The Phase Retrieval Problem

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$\boxed{\text{The Classic Phase Retrieval (PR) Problem}}$

Let $f \in L^2(\mathbb{R}^d)$, f real and $f \geqslant 0$. Can we reconstruct f from $|\hat{f}|$?

Applications X-ray diffraction, wavefront sensing, communication, astronomy, etc.

- **Answer: NO in general!**

Under some constraints, the answer might be YES.

Ambiguities in PR

We have $|\hat{g}| = |\hat{f}|$ if

- $g(x) = f(-x)$.
- $g(x) = f(x-a)$ for any $a \in \mathbb{R}^d$.

$\Big\}$ trivial ones

- Let $f(x) = f_1(x) * f_2(x)$. Set e.g.

$$g(x) = f_1(-x) * f_2(x-a).$$

- The "trivial" ambiguities can be easily overcome through suitable normalization. The third one poses a difficult challenge. There might be other ambiguities.
- These challenges have not detered people from trying PR.

"Projection" Algorithm for PR

- When one attempts PR, there is always an implicit assumption that the problem has enough constraints that there is a unique solution.

- All existing algorithms are heuristics based.

- Most algorithms are "projection" based:

Get an initial guess from constraint A;

↓

Apply constraint B to get an update;

↓

Apply constraint A again;

↓

⋮

General Phase Retrieval in Finite Dimensions

Let $\mathcal{F} = \{f_1, f_2, \cdots, f_N\} \subset \mathbb{R}^d$ or $\mathbb{C}^d$. The Phase Retrieval Problem for $\mathcal{F}$ is :

Let $x \in \mathbf{H} = \mathbb{R}^d$ or $\mathbb{C}^d$. Given $\{|\langle x, f_j \rangle|\}_{j=1}^N$, can we reconstruct x ?

- Clearly, we can only reconstruct x up to cx where $|c|=1$.
- Define $x \sim y$ if $y = cx$ for some $|c|=1$. Let

$$M_{\mathcal{F}} : H/\sim \longrightarrow \mathbb{R}^N$$

$$x \longmapsto \left[|\langle x, f_1 \rangle|^2, |\langle x, f_2 \rangle|^2, \cdots, |\langle x, f_N \rangle|^2 \right]^T.$$

We say $\mathcal{F}$ is Phase Retrieval (PR) if $M_{\mathcal{F}}$ is injective.

Real Case $\mathbf{H} = \mathbb{R}^d$

Let $\mathcal{F} = \{f_1, f_2, \cdots, f_N\} \subset \mathbb{R}^d$. Assume $x \nsim y$, i.e. $x \neq \pm y$ such that $|\langle x, f_j\rangle| = |\langle y, f_j\rangle|$ for all j.

- $\langle x, f_j\rangle = \pm\langle y, f_j\rangle$, $j = 1, 2, \cdots, N$.

  Say
  $$\langle x, f_j\rangle = \langle y, f_j\rangle, \quad j = 1, \cdots, K;$$
  $$\langle x, f_j\rangle = -\langle y, f_j\rangle, \quad j = K+1, \cdots, N.$$

  Hence
  $$\langle x-y, f_j\rangle = 0, \quad j = 1, 2, \cdots, K$$
  $$\langle x+y, f_j\rangle = 0, \quad j = K+1, \cdots, N$$

- There exists a partition of $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ such that neither $\mathcal{F}_1$ nor $\mathcal{F}_2$ spans $\mathbb{R}^d$.

Theorem Let $\mathcal{F} = \{f_1, \cdots, f_N\} \subset \mathbb{R}^d$. Then $\mathcal{F}$ is PR iff for any partition $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, either $\mathcal{F}_1$ spans $\mathbb{R}^d$ or $\mathcal{F}_2$ spans $\mathbb{R}^d$.

Corollary $\mathcal{F}$ is PR $\Rightarrow N \geq 2d-1$.

Furthermore, a generic $\mathcal{F}$ with $N \geq 2d-1$ vectors is PR.

Theorem Let $\mathcal{F}$ be a generic set of $N \geq d+1$ vectors in $\mathbb{R}^d$. Then $M_{\mathcal{F}}$ is injective for almost all $x \in \mathbb{R}^d/\sim$, i.e. $M_{\mathcal{F}}^{-1}(M_{\mathcal{F}}(x)) = \{x\}$ for a.e. $x \in \mathbb{R}^d/\sim$.

- Still, many challenges even in the real case!

Complex Case $\mathbb{H} = \mathbb{C}^d$

The complex case is almost an entirely different game! Many basic questions remain open.

- Let $N(d)$ denotes the minimal number of vectors needed in $\mathcal{F} \subset \mathbb{C}^d$ to have phase retrieval. Then using the real case technique we get $N(d) \geq 2d-1$. But this is far from optimal.

- Balan - Casazza - Edidin 06 : $N(d) \leq 4d-2$.
  - Furthermore, a generic $\mathcal{F}$ with $N \geq 4d-2$ vectors has PR, i.e. $N_g(d) \leq 4d-2$.
  - Conjectured $N(d) = 4d-2$.

Clearly false as $N(2) = 4$.

- Bodmann 2012: $N(d) \leqslant 4d-4$.
  - Doesn't show $4d-4$ generic vectors have PR.

- Finkelstein 2004: $N(d) \geqslant 3d-2$.

- Heinosaari, Mazzarella, Wolf 2012:

$$N(d) \geqslant 4d-4-2\alpha(d-1)$$

where $\alpha(n)$ is the number of 1's in the binary expansion of n.

- Conjecture: $N(d) = 4d-4$. True: $d=3$ (Cahill-Mixon, YW)

- Balan-Casazza-Edidin 06: $N \geqslant 2d$ generic vectors is PR for almost all $x \in \mathbb{C}^d/\sim$.
  - YW 2012: $N < 2d$ generic vectors in $\mathbb{C}^d$ is not PR for almost all $x \in \mathbb{C}^d/\sim$.

Theorem Let $\mathcal{F} = \{f_j\}_{j=1}^N \subset \mathbb{C}^d$. The following are equivalent

(i) $\mathcal{F}$ is not PR.

(ii) There exist non-colinear $u, v \in \mathbb{C}^d$ such that

$$\mathrm{Re}\left(\overline{\langle u, f_j\rangle}\langle v, f_j\rangle\right) = 0 \quad \text{for all } j.$$

(iii) The (real) Jacobian of $M_{\mathcal{F}}$ has rank $< 2d-1$ at some nonzero point on $\mathbb{C}^d/\mathfrak{u}$.

Theorem (YW 2012)

Identify $\mathcal{F} = \{f_j\}_{j=1}^N$ with $F = [f_1, f_2, \cdots, f_N] \in \mathbb{C}^{d\times N}$.

Let $\Pi(d,N) = \{F \in \mathbb{C}^{d\times N} : F \text{ is PR}\}$. Then $\Pi(d,N)$ is open in $\mathbb{C}^{d\times N}$.

PR Algorithms

- The minimality problems are fundamental, but are only a small part of the challenges in PR.

- One big challenge is the computational aspect of PR.

  Efficient algorithm? Robustness?

- The brute force algorithm is useless.

- "Reasonable" PR algorithms fall into one of 3 classes:
  - General $\mathcal{F}$ with $N \gg d$ $(N = O(d^2))$.
  - Specially designed $\mathcal{F}$.
  - Random $\mathcal{F}$ using convex relaxation.

$$\boxed{N = O(d^2)}$$

The basic idea is very simple: Each $|\langle x, f\rangle|^2$ can be written as a linear combination of quadratic monomials.

- Real case: x_j^2, $x_i x_j$.
- Complex: $|x_j|^2$, $\mathrm{Re}(x_i x_j)$, $\mathrm{Im}(x_i x_j)$.

So if $N \geq \frac{d(d+1)}{2}$ (real) or $N \geq d^2$ (complex) one can solve $|\langle x, f_j\rangle|^2 = r_j^2$, $j = 1, \cdots, N$ via solving a system of linear equations.

- Work by Balan et al, "Painless" reconstruction.

Convex Relaxation — Phase Lift

PR in $\mathbb{R}^d/\sim$ or $\mathbb{C}^d/\sim$

$\Longleftrightarrow$ Reconstructing $xx^* = [\bar{x}_i x_j]_{d\times d}$

- We know

$$|\langle x, f_j\rangle|^2 = x^* f_j f_j^* x = tr(f_j f_j^* x x^*).$$

Set $X = xx^*$, $S_j = f_j f_j^*$, and

$$\mathcal{A}(X) = [tr(S_j X)]^T.$$

- View $\mathcal{A}$ as linear map $\mathbb{C}^{d\times d} \xrightarrow{\mathcal{A}} \mathbb{R}^N$.

Knowing $\mathcal{A}(X)$, find X.

$\boxed{\text{PR}}$ $\begin{cases} \text{Solve for } \mathcal{A}(X) = b \\ \text{subject to } X \succeq 0, \ \text{rank}(X) = 1. \end{cases}$

(impossible to solve in general)

$\boxed{\text{Relaxation I}}$

$$\min \ \text{rank}(X)$$
$$\text{subject to } \mathcal{A}(X) = b, \ X \succeq 0.$$

$\boxed{\text{Relaxation II}}$

$$\min \ \text{tr}(X)$$
$$\text{subject to } \mathcal{A}(X) = b, \ X \succeq 0.$$

- Candes, Strohmer & Voroninski – Phase Lift
- Can be solved by semidefinite programming.

Theorem (Candes, Strohmer & Voroninski)

For any x, let $\mathcal{F} = \{f_j\}_{j=1}^{N}$ be $N \geq c_0 d \log d$ randomly chosen unit vectors (uniformly distributed) on the unit sphere. The with high probability the Phase Lift algorithm reconstructs x.

Theorem (Candes & Li)

Improved on the above to

(i) for all x

(ii) $N \geq c_0 d$

Constructing Special $\mathcal{F}$ for PR

Another approach is to construct special $\mathcal{F} = \{f_j\}_{j=1}^N$ so that PR can be done efficiently using a "small" number of vectors. This is easier said than done.

- Balan, Bodmann, Casazza, Edidin 2009: "Painless" frame reconstruction, $N = O(d^2)$, $O(d^4)$ complexity.

- Random $\mathcal{F}$: $N \gtrsim C_0 d \log d$ or $N \gtrsim C_0 d$.

  C_0 large, no precise estimate, slow.

- Alexeev, Afonso, Fickus, Mixon 2012: A graph theoretic construction with $N = Cd$ and $O(N^3)$ complexity, C very large ($\gtrsim 240$).

- Fickus & YW: A class of $\mathcal{F}$ through a recursive construction. $N = Cd\log_2 d$ with C being very small ($C = 2$ works, bigger C yields more robustness), $O(d^2)$ complexity.

Basic idea:

$$\begin{array}{ccc} \mathbb{R}^{\frac{d}{2}} & \times & \mathbb{R}^{\frac{d}{2}} \\ \downarrow & \text{Add links} & \downarrow \\ PR\ \mathcal{F}_1 & \longleftrightarrow & PR\ \mathcal{F}_2 \end{array}$$

- $\mathcal{F}$ with $N = Cd$ with "small" C?

Robustness

All the algorithms mentioned above have some type of robustness, i.e. if the measurements have some error bounded by ε, the error of the reconstruction is within some δ. δ depends on ε, d and $\mathcal{F}$.

- Two noise models

I $\quad b_j = |\langle x, f_j \rangle| + n_j \quad$ (b_j: measurement)

II $\quad b_j = |\langle x, f_j \rangle|^2 + n_j$

- There have been some estimates on the robustness under model II, e.g. Eldar & Mendelson 2012.

- Precise estimate in Model I (Balan & YW)

Let $F = [f_1, \cdots, f_N]$. For any $S \subset \{1, \cdots, N\}$ let F_S be the submatrix consisting of columns f_j, $j \in S$

We now introduce two quantities $\tau(F)$ and $\omega(F)$,

$$\tau(F) := \min_{S \subset \{1,2,\cdots,N\}} \max \{\sigma_d(F_S), \sigma_d(F_{S^c})\},$$

$$\omega(F) := \min_{S \subset \{1,2,\cdots,N\}} \{\sigma_d(F_S) : \operatorname{rank}(F_{S^c}) < d\}.$$

$$\omega(F) \geq \tau(F)$$

Theorem (Balan & YW).

Assume that $\varepsilon = \left(\sum |n_j|^2\right)^{\frac{1}{2}}$. For each $x \in \mathbb{R}^d$ let $\hat{x}$ be the reconstruction. Then

(i) $$\sup_{\|x\|=1} \|x - \hat{x}\| \leq \frac{\varepsilon}{\tau(F)};$$

(ii) $$\sup_{\|x\|=1} \|x - \hat{x}\| \geq \frac{\varepsilon}{\omega(F)}.$$

- For "really small" $\varepsilon > 0$, we've precisely

$$\sup_{\|x\|=1} \|x - \hat{x}\| = \frac{\varepsilon}{\omega(F)}.$$

- For Gaussian $\mathcal{F}$, with $N \geq \lambda_0 d$, $\lambda_0 > 2$, with "high" probability, $\tau(F) \geq \mu_0 \sqrt{d}$, μ_0 depends only on λ_0.