Presburger modules: quasi-polynomials and tameness*

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ACD2025 - IPAM

February 13, 2025

Key takeaway: Finiteness conditions on families of algebraic, combinatorial, or geometric objects lead to (somewhat) predictable growth behavior.

Hilbert polynomial

$R = \oplus_{n\in\mathbb{N}} R_n$, a Noetherian standard graded algebra over a field $R_0 = \Bbbk$.

$$\dim_{\Bbbk}(R_n) \underset{n\gg 0}{=} a_d n^d + a_{n-1} n^{d-1} + \cdots + a_0 \in \mathbb{Q}[n].$$

Ehrhart polynomial

$\mathcal{P} \subset \mathbb{Q}^d$ a lattice polytope.

$$\#(n\mathcal{P} \cap \mathbb{Z}^d) = b_d n^d + b_{n-1} n^{d-1} + \cdots + b_0 \in \mathbb{Q}[n].$$

Snapper polynomial

Let X be a d-dimensional projective scheme over a field $\Bbbk$, $\mathcal{L}$ an invertible sheaf.

$$\mathcal{X}_{\Bbbk}(\mathcal{L}^{\otimes n}) = c_d n^d + c_{n-1} n^{d-1} + \cdots + c_0 \in \mathbb{Q}[n].$$

If "standard" and/or "lattice" are removed, then one obtains *quasi-polynomials*.

$P : \mathbb{Z} \to \mathbb{Q}$ is a **quasi-polynomial** if there exist $P_1, \ldots, P_\pi \in \mathbb{Q}[n]$, such that

$$P(n) = P_i(n), \qquad \text{for} \qquad n \equiv i \pmod{\pi}.$$

More (quasi-)polynomials in commutative algebra

- (**Hilbert-Samuel function**) $\text{length}(R/I^n)$ is a polynomial for $n \gg 0$, where I and $\mathfrak{m}$-primary ideal.
- (**Number of generators**) $\mu(I^n)$ is a polynomial for $n \gg 0$, for any ideal I.
- (**Ext, Tor, Betti and Bass numbers**) $\text{length}(\text{Tor}_i^R(R/I^n, M))$, $\text{length}(\text{Ext}_R^i(M, R/I^n))$, $\mu(\text{Tor}_i^R(R/I^n, M))$, and $\mu(\text{Ext}_R^i(M, R/I^n))$ are polynomials for $n \gg 0$. (Kodiyalam, 1993)
- (**Castelnuovo-Mumford regularity**) $\text{reg}(R/I^n)$ is a linear function for $n \gg 0$, where I a homogeneous ideal. (Kodiyalam, 2000), (Cutkosky, Herzog, Trung, 1999)
- (v-**invariant**) $v_P(I^n)$ is a linear function for $n \gg 0$, where I a homogeneous ideal. (Conca, 2023)

$$\vdots$$

These results use in a fundamental way that the **Rees algebra** $R[It] = \oplus_{n\in\mathbb{N}} I^n t^n \subset R[t]$ is Noetherian.

One obtains quasi-polynomials if $\{I^n\}_{n\in\mathbb{N}}$ is replaced by a **graded family** $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$, $I_n I_m \subset I_{n+m}$, such that $R[\mathcal{I}t] = \oplus_{n\in\mathbb{N}} I_n t^n \subset R[t]$ is Noetherian.

Local cohomology of monomial ideals

$R = \Bbbk[x_1, \dots, x_d]$, $\mathfrak{m} = (x_1, \dots, x_d)$, $I = (\mathbf{x}^{\mathbf{n}_1}, \dots, \mathbf{x}^{\mathbf{n}_u})$ monomial ideal, M an R-module.

$$\check{C}^{\cdot}(I) = 0 \to R \to \bigoplus_{i=1}^{u} R_{\mathbf{x}^{\mathbf{n}_1}} \to \bigoplus_{1 \leqslant i < j \leqslant u}^{n} R_{\mathbf{x}^{\mathbf{n}_i}\mathbf{x}^{\mathbf{n}_j}} \to \cdots \to R_{\mathbf{x}^{\mathbf{n}_1} \cdots \mathbf{x}^{\mathbf{n}_u}} \to 0$$

$\mathrm{H}^i_I(M) := H^i(\check{C}^{\cdot}(I) \otimes_R M)$ is the **i-th local cohomology** of M with support I

Theorem (Dao - M, 2019)

If $\operatorname{length}(\mathrm{H}^i_{\mathfrak{m}}(R/I^n)) < \infty$ *for* $n \gg 0$*, then* $\operatorname{length}(\mathrm{H}^i_{\mathfrak{m}}(R/I^n))$ *is a quasi-polynomial for* $n \gg 0$*.*

Example: $I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5) \subseteq \Bbbk[x_1, \dots, x_5]$.

$$\operatorname{length}(\mathrm{H}^1_{\mathfrak{m}}(R/I^n)) = \begin{cases} \frac{n^5}{240} + \frac{n^4}{96} - \frac{n^3}{48} - \frac{n^2}{24} + \frac{n}{60}, & \text{if } n \equiv 0 \pmod 2 \\ \frac{n^5}{240} + \frac{n^4}{96} - \frac{n^3}{48} - \frac{n^2}{24} + \frac{n}{60} + \frac{1}{32}, & \text{if } n \equiv 1 \pmod 2); \end{cases}$$

$$\sum_{n=0}^{\infty} \operatorname{length}(\mathrm{H}^1_{\mathfrak{m}}(R/I^n)) z^n = \frac{z^3}{(1-z)^5(1-z^2)}.$$

Main ingredient: Presburger arithmetic

A **Presburger formula** F is a first order formula that can be written using quantifiers $\exists$, $\forall$, boolean operations $\wedge$, $\vee$, $\neg$, and affine integer linear inequalities in the variables.

A set $S \subseteq \mathbb{Z}^d$ is **Presburger definable** if it can be defined via a Presburger formula $F(\mathbf{u})$, i.e, $S = \{\mathbf{u} \in \mathbb{Z}^d \mid F(\mathbf{u})\}$.

Equivalently, S is a finite union of sets $P \cap (\mathbf{q} + \Lambda)$, where $P \subseteq \mathbb{Q}^d$ is a polyhedron, $\mathbf{q} \in \mathbb{Z}^d$, and $\Lambda \subset \mathbb{Z}^d$ is a lattice. (Woods, 15)

Example:

- $F(u) = (\exists c \in \mathbb{Z})(c \geqslant 0 \;\wedge\; u = 3c + 1)$ defines the positive integers with remainder 1 mod 3.
- $G(u_1, u_2) = (2u_1 + u_2 \geqslant 3 \;\wedge\; 3u_1 - u_2 \geqslant 2)$ defines the set of integer points in the cone $(1,1) + \mathbb{R}_{\geqslant 0}\operatorname{conv}\big(\{(1,3),(1,-2)\}\big)$.
- $Y_I(\mathbf{a}, n) = (\exists t_p \in \mathbb{Z}_{\geqslant 0}, 1 \leqslant p \leqslant u)\Big(\sum t_p = n \wedge \mathbf{a} \succeq \sum t_p \mathbf{n}_p\Big)$ defines the monomials $\mathbf{x}^{\mathbf{a}}$ in the ideal $I^n = (\mathbf{x}^{\mathbf{n}_1}, \ldots, \mathbf{x}^{\mathbf{n}_u})^n$.

Wood's Theorem

Let $F(\mathbf{a}, n)$ be a Presburger formula. Set

$$f(n) = \#\{\mathbf{a} \in \mathbb{Z}^d \mid F(\mathbf{a}, n)\},$$

this function is called a **Presburger counting function**.

Theorem (Woods '15)

Let $f : \mathbb{N} \to \mathbb{N}$ be a function, then the following conditions are equivalent:

1. *f is a Presburger counting function.*
2. *f is a quasi-polynomial for $n \gg 0$.*
3. *The generating function $\sum_{n\in\mathbb{N}} f(n)z^n$ is rational with denominator of the form $\prod_i (1 - z^{n_i})$, $n_i \in \mathbb{Z}_{>0}$.*

Example: Let $F(\mathbf{a}, n) = (\mathbf{a} \in \mathbb{N}^d) \wedge (\neg Y_I(\mathbf{a}, n))$. Then

$$\text{length}(R/I^n) = \#\{\mathbf{a} \in \mathbb{Z}^d \mid F(\mathbf{a}, n)\}$$

is a quasi-polynomial for $n \gg 0$.

Sketch of proof for Dao-M's Theorem

For each $\mathbf{a} \in \mathbb{N}^d$ there exists a subcomplex $\Delta_{\mathbf{a}}$ of $\Delta(I)$, the Stanley-Reisner complex of $\sqrt{I}$, such that

$$\dim_{\Bbbk} \mathrm{H}^i_{\mathfrak{m}}(R/I^n)_{\mathbf{a}} = \dim_{\Bbbk} \tilde{H}_{i-1}(\Delta_{\mathbf{a}}(I), \Bbbk).$$

(Hochster '77, Takayama '05)

Given $\Delta' \subseteq \Delta(I)$, the $\mathbf{a}$ such that $\Delta_{\mathbf{a}}(I^n) = \Delta'$ are characterized by simultaneous membership and non-membership in powers of monomial ideals, which is a Presburger counting function $f_{\Delta'}(n)$. Thus

$$\begin{aligned}
\mathrm{length}(\mathrm{H}^i_{\mathfrak{m}}(R/I^n)) &= \sum_{\mathbf{a}\in\mathbb{N}^d} \dim_{\Bbbk} \mathrm{H}^i_{\mathfrak{m}}(R/I^n)_{\mathbf{a}} \\
&= \sum_{\mathbf{a}\in\mathbb{N}^d} \dim_{\Bbbk} \tilde{H}_{i-1}(\Delta_{\mathbf{a}}(I^n), \Bbbk) \\
&= \sum_{\Delta' \subseteq \Delta(I)} \sum_{\{\mathbf{a} \mid \Delta_{\mathbf{a}}(I^n) = \Delta'\}} \dim_{\Bbbk} \tilde{H}_{i-1}(\Delta', \Bbbk) \\
&= \sum_{\Delta' \subseteq \Delta(I)} \dim_{\Bbbk} \tilde{H}_{i-1}(\Delta', \Bbbk) f_{\Delta'}(n), \text{ a quasi-polynomial by Wood's.}
\end{aligned}$$

Kevin Woods and his coauthors expanded the Presburger methods to prove quasi-polynomial behavior in a wide and diverse range of situations.

The unreasonable ubiquitousness of quasi-polynomials*

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Submitted: Sep 25, 2013; Accepted: Feb 17, 2014; Published: Feb 28, 2014
Mathematics Subject Classifications: 05A15, 52C07, 03B10

Abstract

A function g, with domain the natural numbers, is a quasi-polynomial if there exists a period m and polynomials $p_0, p_1, \ldots, p_{m-1}$ such that $g(t) = p_i(t)$ for $t \equiv i \bmod m$. Quasi-polynomials classically – and "reasonably" – appear in Ehrhart theory and in other contexts where one examines a family of polyhedra, parametrized by a variable t, and defined by linear inequalities of the form $a_1x_1 + \cdots + a_dx_d \leqslant b(t)$.

Recent results of Chen, Li, Sam; Calegari, Walker; and Roune, Woods show a quasi-polynomial structure in several problems where the a_i are also allowed to vary with t. We discuss these "unreasonable" results and conjecture a general class of sets that exhibit various (eventual) quasi-polynomial behaviors: sets $S_t \subseteq \mathbb{N}^d$ that are defined with quantifiers $(\forall, \exists)$, boolean operations (and, or, not), and statements of the form $a_1(t)x_1 + \cdots + a_d(t)x_d \leqslant b(t)$, where $a_i(t)$ and $b(t)$ are polynomials in t. These sets are a generalization of sets defined in the Presburger arithmetic. We prove several relationships between our conjectures, and we prove several special cases of the conjectures. The title is a play on Eugene Wigner's "The unreasonable effectiveness of mathematics in the natural sciences".

Keywords: Ehrhart polynomials; generating functions; Presburger arithmetic; quasi-polynomials; rational generating functions

1 Reasonable Ubiquitousness

In this section, we survey classical appearances of quasi-polynomials (though Section 1.3 might be new even to readers already familiar with Ehrhart theory). In Section 2, we

*With apologies to Wigner [18] and Hamming [9]. Extended abstract appeared in FPSAC 2013.

K. Woods, *The unreasonable ubiquitousness of quasi-polynomials*, The Electronic Journal of Combinatorics **21** (2014), #P1.44.

Goal: Describe a category that exploits the finiteness properties of Presburger to prove quasi-polynomial behavior of multigraded modules and functors.

From Dao-M's proof, the idea is to divide supports of modules in finitely many "Presburger describable" regions.

The issue: What morphisms preserve the Presburger structure? Tameness from persistent homology developed by Ezra Miller.

HOMOLOGICAL ALGEBRA OF MODULES OVER POSETS

EZRA MILLER

ABSTRACT. Homological algebra of modules over posets is developed, as closely parallel as possible to that of finitely generated modules over noetherian commutative rings, in the direction of finite presentations and resolutions. Centrally at issue is how to define finiteness to replace the noetherian hypothesis which fails. The tameness condition introduced for this purpose captures finiteness for variation in families of vector spaces indexed by posets in a way that is characterized equivalently by distinct topological, algebraic, combinatorial, and homological manifestations. Tameness serves both theoretical and computational purposes: it guarantees finite presentations and resolutions of various sorts, all related by a syzygy theorem, amenable to algorithmic manipulation. Tameness and its homological theory are new even in the finitely generated discrete setting of $\mathbb{N}^n$-gradings, where tame is materially weaker than noetherian. In the context of persistent homology of filtered topological spaces, especially with multiple real parameters, the algebraic theory of tameness yields topologically interpretable data structures in terms of birth and death of homology classes.

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Date: 10 August 2020.
2020 *Mathematics Subject Classification*. Primary: 05E40, 13E99, 06B15, 13D02, 55N31, 06A07, 32B20, 14P10, 52B99, 13A02, 13P20, 68W30, 13P25, 62R40, 06A11, 06F20, 06F05, 68T09; Secondary: 13C99, 05E16, 32S60, 14F07, 62R01, 62H35, 92D15, 92C15, 13F99, 20M14, 14P15, 06B35, 22A25.

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Q-modules

Let $Q \cong \mathbb{Z}^d$ with a generating semigroup $Q_+ \subset Q$, such that Q_+ has trivial unit group. Q has a partial order given by:

$$\mathbf{q} \preceq \mathbf{q}' \Leftrightarrow \mathbf{q}' - \mathbf{q} \in Q_+.$$

A **Q-module** M is a Q-graded $\Bbbk$-vector space $M = \oplus_{\mathbf{q} \in Q} M_q$ with a $\Bbbk$-linear map $M_\mathbf{q} \to M_{\mathbf{q}'}$ for every pair $\mathbf{q} \preceq \mathbf{q}'$, such that $M_\mathbf{q} \to M_{\mathbf{q}''}$ equals the composition $M_\mathbf{q} \to M_{\mathbf{q}'} \to M_{\mathbf{q}''}$ if $\mathbf{q} \preceq \mathbf{q}' \preceq \mathbf{q}''$.

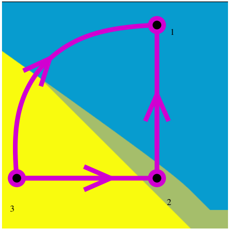
Example: $R = \Bbbk[Q_+] \subset \Bbbk[Q] \cong \Bbbk[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ is an **affine semigroup ring** and $M = \oplus_{\mathbf{q} \in \mathbb{Z}^d} M_\mathbf{q}$ a multigraded R-module.

The maps are given by $\mathbf{x}^\mathbf{a} M_\mathbf{q} \subset M_{\mathbf{a}+\mathbf{q}}$.

For example, R, monomial ideals I, quotients R/I,... are all Q-modules.

Tameness: the finiteness condition

A Q-module M admits a **constant subdivision** if Q is partitioned into constant regions A, each with vector space $M_A \xrightarrow{\sim} M_{\mathbf{a}}$ for all $\mathbf{a} \in A$, with **no monodromy**, i.e., all $\mathbf{a} \preceq \mathbf{b}$ with $\mathbf{a} \in A$ and $\mathbf{b} \in B$ induce the same composite $M_A \to M_{\mathbf{a}} \to M_{\mathbf{b}} \to M_B$.



M is **tame** if it admits a finite constant subdivision and $\dim_{\Bbbk} M_{\mathbf{q}} < \infty$ for all $\mathbf{q}$.

M is a **Presburger module** if M is tame and the finitely many constant regions are Presburger definable.

Figure credit: Ezra Miller.

Presburger families

A **Presburger Rees monoid** over Q is the G_+ of a partially ordered group $G \cong Q \times \mathbb{Z}$ such that $G_+ \subseteq Q_+ \times \mathbb{N}$ and $G_+ \cap \big(Q \times \{0\}\big) = Q_+ \times \{0\}$, and that the generators of G_+ are Presburger definable.

A **Presburger family** is a family $\{M_n\}_{n\in\mathbb{Z}}$ of Q-modules whose direct sum is a Presburger G-module $M = \bigoplus_{\mathbf{g}\in G} M_{\mathbf{g}} = \bigoplus_{n\mathbf{q}} M_{n\mathbf{q}}$.

Example: The following are Presburger families of Q-modules (all ideals are monomials).

1. Powers $\{I^n\}_{n\in\mathbb{N}}$, integrally closures $\{\overline{I^n}\}_{n\in\mathbb{N}}$, symbolic powers $\{I^{(n)}\}_{n\in\mathbb{N}}$, saturations $\{I^n : J^\infty\}_{n\in\mathbb{N}}$. We note, however, that these families are Noetherian graded families.
2. $\{I^{an+b} : J^{cn+d}\}_{n\in\mathbb{N}}$ for $a, c \in \mathbb{N}$, $b, d \in \mathbb{Z}$ do not necessarily form a graded family but it is a Presburger family.
3. If $Q_+ = \mathbb{N}^d$, the multiplier ideals $\{\mathcal{J}(I^n)\}_{n\in\mathbb{N}}$ form a Presburger family.

Presburger families (continued)

4. If $\cap_{n\in\mathbb{N}}(I_{n+1} : I_n) \neq (0)$ and membership in the ideals I_n is Presburger, then $\{I_n\}_{n\in\mathbb{N}}$ is a Presburger family.

   Therefore, integral closures $\{\overline{I_n}\}_{n\in\mathbb{N}}$, colons $\{I_n : J^{cn+d}\}_{n\in\mathbb{N}}$ with $c > 0$, saturations $\{I_n : J^\infty\}_{n\in\mathbb{N}}$, and over $Q_+ = \mathbb{N}^d$ the multiplier ideals $\{\mathcal{J}(I_n)\}_{n\in\mathbb{N}}$, are all Presburger families.

   One can combine all these operations to find elaborated examples. For example, in $\Bbbk[\mathbb{N}^d]$ the following is a Presburger family

$$\{\overline{\mathcal{J}(I^n : J^{3n-2}) : K^\infty}\}_{n\in\mathbb{N}}.$$

5. Presburger families are closed under direct sums and quotients, so we can create other families from the ones above. For example

$$\{M_n\}_{n\in\mathbb{N}}; \qquad M_n = \mathcal{J}(I^n : J^n)/\overline{I^n} \oplus I^{(n)}/I^n \oplus I^n : J^\infty.$$

Homomorphism of tame modules

A **homomorphism** of Q-modules $\varphi : M \to N$ is a collection of $\Bbbk$-linear maps $M_{\mathbf{q}} \to N_{\mathbf{q}}$, $\mathbf{q} \in Q$, making the following diagram commute for every $\mathbf{q} \preceq \mathbf{q}'$

$$\begin{array}{ccc} M_{\mathbf{q}} & \to & N_{\mathbf{q}'} \\ \downarrow & & \downarrow \\ M_{\mathbf{q}'} & \to & N_{\mathbf{q}'} \end{array}$$

$\varphi : M \to N$ **tame** if Q admits a finite constant subdivision that tames both M and N, such that for each region I the composition $M_I \to M_{\mathbf{i}} \to N_{\mathbf{i}} \to N_I$ does not depend on $\mathbf{i} \in I$. If the constant subdivision is Presburger definable, we say φ is **Presburger**.

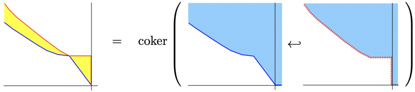
Theorem (Miller, '17)

Kernel and cokernels of tame (Presburger) homomorphisms are tame (Presburger).

Thus, the cohomology of a complex $C^\bullet$ of Q-modules with Presburger modules and morphisms is again Presburger.

Resolutions of tame modules

$U \subset Q$ is an **upset** if $U + Q_+ = U$. We call $\Bbbk[U] = \oplus_{\mathbf{q} \in U} \Bbbk$ an **upset module**.



A **upset resolution** of M is a homology isomorphism $F_\bullet \to M$, where each module in $F_\bullet$ is a direct sum of upset modules. Likewise define **downset resolutions**.

These resolutions are **finite** if there are finitely many upsets or downsets involved. They are **Presburger** if all the objects involved (modules, upsets, downsets) are Presburger definable.

Theorem (Syzygy Theorem, Miller, '17)

The following are equivalent.

1. *M is tame (Presburger).*
2. *M has a finite (Presburger) upset resolution.*
3. *M has a finite (Presburger) downset resolution.*

Figure credit: Ezra Miller.

Presburger functors

$Q \cong \mathbb{Z}^d$ with generating semigroup $Q_+ \subset Q$ that has trivial unit group.

Theorem (Dao-Miller-M-O'Neil-Woods)

Let $\{M_n\}_{n\in\mathbb{Z}}$ be a Presburger family of Q-modules. The following are again Presburger families:

1. *Localizations $\left\{(M_n)_P\right\}_{n\in\mathbb{Z}}$ for <u>any</u> monomial prime ideal $P \subset R = \Bbbk[Q_+]$.*
2. $\left\{\mathrm{H}^i_I(M_n)\right\}_{n\in\mathbb{Z}}$ *for <u>any</u> monomial ideal $I \subset R = \Bbbk[Q_+]$.*
3. $\left\{\mathrm{Tor}_i^R(M_n, L)\right\}_{n\in\mathbb{Z}}$ *for L a Noetherian R-module.*
4. $\left\{\mathrm{Ext}^i_R(L, M_n)\right\}_{n\in\mathbb{Z}}$ *for L a Noetherian R-module.*
5. $\left\{\mathrm{Ext}^i_R(M_n, L)\right\}_{n\in\mathbb{Z}}$ *for L a Noetherian or Artinian R-module.*

Presburger functors (continued)

Example: The following are Presburger families (all ideals are monomials).

1. $\left\{\mathrm{H}^i_J(R/I^n)\right\}_{n\in\mathbb{Z}}$.
2. $\left\{\mathrm{Ext}^{i_1}_R(\mathrm{Ext}^{i_2}_R(\cdots(\mathrm{Ext}^{i_r}_R\left(I^{(n)}/I, R\right), R), \cdots, R)\right\}_{n\in\mathbb{Z}}$
3. $\left\{\mathrm{Tor}^R_i\left(\mathrm{H}^i_J(\mathcal{J}(I^n)/\overline{I^n}), \Bbbk\right)\right\}_{n\in\mathbb{Z}}$.
4. $\left\{\mathrm{Ext}^i_R\left(R/P, \mathrm{H}^i_J(\mathcal{J}(I^n)/\overline{I^n})\right)_P\right\}_{n\in\mathbb{Z}}$.

A word about the proof

$\left\{\mathrm{Tor}_i^R(M_n, L)\right\}_{n\in\mathbb{Z}}$ is a Presburger family, for L a Noetherian R-module.

Tensor over R a free resolution of L and a finite Presburger upset presentation of $M = \oplus_{n\in\mathbb{Z}} M_n$ as a G-module to obtain

$$\begin{array}{ccccc}
\to & Z_1 \otimes F_{i+1} & \to & Z_1 \otimes F_i & \to \\
 & \downarrow & & \downarrow & \\
\to & Z_0 \otimes F_{i+1} & \to & Z_0 \otimes F_i & \to \\
 & \downarrow & & \downarrow & \\
\to & M \otimes F_{i+1} & \to & M \otimes F_i & \to \\
 & \downarrow & & \downarrow & \\
 & 0 & & 0 &
\end{array}$$

The four maps in the top square have components of the form $\Bbbk[U] \otimes \Bbbk[\mathbf{a} + Q_+] \to \Bbbk[U'] \otimes \Bbbk[\mathbf{b} + Q_+]$, which is 0 if $U + \mathbf{a} \not\subset U' + \mathbf{b}$, and multiplication by elements in $\Bbbk$ otherwise. This implies these maps and its vertical cokernel are Presburger. The conclusion follows by taking homology.

Tensor products of general Presburger upsets need not be tame, so one needs L to be Noetherian.

The unreasonable ubiquitousness of quasi-polynomials

General idea: *For Presburger families, finite invariants grow quasi-polynomially.*

Theorem/Example (Dao-Miller-M-O'Neil-Woods)

Let $\{M_n\}_{n\in\mathbb{Z}}$ be a Presburger family of Q-modules. The following are quasi-polynomials for $n \gg 0$.

1. $$\operatorname{length}\Big(\bigoplus_{an+b\leqslant|\mathbf{a}|\leqslant cn+d} \mathrm{H}^i_J(M_n)_{\mathbf{a}}\Big), \qquad \operatorname{length}\left(\mathrm{H}^i_{\mathfrak{m}}(M_n)_{\geqslant an+b}\right).$$

   $R = \Bbbk[x,y,z,w]$; $\operatorname{length}\Big(\mathrm{H}^i_{(x,y)}(R/I^n : \mathfrak{m}^n)_{an+b\leqslant|\mathbf{a}|\leqslant cn+d}\Big)$

2. $$\operatorname{length}\left(\operatorname{Tor}^R_i(M_n, L)\right), \qquad \operatorname{length}\left(\operatorname{Ext}^i_R(L, M_n)\right)$$

   $$\operatorname{length}\left(\operatorname{Ext}^i_R(M_n, L)\right),$$ *for L a Noetherian R-module.*

3. *Replacing* length *by μ in any of the above.*

The unreasonable ubiquitousness of quasi-polynomials (continued)

4. $\beta_i(M_n) = \dim_{\Bbbk} \operatorname{Tor}_i^R(M_n, \Bbbk)$, *Betti numbers.*

5. $\mu_i(P, M_n) = \dim_{(R/P)_P} \operatorname{Ext}_R^i \left(R/P, M_n\right)_P$, *Bass numbers.*

6. $a_i\,(M_n) = \max\{|\mathbf{a}| \mid \mathrm{H}^i_{\mathfrak{m}}(M_n)_{\mathbf{a}} \neq 0\}$, *a-invariants*, quasi-linear.

7. $\operatorname{reg}(M_n) = \max\{a_i(M_n) + i\}$, *Castelnuovo-Mumford regularities*, quasi-linear.
   $\operatorname{reg}\left(\overline{\mathcal{J}(I^n : J^{3n-2}) : K^\infty}\right)$

8. $v_P(M_n) = \min\{|\mathbf{a}| \mid (\exists f \in R_{\mathbf{a}})(P = 0 :_{M_n} f)\}$, *v-invariants*, quasi-linear.

9. $Q = \mathbb{N}^d$, $\deg(M_n)$, *degrees*.

10. $Q = \mathbb{N}^d$, $\operatorname{hdeg}(M_n)$,*homological degrees*, (Vasconcelos, 1998).
    The latter is a linear combination of $\deg(M_n)$ and degrees of modules of the form $\operatorname{Ext}_R^{i_1}(\operatorname{Ext}_R^{i_2}(\cdots(\operatorname{Ext}_R^{i_r}\left(I^{(n)}/I, R\right), R), \cdots, R)$. This was not know to grow polynomially, even for $\{R/I^n\}_{n\in\mathbb{N}}$.

Thank you!

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