Vertex order shellings

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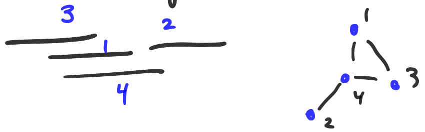
From graphs . . .

Interval graph intersection graph of intervals in $\mathbb{R}$



Unit interval graph Same, with unit intervals

Ex: see above

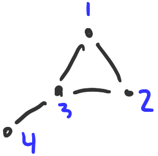
non-ex:

. . . through vertex orders. . .

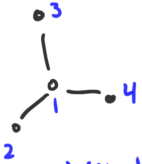
Interval graph order vertices s.t. $ac \in G$ and $a < c$
then $ab \in G$ for all $a < b < c$

Unit interval graph order vertices such that if
$ac \in G$ and $a < c$
then $ab, bc \in G$ for all
$a < b < c$

Ex



unit interval



interval, not unit

. . . to complexes

INTERVAL

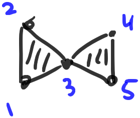
(Under-closed) order Order on the vertices of Δ such that pure

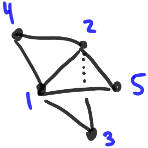
if $v_0 v_1 \dots v_d$ is a facet of Δ $v_0 < v_1 < \dots < v_d$

then $v_0 w_1 \dots w_d$ is a facet of Δ

whenever $w_i \leq v_i$ for all $i \in [d]$.

Ex $\dim(\Delta) = 2$





Complexes continued

$\dim(\Delta) = d$

Unit interval order Order on the vertices of Δ such that

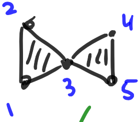
if $v_0 v_1 \dots v_d$ is a facet of Δ

complete d-skeleton on $\{v_0, \dots, v_d\}$

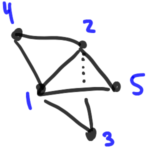
then $w_0 w_1 \dots w_d$ is a facet of Δ

whenever $w_i \in \{v_0, v_0 + 1, \dots, v_d\}$ for all $i \in [d]$.

Ex $\dim(\Delta) = 2$



✓



✗ NO

Computation

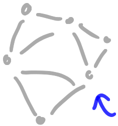
Marta's code: `https://detector.martapavelka.com/`

Boxicity for graphs: Smallest k s.t. G is an intersectn graph of axis-parallel boxes in $\mathbb{R}^k$

(cubicity)

- $k=1$: interval ← can be determined in polynomial time
- $k \geq 2$: NP-hard

finite for all graphs



boxicity 2

Higher dimensions?

intersecting for simplicial complexes?

??

complexity for determining?

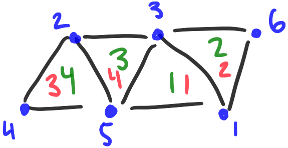
Shellability

A pure d-dimensional complex Δ is **shellable** if there exists an order on its facets $F_1, F_2, \ldots, F_k$ such that

$$\langle F_1, \ldots, F_i \rangle \cap \langle F_{i+1} \rangle$$

is pure and $(d-1)$-dimensional for each $1 \leq i \leq k-1$.

$\dim(\Delta) = 2$



Shelling :) ✓

Not :(✗

Shellable: very nice!

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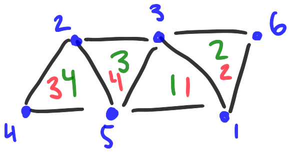
Lex shellings

A shelling $F_1, F_2, \ldots, F_k$ of Δ is a **lex shelling** if it is induced by the lexicographic order on the vertices of Δ.

$\dim(\Delta) = 2$



$\underline{1 < 2 < 3 < 4 < 5 < 6}$

135, 136, 235, 245

$\underline{1 < 2 < 4 < 3 < 5 < 6}$

135, 136, 245, 235

Shellability is hard

Determining shellability : NP-hard in $\dim(\Delta) \geq 2$
(Goaoc et. al 19)

Matroids : Björner '88 Pure complex is matroid independence complex iff ALL vertex orders induce shellings!

(can get away with checking $\approx 3^n$ permutations if you know which to check)

$n = \#$ vertices

The question of the day

What is the relationship between these

vertex orders and **lex shellings**?

Back to graphs

Proposition (G–Pavelka)

Assume Δ is 1-dimensional and connected. ← shellable!

An internal order induces a shelling.

Idea:

G' : partially shelled

ij : edge to add $(i<j)$

$\Rightarrow \quad k < i < \ell$

$\Rightarrow \quad ki \in G$

$\Rightarrow \quad ki \in G'$ ☺

Higher dimensions

dual graph is connected

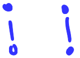
Proposition (G–Pavelka)

Let $\dim \Delta > 1$. *Assume* Δ *is pure and* ***strongly connected***.

INTERVAL $\not\Rightarrow$ shellable

Ex:

$\Delta = \langle 123,\ 124,\ \underline{125},\ 234,\ \underline{345} \rangle$

- $\mathrm{link}(5) = \langle 12, 34 \rangle \rightsquigarrow$  not shellable

$\Rightarrow \Delta$ is not shellable!

A *little bit* stronger

Theorem (G–Pavelka)

Let $\dim \Delta \geq 1$*. Assume* Δ *is pure and strongly connected.*

Any ***unit interval order*** *induces a* ***lex shelling*** *on* Δ*.*

Idea:

- Add facets w/ gaps is easy
  ($F = 12\ 45$)
- Gap-free facets: simila to 1-dim case

Vertex decomposability

Definition

A pure simplicial complex Δ is **vertex decomposable** if

(i) either Δ is a simplex

(ii) or there exists a vertex v such that $\mathrm{link}_\Delta(v)$ and $\mathrm{del}_\Delta(v)$ are vertex decomposable, and every facet of $\mathrm{del}_\Delta(v)$ is a facet of Δ.

$$\begin{tikzcd}
 & \text{vertex decomp} \arrow[d, Rightarrow, shift right, "/" marking] \arrow[dr, Rightarrow] & \\
\text{matroid} \arrow[ur, Rightarrow] \arrow[r, Rightarrow] & \text{lex shelling} \arrow[u, Rightarrow, shift right, "/" marking] \arrow[r, Rightarrow] & \text{shellable}
\end{tikzcd}$$

Doolittle - G - Lazar

Vertex decomposability

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(ii) or there exists a vertex v such that $\mathrm{link}_\Delta(v)$ and $\mathrm{del}_\Delta(v)$ are vertex decomposable, and every facet of $\mathrm{del}_\Delta(v)$ is a facet of Δ.

Theorem (G–Pavelka)

Let $\dim \Delta > 1$. *Assume* Δ *is pure and strongly connected.*

If Δ *has a* ***unit interval order*** *then* Δ *is* ***vertex decomposable***.

Simon's Conjecture

dim(Δ)=d, n vertices

Shelling completable: Any shelling of Δ can be extended to complete $\underbrace{d\text{-skeleton on } [n]}_{\text{rank } d+1 \text{ uniform matroid}}$.

Conjecture (Simon 93)

Every shellable complex is shelling completable.

Theorem (Coleman et. al 22)

Every vertex decomposable complex is shelling completable.

Cor: VI + SC ⇒ SC

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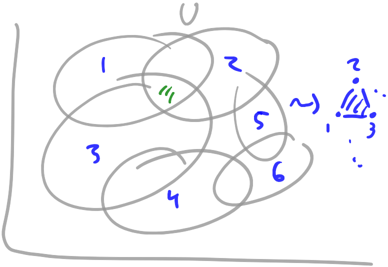
Questions



Efficiency (and boxicity?) ??

Lex shellings

If Δ has a lex shelling,
is it shelling completable?

Billera–Myers: Shellability of interval orders