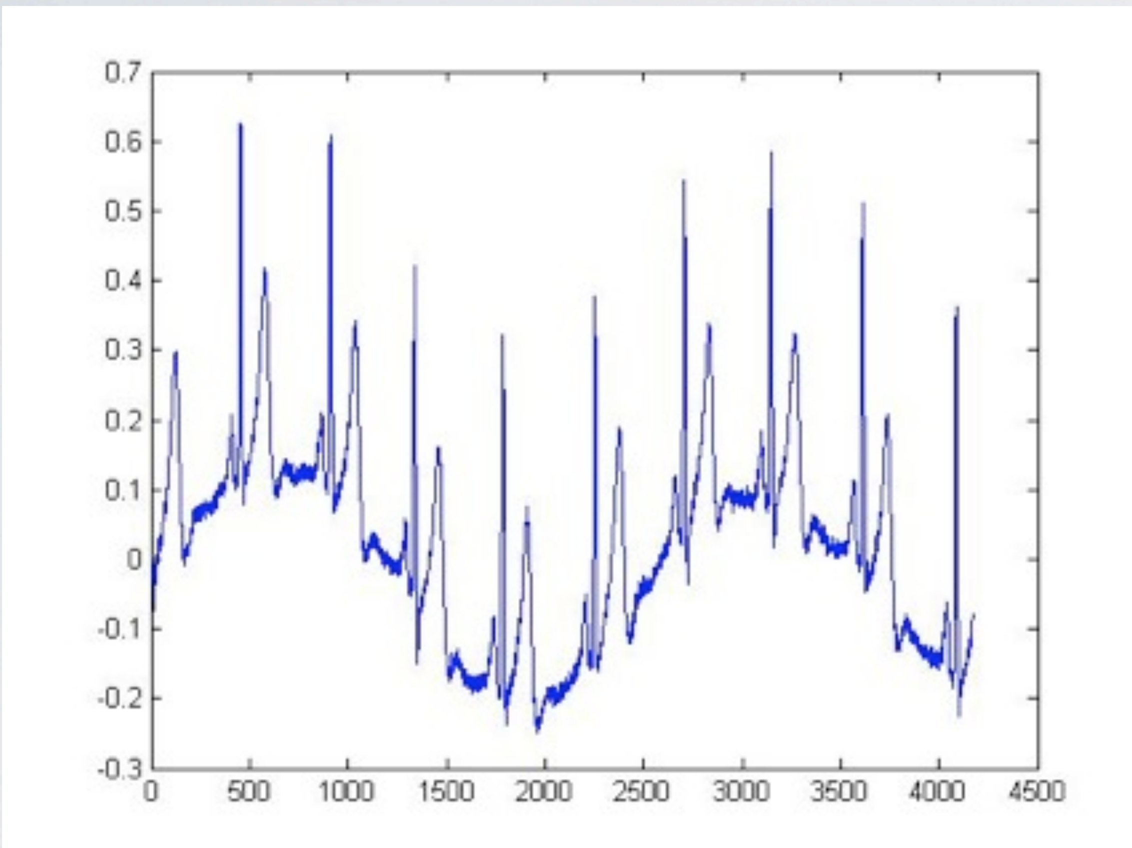
Iterative filters without one-side adaption

One-Side (ENO) Adaptive Filters

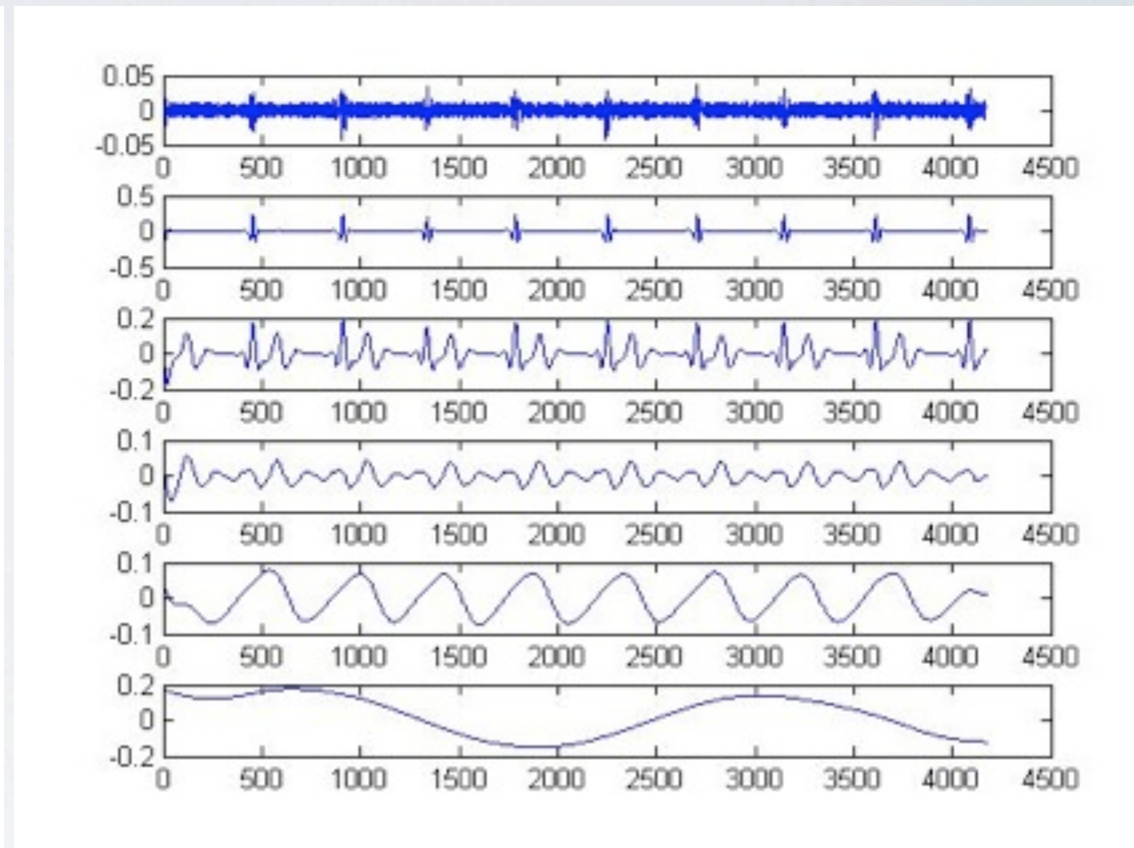


Iterative filters with one-side adaption

An Example on Real Data



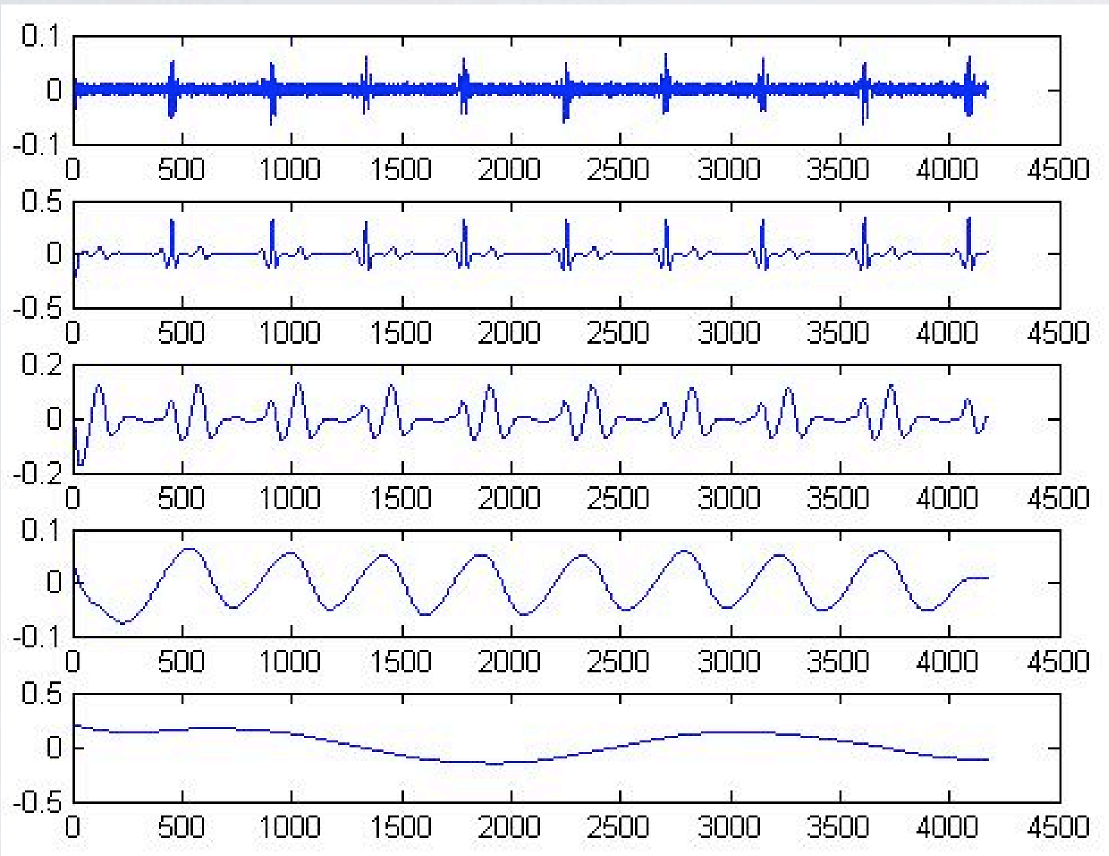
Original ECG data



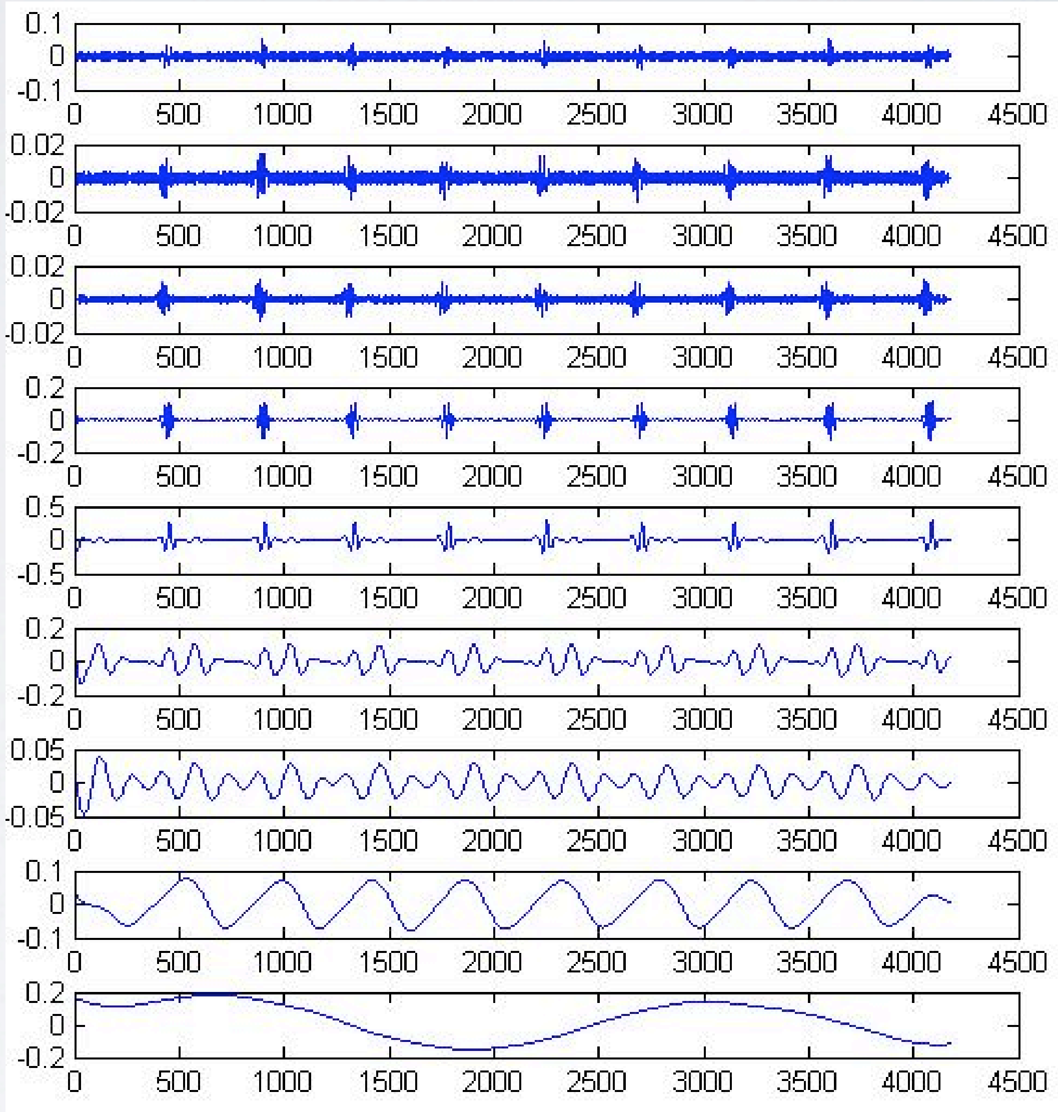
IMF's by double average filters

Iterative Filters clearly separate the original ECG data into transient parts and regular oscillations. Note the 5th IMF is a oscillation can not be captured easily.

An Example on Real Data

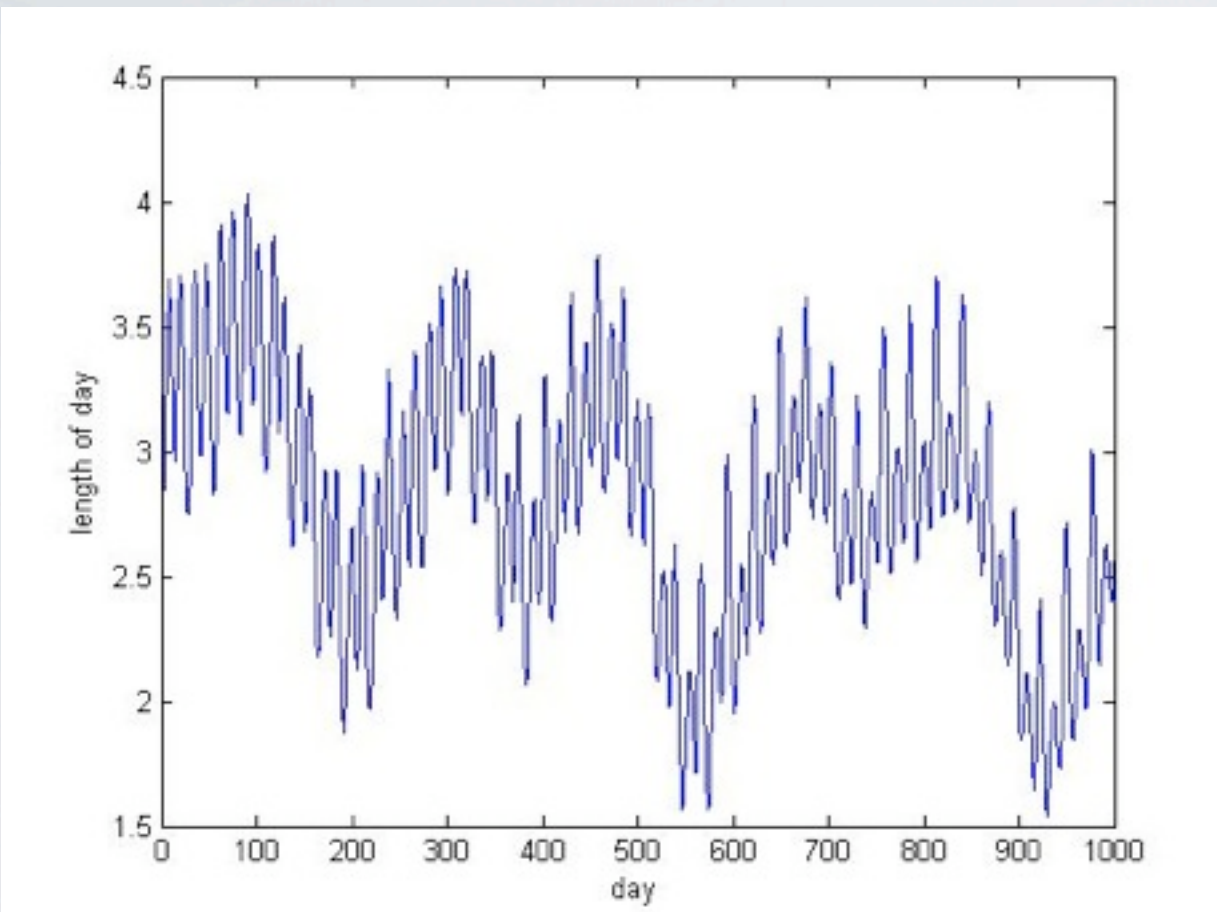


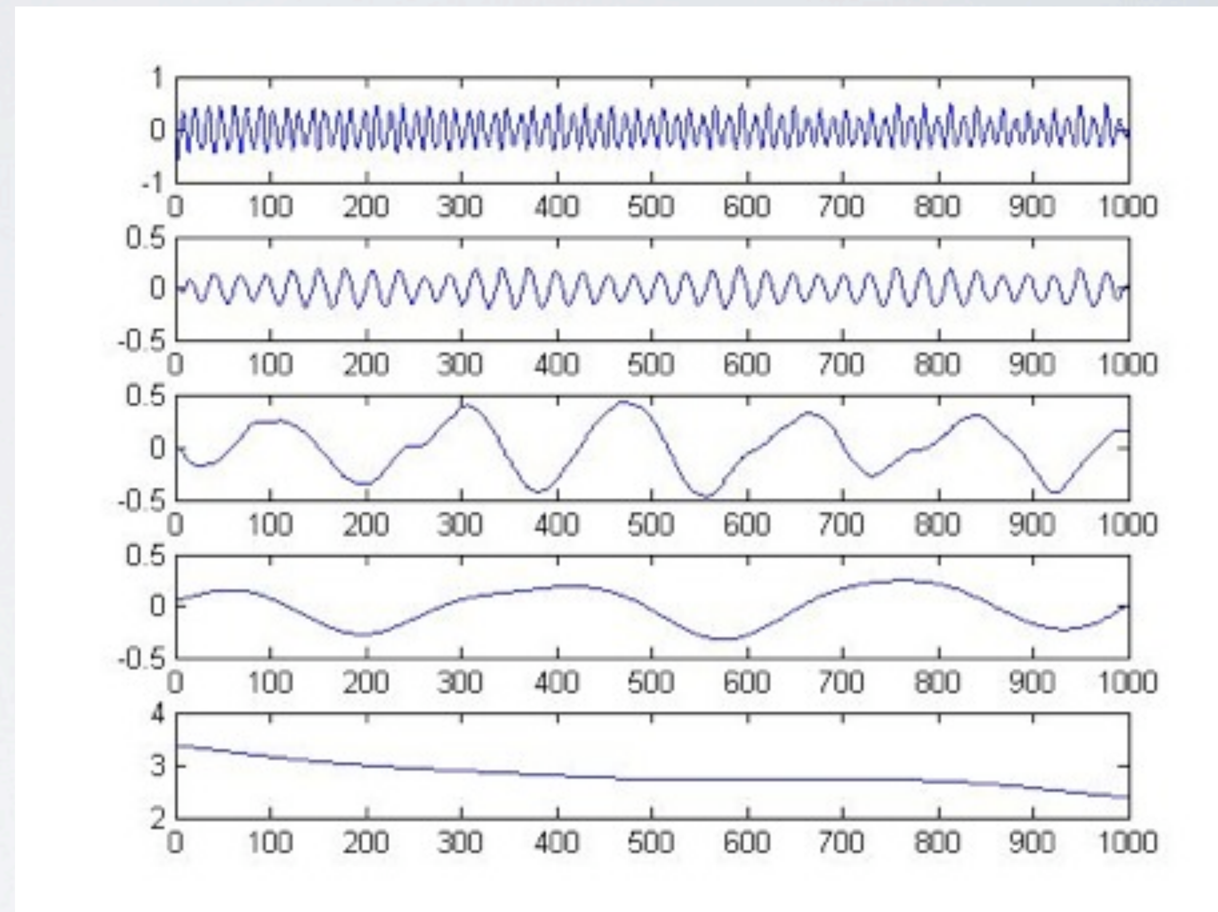
Gaussian Filters



Fokker-Planck Filters

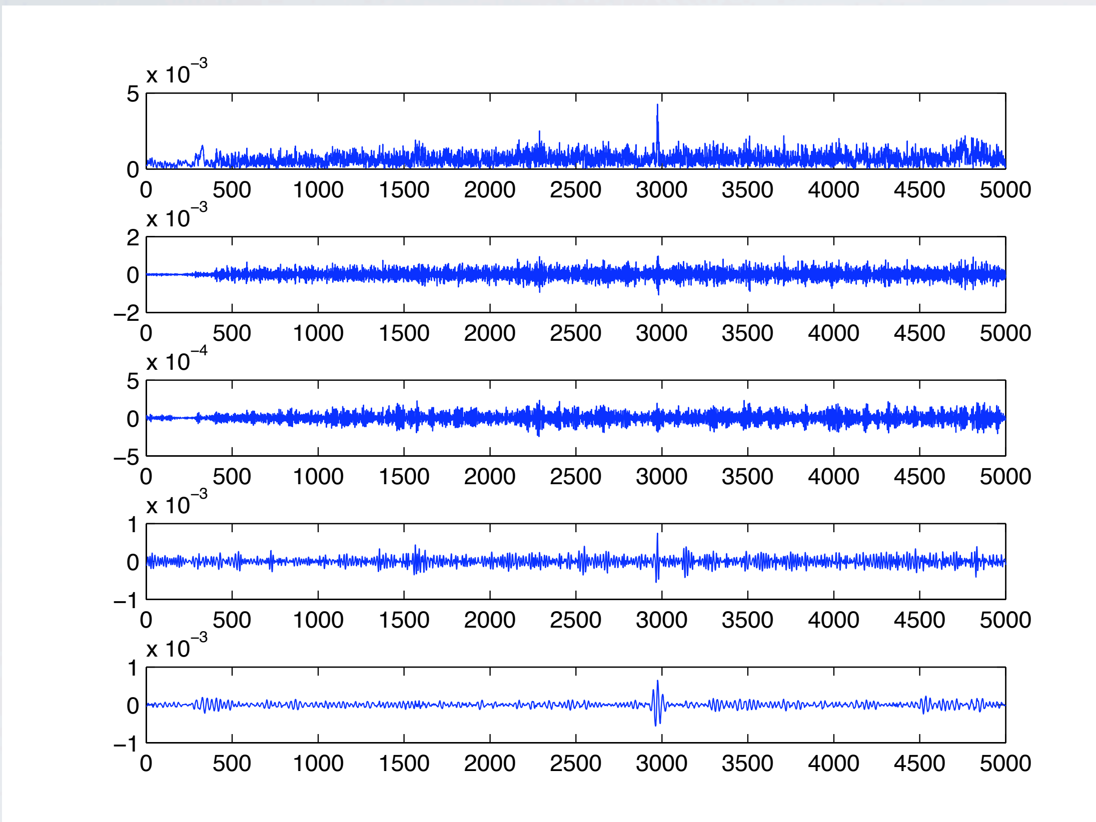
An Example on Real Data

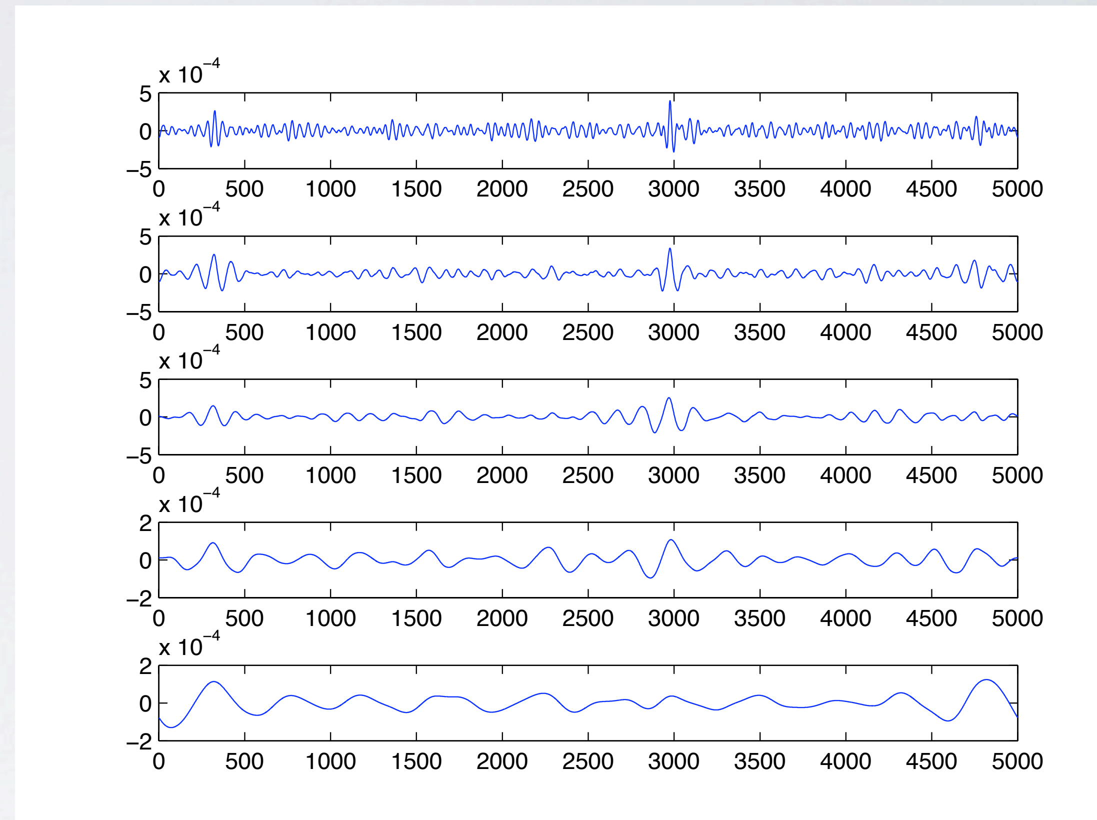




Length of days

An Example on Real Data





The original sonar data (top-left) and its 9 IMF's. It is clear that a transient component, especially singled out in the 4th-6th IMF's, is picked up around t=3000. It actually corresponding to a real underwater target at that location. (jointly with F. Crosby, Q. Huynh and R. Goroshin, working in progress.)

Conclusions and Future Work

- EMD and Iterative Filters are robust to handle nonlinear and non-stationary signals, with great application potentials.

- The results can be viewed as sparse representations of data sets.

- Some theoretical progress has been made.

- There are more questions than answers in the subject.

- Possible applications in many other areas.