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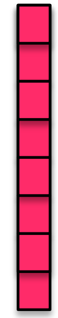
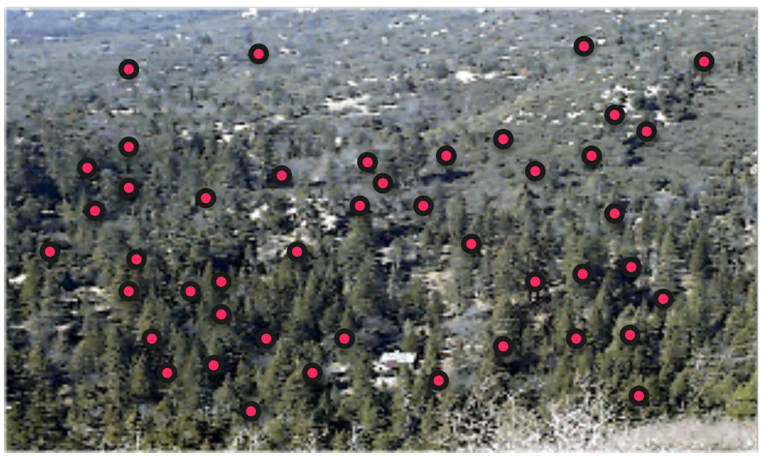
Local Convergence of an Incremental Algorithm for Subspace Identification

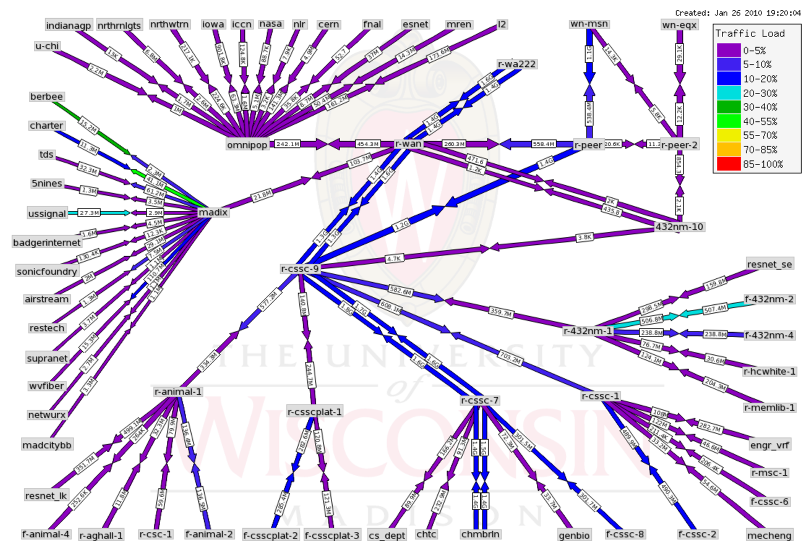
Subspace Representations

Monitor/sense with n nodes





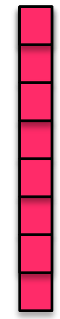
$v \in \mathbb{R}^n$ is a snapshot of the system state (e.g., temperature at each node)

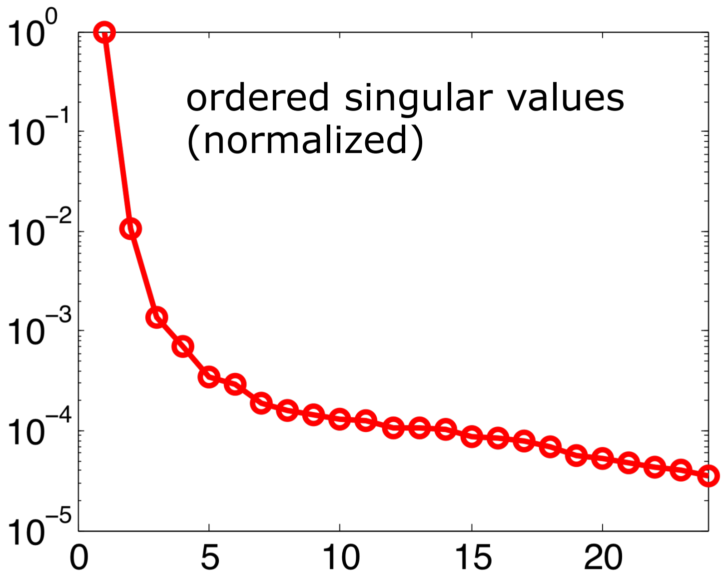


$v \in \mathbb{R}^n$ is a snapshot of the system state (e.g., traffic rates at each monitor)

Subspace Representations

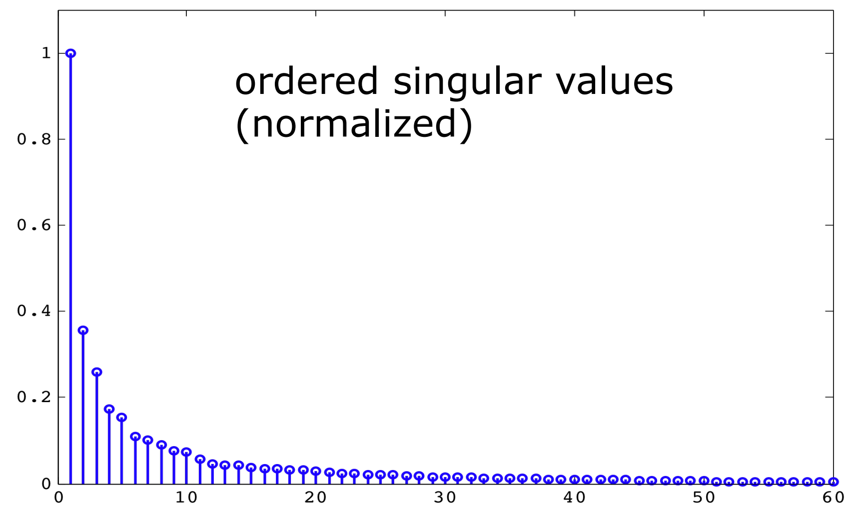
Monitor/sense with n nodes





Temperature data from UCLA Sensornet

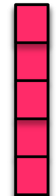
$v \in \mathbb{R}^n$ is a snapshot of the system state (e.g., temperature at each node)



Byte Count data from UW network

$v \in \mathbb{R}^n$ is a snapshot of the system state (e.g., traffic rates at each monitor)

Monitor/sense with n nodes



Each snapshot lies near a low-dimensional subspace

$$S \subset \mathbb{R}^n$$

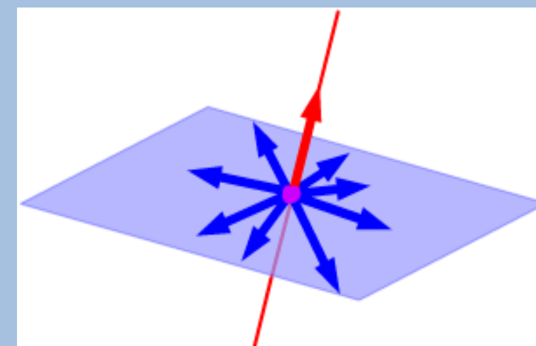


image courtesy of theJuniverse.org

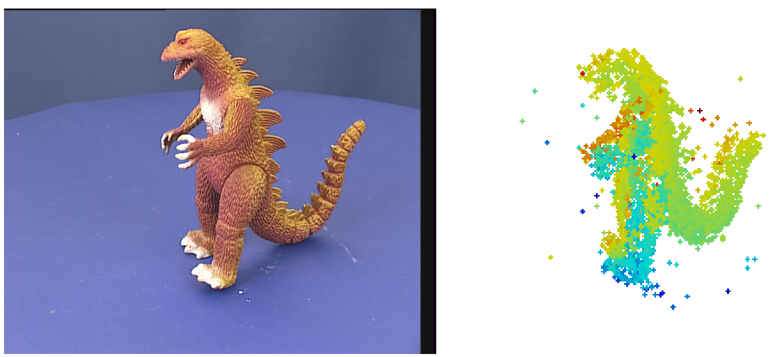
Using the **subspace as a model** for the data, we can leverage these dependencies for detection, estimation and prediction.

Byte Count data from UW network

Subspace Representations

Image with n pixels

$t = 1$ $t = 230$ $t = 1400$



(a) Dinosaur

(b) Teddy Bear

Capture n 3-d object features with a 2-d image

Missing Data

In every problem mentioned, we have missing data.

In networks, communication links fail.

In the 3d object imaging problem, some features are not visible from some perspectives

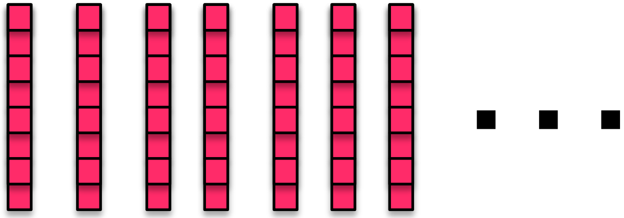
In background and foreground separation, foreground pixels obscure low-rank background pixels

In all problems, subsampling can improve processing speeds

Subspace Identification: Full Data

Suppose we receive a sequence of length-n vectors that lie in a d-dimensional subspace S:

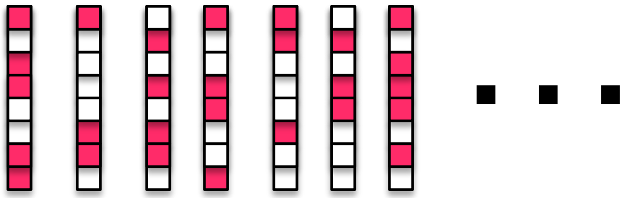
$$v_1, v_2, \ldots, v_t, \ldots, \in S \subset \mathbb{R}^n$$



Subspace Identification: Missing Data

Suppose we receive a sequence of incomplete length-n vectors that lie in a d-dimensional subspace S, and $\Omega_t \subset \{1, \ldots, n\}$ refers to the observed indices:

$$[v_1]_{\Omega_1}, [v_2]_{\Omega_2}, \ldots, [v_t]_{\Omega_t}, \ldots, \in S \subset \mathbb{R}^n$$



Problem Set-Up

- Seek subspace $S \subset \mathbb{R}^n$ of known dimension $d \ll n$.
- Know certain components $\Omega_t \subset \{1, 2, \ldots, n\}$ of vectors $v_t \in S$, $t = 1, 2, \ldots$ — the subvector $[v_t]_{\Omega_t}$.
- Assume that $\mathcal{S}$ is incoherent w.r.t. the coordinate directions.

We'll also assume for purposes of analysis that

- $v_t = \bar{U} s_t$, where $\bar{U}$ is an $n \times d$ orthonormal spanning $\mathcal{S}$ and the components of $s_t \in \mathbb{R}^d$ are i.i.d. normal with mean 0.
- Sample set Ω_t is independent for each t with $|\Omega_t| \geq q$, for some q between d and n.
- Observation subvectors $[v_t]_{\Omega_t}$ contain no noise.

Problem Set-Up

We take a stochastic gradient approach to minimizing over $\mathcal{S}$ the function

$$F(\mathcal{S}) = \sum_{i=1}^{T} \| [v_i - P_{\mathcal{S}} v_i]_{\Omega_i} \|_2^2 \; .$$

Since the variable is a subspace we optimize on the Grassmannian.

GROUSE

Given current estimate U_t and partial data vector $[v_t]_{\Omega_t}$, where $v_t = \bar{U} s_t$:

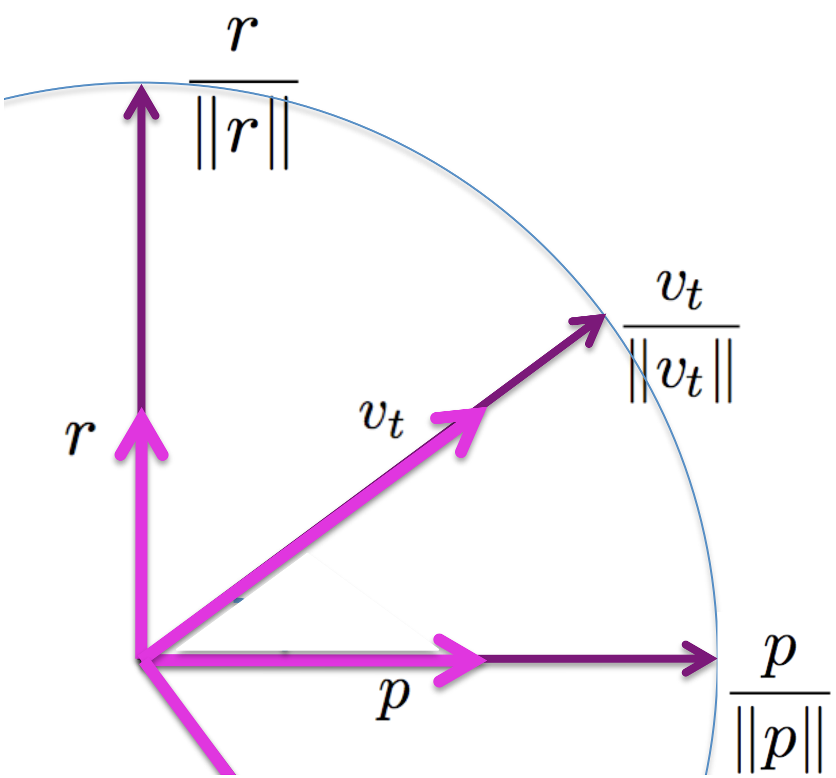
$$
\begin{aligned}
w_t &:= \arg\min_w \|[U_t w - v_t]_{\Omega_t}\|_2^2; \\
p_t &:= U_t w_t; \\
[r_t]_{\Omega_t} &:= [v_t - U_t w_t]_{\Omega_t}; \quad [r_t]_{\Omega_t^c} := 0; \\
\sigma_t &:= \|r_t\| \|p_t\|; \\
\text{Choose} \quad \eta_t &> 0; \\
U_{t+1} &:= U_t + \left[(\cos \sigma_t \eta_t - 1) \frac{p_t}{\|p_t\|} + \sin \sigma_t \eta_t \frac{r_t}{\|r_t\|} \right] \frac{w_t^T}{\|w_t\|};
\end{aligned}
$$

We focus on the (locally acceptable) choice

$$
\eta_t = \frac{1}{\sigma_t} \arcsin \frac{\|r_t\|}{\|p_t\|}, \quad \text{which yields} \quad \sigma_t \eta_t = \arcsin \frac{\|r_t\|}{\|p_t\|} \approx \frac{\|r_t\|}{\|p_t\|}.
$$

GROUSE

$$U_{t+1} := U_t + \left[(\cos \sigma_t \eta_t - 1) \frac{p_t}{\|p_t\|} + \sin \sigma_t \eta_t \frac{r_t}{\|r_t\|} \right] \frac{w_t^T}{\|w_t\|};$$



To measure the discrepancy between the current estimate span(U_t) and $\mathcal{S}$, we use the angles between the two subspaces. There are d angles between two d-dimensional subspaces, and we call them $\phi_{t,i}$, $i = 1, \ldots, d$, where

$$\cos \phi_{t,i} = \sigma_i(U_t^T \bar{U}) \ ,$$

where σ_i denotes the i^{th} singular value. Define

$$\epsilon_t := \sum_{i=1}^{d} \phi_{t,i} = d - \sum_{i=1}^{d} \sigma_i(U_t^T \bar{U})^2 = d - \|U_t^T \bar{U}\|_F^2 \ .$$

We seek a bound for $\mathbb{E}[\epsilon_{t+1} | \epsilon_t]$, where the expectation is taken over the random vector s_t for which $v_t = \bar{U} s_t$.

Outline

- ✧GROUSE algorithm convergence rate in the full-data case
- ✧GROUSE algorithm convergence rate with missing data
- ✧Equivalence of grouse to a kind of missing-data incremental SVD

Full-data case vastly simpler to analyze than the general case. Define

- $\theta_t := \arccos(\|p_t\|/\|v_t\|)$ is the angle between $R(U_t)$ and $\mathcal{S}$ that is revealed by the update vector v_t;
- Define $A_t := U_t^T \bar{U}$, $d \times d$, nearly orthogonal when $R(U_t) \approx \mathcal{S}$. We have $\epsilon_t = d - \|A_t\|_F^2$.

Lemma

$$\epsilon_t - \epsilon_{t+1} = \frac{\sin(\sigma_t \eta_t) \sin(2\theta_t - \sigma_t \eta_t)}{\sin^2 \theta_t} \left(1 - \frac{s_t^T A_t^T A_t A_t^T A_t s_t}{s_t^T A_t^T A_t s_t}\right),$$

The right-hand side is nonnegative for $\sigma_t \eta_t \in (0, 2\theta_t)$, *and zero if* $v_t \in R(U_t) = \mathcal{S}_t$ *or* $v_t \perp \mathcal{S}_t$.

Theorem

Suppose that $\epsilon_t \le \bar{\epsilon}$ *for some* $\bar{\epsilon} \in (0, 1/3)$. *Then*

$$E\left[\epsilon_{t+1} \,|\, \epsilon_t\right] \le \left(1 - \left(\frac{1-3\bar{\epsilon}}{1-\bar{\epsilon}}\right)\frac{1}{d}\right)\epsilon_t.$$

Since the sequence $\{\epsilon_t\}$ is decreasing, by the earlier lemma, we have $\epsilon_t \downarrow 0$ with probability 1 when started with $\epsilon_0 \le \bar{\epsilon}$.

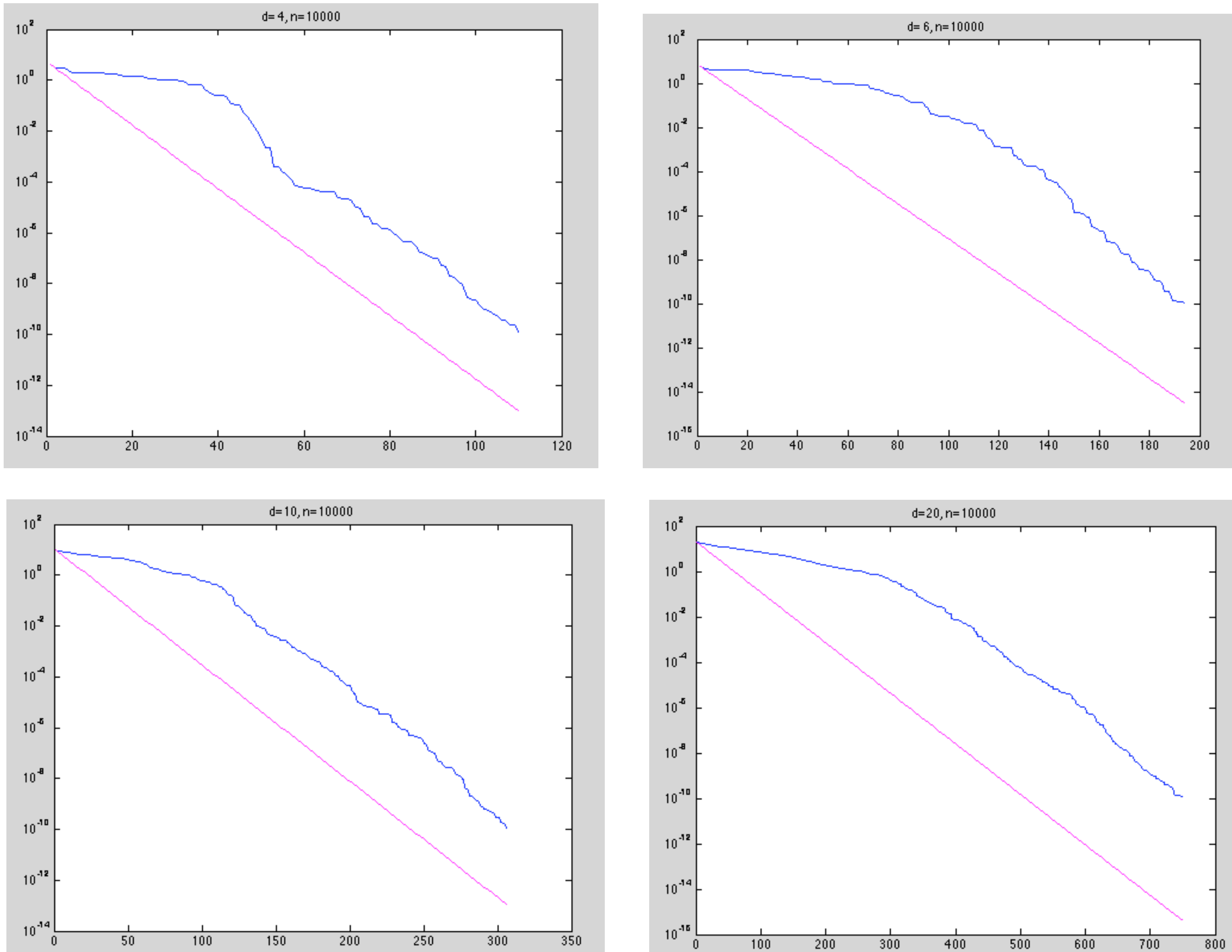
Linear convergence rate is asymptotically $1 - 1/d$.

- For $d = 1$, get near-convergence in one step (thankfully!)
- Generally, in d steps (which is number of steps to get the exact solution using SVD), improvement factor is

$$(1 - 1/d)^d < \frac{1}{e}.$$

ε_t versus 1-1/d

n=10000
d=4, 6, 10, 20



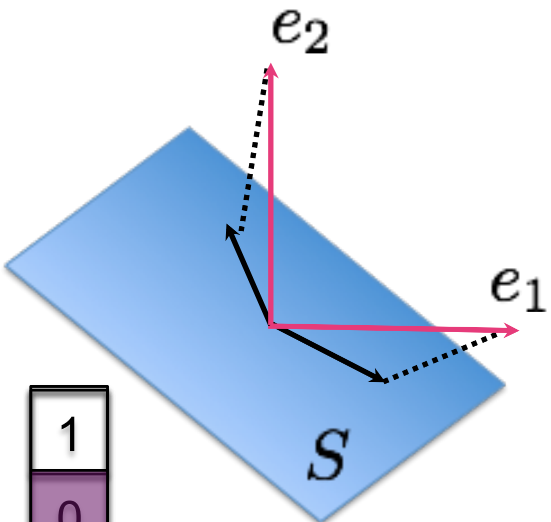
Outline

- GROUSE algorithm convergence rate in the full-data case
- GROUSE algorithm convergence rate with missing data
- Equivalence of grouse to a kind of missing-data incremental SVD

Coherence

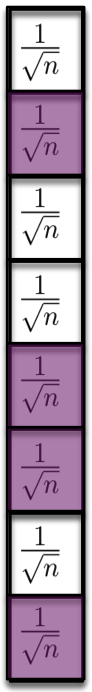
A fundamental problem with subsampling is that we may miss the important information.

How aligned are the subspace S and the vector v to the canonical basis?



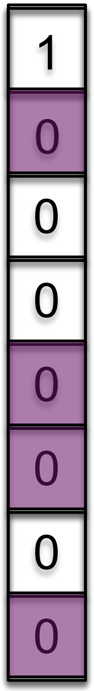
Examples of bases that form an Incoherent Subspace:

- Orthonormalize Gaussian random vectors.
- Fourier basis.



$$\mu := \frac{n}{d} \max_j \|P_S e_j\|_2^2$$

$$1 \leq \mu(v) \leq n$$



Examples of bases that form a Coherent Subspace:

- Identity basis.
- Any basis where the vectors are very sparse.

Our Result for the General Case

Recall, n is the ambient dimension, d the inherent dimension, we have $|\Omega| > q$ samples per vector. We have assumptions on the number of samples, the coherence in the subspaces and in the residual vectors, and we require that these assumptions hold with probability $1-\delta$ for $\delta \in (0, .6)$. Then for

$$\epsilon_t \leq (8 \times 10^{-6})(.6-\delta)^2 \frac{q^3}{n^3 d^2}$$

we have

$$\mathbb{E}[\epsilon_{t+1} | \epsilon_t] \leq \left(1 - (.16)(.6-\delta)\frac{q}{nd}\right) \epsilon_t \ .$$

Comments

$$\epsilon_t \leq (8 \times 10^{-6})(.6 - \delta)^2 \frac{q^3}{n^3 d^2}$$

$$\mathbb{E}[\epsilon_{t+1} | \epsilon_t] \leq \left(1 - (.16)(.6 - \delta)\frac{q}{nd}\right) \epsilon_t \ .$$

The decrease constant is not too far from that observed in practice; we see a factor of about

$$1 - X\frac{q}{nd}$$

where X is not much less than 1.

The threshold condition on ϵ_t, however, is quite pessimistic. Linear convergence behavior is seen at much higher values.

Outline

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- GROUSE algorithm convergence rate in the full-data case
- GROUSE algorithm convergence rate with missing data
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The standard iSVD

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Algorithm 2 iSVD: Full Data

Given U_0, an arbitrary $n \times d$ orthonormal matrix, with $0 < d < n$; Σ_0, a $d \times d$ diagonal matrix of zeros which will later hold the singular values, and V_0, an arbitrary $n \times d$ orthonormal matrix.

for $t = 0, 1, 2, \ldots$ **do**

Take the current data column vector v_t;

Define $w_t := \arg\min_w \|U_t w - v\|_2^2 = U_t^T v_t$;

Define

$$p_t := U_t w_t; \quad r_t := v_t - p_t;$$

Noting that

$$\begin{bmatrix} U_t \Sigma_t V_t^T & v_t \end{bmatrix} = \begin{bmatrix} U_t & \frac{r_t}{\|r_t\|} \end{bmatrix} \begin{bmatrix} \Sigma_t & w_t \\ 0 & \|r_t\| \end{bmatrix} \begin{bmatrix} V_t & 0 \\ 0 & 1 \end{bmatrix}^T,$$

we compute the SVD of the update matrix:

$$\begin{bmatrix} \Sigma_t & w_t \\ 0 & \|r_t\| \end{bmatrix} = \hat{U}\hat{\Sigma}\hat{V}^T,$$

and set

$$U_{t+1} := \begin{bmatrix} U_t & \frac{r_t}{\|r_t\|} \end{bmatrix} \hat{U}, \quad \Sigma_{t+1} = \hat{\Sigma}, \quad V_{t+1} = \begin{bmatrix} V_t & 0 \\ 0 & 1 \end{bmatrix} \hat{V}.$$

end for

How do we incorporate missing data?

- We could put zeros into the matrix
  - Very interesting recent results from Sourav Chatterjee on one-step "Universal Singular Value Thresholding" show that zero-filling followed by SVD reaches the minimax lower bound on MSE.
  - But in the average case, we see that convergence of the zero-filled SVD is very very slow.
- Let's instead replace the missing entries with our prediction using the existing model

iSVD with missing data 2

Algorithm 4 iSVD: Partial Data, Forget singular values

Given U_0, an $n \times d$ orthonormal matrix, with $0 < d < n$;

for $t = 0, 1, 2, \ldots$ **do**

Take Ω_t and v_{Ω_t} from (2.1);

Define $w_t := \arg\min_w \|U_{\Omega_t} w - v_{\Omega_t}\|_2^2$;

Define vectors $\tilde{v}_t$, p_t, r_t:

$$(\tilde{v}_t)_i := \begin{cases} v_i & i \in \Omega_t \\ (U_t w_t)_i & i \in \Omega_t^C \end{cases}; \quad p_t := U_t w_t; \quad r_t := \tilde{v}_t - p_t;$$

Noting that

$$\begin{bmatrix} U_t & \tilde{v}_t \end{bmatrix} = \begin{bmatrix} U_t & \frac{r_t}{\|r_t\|} \end{bmatrix} \begin{bmatrix} I & w_t \\ 0 & \|r_t\| \end{bmatrix},$$

we compute the SVD of the update matrix:

$$\begin{bmatrix} I & w_t \\ 0 & \|r_t\| \end{bmatrix} = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^T,$$

and set $U_{t+1} := \begin{bmatrix} U_t & \frac{r_t}{\|r_t\|} \end{bmatrix} \widetilde{U}_{:,1:d} W_t$, where W_t is an arbitrary $d \times d$ orthogonal matrix.

end for

GROUSE and iSVD equivalence

Theorem

Suppose we have the same U_t and $[v_t]_{\Omega_t}$ at the t-th iterations of iSVD and GROUSE. Then there exists $\eta_t > 0$ in GROUSE such that the next iterates U_{t+1} of both algorithms are identical, to within an orthogonal transformation by the $d \times d$ matrix

$$W_t := \left[\frac{w_t}{\|w_t\|} \mid Z_t \right],$$

where Z_t is a $d \times (d-1)$ orthonormal matrix whose columns span $N(w_t^T)$.

The precise values for which GROUSE and iSVD are identical are:

$$\lambda = \frac{1}{2}\left[(\|w_t\|^2 + \|r_t\|^2 + 1) + \sqrt{(\|w_t\|^2 + \|r_t\|^2 + 1)^2 - 4\|r_t\|^2} \right]$$

$$\beta = \frac{\|r_t\|^2 \|w_t\|^2}{\|r_t\|^2 \|w_t\|^2 + (\lambda - \|r_t\|^2)^2}$$

$$\eta_t = \frac{1}{\sigma_t} \arcsin \beta.$$

Future Directions

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- ✧ Apply GROUSE analysis to ell-1 version, GRASTA
- ✧ Re-think the proof from new angles.
  - ✧ We see convergence at higher ε.
  - ✧ We see monotonic decrease at any random initialization.
  - ✧ We see convergence even without incoherence (but good steps are only made when the samples align).

Thank you!

Questions?