Interval complexes, linear resolutions, and spaces of homomorphisms

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The setup - Ideals and Resolutions

Fix a field $\mathbb{K}$ and let $R = \mathbb{K}[x_1, x_2, \dots, x_n]$ denote the polynomial ring. Let $I \subset R$ be a homogenous ideal with quotient ring R/I.

- Recall that a *free resolution* of I is a long exact sequence

$$\mathcal{F} = 0 \leftarrow I \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_d \leftarrow 0.$$

  where each $F_i = \oplus_j R(-j)^{\beta_{i,j}}$

- The resolution is *minimal* if the $\beta_{i,j}$ are minimal among all resolutions, in which case $\beta_{i,j} = \mathrm{Tor}_i^R(I, \mathbb{K})_j$ are called the *(graded) Betti numbers* of I.
- Replace last map with $0 \leftarrow R/I \leftarrow R \leftarrow F_0$ to get a resolution of R/I.

Constructing a resolution

A free resolution of I:

$$\mathcal{F} : 0 \leftarrow I \underset{\partial_0}{\leftarrow} F_0 \underset{\partial_1}{\leftarrow} F_1 \leftarrow \cdots \leftarrow F_d \leftarrow 0.$$

where each $F_i = \oplus_j R(-j)^{\beta_{i,j}}$

How to construct $\mathcal{F}$:

- If $I = \langle f_1, f_2, \ldots, f_j \rangle$, say all of degree d, let $F_0 = R(-d)^j$ and a map $\partial_0 : F_0 \to I$.
- The module $\ker(\partial_0)$ is the (first) syzygy module, say it has generators $\{g_1, g_2, \ldots, g_k\}$. Let $F_1 = R^k$ and map $\partial_1 : F_1 \to F_0$.
- Continue this process. The *Hilbert Syzygy Theorem* says that eventually (at most by the nth step) we'll get an injective map. That's your resolution!

Combinatorial invariants

- The Betti numbers of I can be collected in its *Betti table*, (j, i)-entry given by $\beta_{i,i+j}$.

$$0 \leftarrow I \leftarrow R[-2]^8 \leftarrow R[-3]^{13} \oplus R[-4] \leftarrow R[-5]^7 \oplus R[-6]^2 \leftarrow R[-5] \oplus R[-6] \leftarrow 0$$

	0	1	2	3	4
total:	1	8	14	9	2
0:	1	.	.	.	.
1:	.	8	13	7	1
2:	.	.	1	2	1

- The *projective dimension* and *(Castelnuovo–Mumford) regularity* are:

$$\operatorname{pdim}(I) = \max\{i : \beta_{i,i+j} \neq 0 \text{ for some } j\}$$

$$\operatorname{reg}(I) = \max\{j : \beta_{i,i+j} \neq 0 \text{ for some } i\}$$

- The ideal I has a *linear resolution* if all entries in the matrices representing the boundary maps are linear forms.

Resolutions - who cares?

The minimal free resolution of a graded R-module M is a way to extract algebraic/geometric information about M.

- A way to study the *Hilbert function* of M, dimension of the dth graded component.
- Coefficents of the Hilbert polynomial are determined by the graded Betti numbers $\beta_{i,j}$.
- Minimal free resolutions detect *depth*, Cohen-Macaulay properties, etc. (via Auslander-Buchsbaum):

$$\mathrm{pdim}(M) + \mathrm{depth}(M) = n$$

- A measure of the *complexity* of the ideal.

Monomial is good enough

Gröbner theory and the monotonicity theorem allow us to focus on *monomial* ideals.

- By example: If $I = \langle x^2 - y, x^3 - x \rangle$,

  $$G = \langle x^2 - y, x^3 - x, y^2 - y \rangle,$$
  $$\mathrm{in}_< I = \langle x^2, x^3, y^2 \rangle.$$

- Monotonicity of Betti numbers:

  $$\beta_{i,j}(R/I) \leq \beta_{i,j}(R/\mathrm{in}_< I)$$

  with equality in many interesting cases!

- Gröbner degeneration also detects desirable algebraic properties (eg Cohen-Macaulayness).
- If I has a square-free initial ideal $\mathrm{in}(I)$, then the *extremal* Betti numbers of I and $\mathrm{in}(I)$ are the same [Conca, Varbaro].

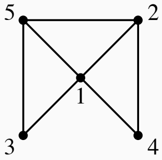
Facet ideals

We will consider monomials defined by (pure) simplicial complexes.

- Suppose Δ is a pure d-dimensional simplicial complex on vertex set $[n] = 1, 2, \ldots, n$. From this we construct the *facet ideal* in $\mathbb{K}$ generated by degree d monomials:

$$I_\Delta = \langle x_{i_0} x_{i_1} \cdot \cdots \cdot x_{i_d} : (i_0, i_1, \ldots, i_d) \in \Delta \rangle$$

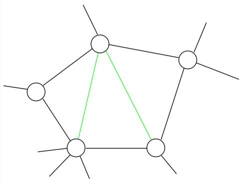
- Example $(d = 1,\ n = 5)$



$$I_\Delta = \langle x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_4, x_2x_5, x_3x_5 \rangle.$$

Edge ideals and linear resolutions

- For $d = 1$ we have that Δ is a graph and these are often called *edge ideals*.
- Recall that a graph G is *chordal* if G has no induced cycles of length ≥ 4.



- For a graph G on vertex set $[n]$, the complement is the graph $\overline{G}$ with edges given by all missing 2-sets.

Theorem (Fröberg)

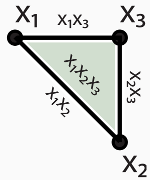
The edge ideal I_G has a linear resolution if and only if $\overline{G}$ is a chordal graph.

- Considerable efforts to generalize to d-dimensional simplicial complexes.

Cellular resolutions

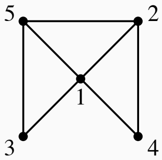
- Explicit minimal resolutions are known for some classes of ideals (Eliahou-Kervaire, Eagon-Northcott), but they are often cumbersome to work with, or at least to ‘picture’.
- Often we can encode syzygy modules and differential maps in face structure of polyhedral (CW-) complex, via the chain complex that computes cellular (co)homology.
- The resulting *cellular resolutions* were introduced by Bayer and Sturmfels in their study of the *Scarf complex* of a generic ideal.
- For example the simplex supports a resolution of the ideal $\mathfrak{m} = \langle x_1, x_2, \ldots, x_n \rangle$.

$$0 \leftarrow \mathfrak{m} \leftarrow R[-1]^3 \leftarrow R[-2]^3 \leftarrow R[-3] \leftarrow 0.$$



Resolutions of facet ideals

- Suppose Δ is a d-dimensional simplicial complex on vertex set $[n]$ and let I_Δ denote its facet idea (equivalently, I_Δ is squarefree monomial and equigenerated in degree $d+1$).
- Example (again): Let $d = 1$, $n = 5$, and consider the graph G:



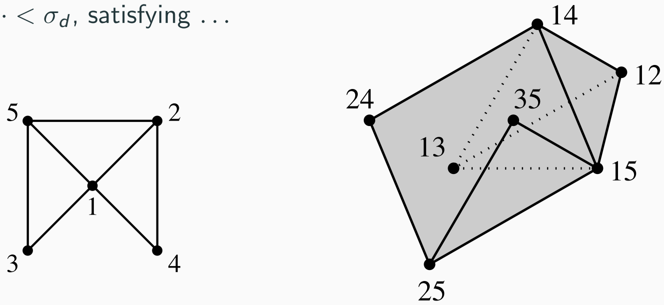
$$I_G = \langle x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_4, x_2x_5, x_3x_5 \rangle.$$

- In what follows we write a facet of Δ as $F = (v_0 v_1 \cdots v_d)$ where $v_0 < v_1 < \cdots < v_d$.
- The facets of Δ are generators of an ideal, we look for a 'space of (directed) facets' to describe syzygies of this ideal.

A space of facets

Suppose Δ is a d-dim complex on vertex set $[n]$. Then X_Δ is the polyhedral complex (subcomplex of $\prod_{i=1}^{d} \Delta_{n-1}$) satisfying:

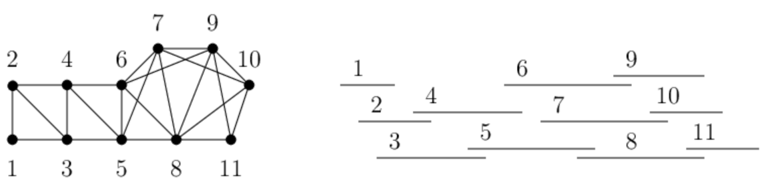
- The vertices of X_Δ are $(v_0 v_1 \cdots v_d)$, where $(v_0 v_1 \cdots v_d) \in \Delta$.
- The higher dimensional cells are of the form $\sigma_0 \times \sigma_1 \times \cdots \times \sigma_d$, where $\sigma_i \subset [n]$ and $\sigma_0 < \sigma_1 < \cdots < \sigma_d$, satisfying ...



- Label each face by the LCM of the monomials labels of the vertices it contains.

Interval graphs

- The construction of X_Δ construction makes sense for any (pure) simplicial complex Δ. When does it support a resolution of I_Δ?
- A graph G is *interval* if its vertices can be represented as intervals in real line $I_1, \ldots, I_n$ such that $i \sim j$ if and only if $I_1 \cap I_2 \neq \emptyset$.



- Interval graphs are chordal.
- Label the left endpoint of each interval moving L to R, then G is interval $\Leftrightarrow$

$$\text{If } e = ab \text{ is an edge (for } a < b\text{), we have } ac \text{ is an edge for all } a < c < b.$$

Interval complexes

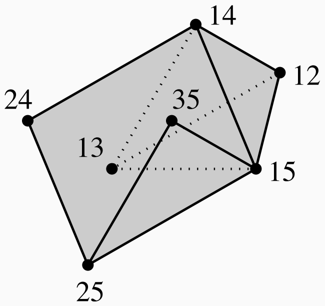
This inspires the following definition. Suppose Δ is a pure d-dimensional simplicial complex on vertex set $[n]$.

- Δ is *interval* if there exists a labeling of it vertex set such that for any facet $F = a_0a_1 \cdots a_d$, we have that $a_0i_1i_2 \cdots i_d \in \Delta$, for all $i_1 \leq a_1, i_2 \leq a_2, \ldots, i_d \leq a_d$.
- For example $\Delta = \{1234, 1235, 2345, 2346, 2356, 2456\}$ are the facets of a 3-dim interval complex on vertex set [6].
- Strictly includes the class of *shifted* complexes.
- Such complexes were used (and defined this way) by Benedetti, Seccia, and Varbaro: the determinantal facet ideals of such complexes have square-free initial ideals.

A minimal resolution

Theorem (D-, Engström, 2012)

Suppose Δ is a d-dimensional interval complex, and let $\overline{\Delta}$ denote its complement. Then the complex $X_{\overline{\Delta}}$ supports a minimal resolution of the facet ideal $I_{\overline{\Delta}}$.



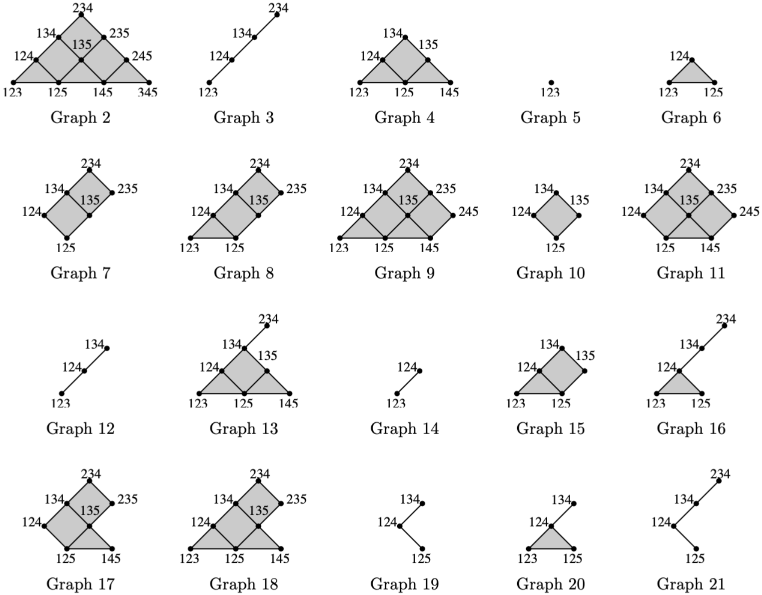
- Betti numbers of I_G are given by the f-vector of $X_{\overline{\Delta}}$.

$$0 \leftarrow I \leftarrow R[-2]^7 \leftarrow R[-3]^{11} \leftarrow R[-4]^6 \leftarrow R[-5] \leftarrow 0$$

Linear resolutions and chordal complexes

- Our construction generalizes the *box of complexes* resolution of Nagel and Reiner (they considered shifted complexes).
- Since the dimension of a face of X_Γ can be read off from the monomial label, we conclude that I_Γ has a *linear* resolution when Γ is the complement of an interval complex.
- Recall that interval graphs are chordal, and hence this provides a generalization/specialization of Fröberg's Theorem.
- There exist chordal graphs G such that $\Delta_{\overline{G}}$ does *not* support a resolution of $I_{\overline{G}}$.
- Question: How do interval complexes relate to other notions of *chordal* simplicial complexes?

Examples $d = 2$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Graph 20
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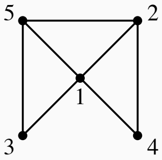
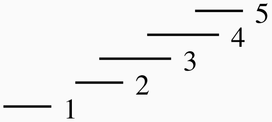
Graph 21

Cointerval complexes

In our old paper, we gave a definition of a *cointerval* complex.

- The definition is recursive, we define d-dim cointerval complexes in terms of $(d-1)$-dim cointerval complexes. Helpful for our proofs.
- Base case of $d = 1$ corresponds to the *complements of interval* graphs: vertices given by intervals, adjacent if *non*-overlapping.





Proposition (D-, Goeckner, Pavelka)
A complex Δ is interval if and only if $\overline{\Delta}$ is cointerval.

Tools - topology of posets

How do we prove that the labeled complex supports a resolution?

- To prove that $X = X_{\overline{\Delta}}$ supports a minimal resolution of the edge ideal I_G we must study the topology of certain induced subcomplexes $X_{\leq \alpha}$.
- Here $X_{\leq \alpha}$ denotes the sub complex of X generated by all vertices (=monomials) which divide α.
- In the case of X. we can utilize tools from *poset topology* and Quillen type fiber lemmas.

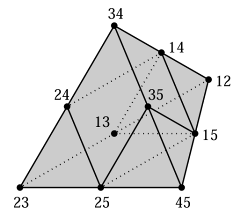
A nice embedding

Recall that X_Δ is defined as a subcomplex of $\prod_{i=1}^{d} \Delta_{n-1}$ - that's big!

Turns out we can do better.

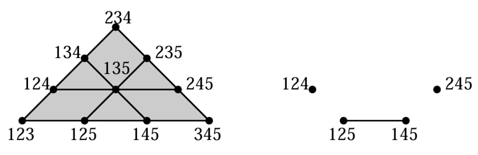
- Let $\Delta_{d,n}$ denote the d-skeleton of the simplex on n vertices.

Proposition (D-, Engström)
The ideal $I_{\Delta_{d,n}}$ has a minimal resolution supported on a certain fine mixed subdivision of the dilated simplex $(d+1)\Delta_{n-d}$.



A nice embedding

- As a corollary, for any cointerval complex Δ, the ideal I_Δ has minimal cellular resolution supported on a subcomplex of $(d+1)\Delta_{n-d}$.
- We use the mixed subdivision corresponding to the *staircase triangulation* of $\Delta_d \times \Delta_{n-d}$.
- Turns out not all regular mixed subdivisions work!



- Side note: *any* regular mixed subdivision (of certain classes of generalized permutohedra) can be used to produce cellular resolutions (of the ideal generated by the 0-cells, eg $\mathfrak{m}^k$).

Related work

Can think of our space of facets as moduli space of homomorphisms $E \to \Delta$. Other classes of ideals are defined similarly:

- Ideals defined by nondegenerate simplicial maps [Braun, Browder, Klee]
- Ideals of poset homomorphisms [Flöystad, Greve, Herzog, Juhnke-Kubitzke, Katthän, Madani]
- Notion of *path ideals* of a graph, as a generalization of an edge ideal (cellular resolutions for certain classes described by [Chau, Kara, Wang])

Homomorphism complexes of digraphs studied more recently with Anurag Singh:

- If T_n is the acyclic tournament, then $\mathrm{Hom}(G, T_n)$ is contractible for *any* digraph G.

Further thoughts

- Does X_Δ support resolutions for a larger class of complexes? Can we characterize the class that works?
- Can use an interval decomposition of a graph/complex G to produce (non-minimal) cellular resolutions of I_G?
- Is the Alexander dual (appropriately defined) of an interval complex shellable? Vertex decomposable?
- Can we get cellular resolutions by considering other *homomorphism* complexes?
- How do interval complexes relate to other notions of *chordal* simplicial complexes?
- Interval graphs (and chordal graphs) can be recognized in polynomial time. Is the same true for interval complexes?

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Thanks for listening!