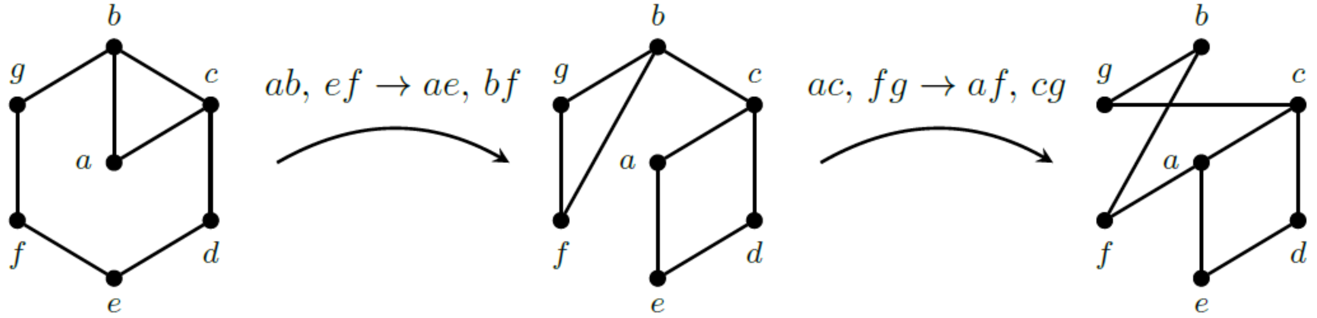
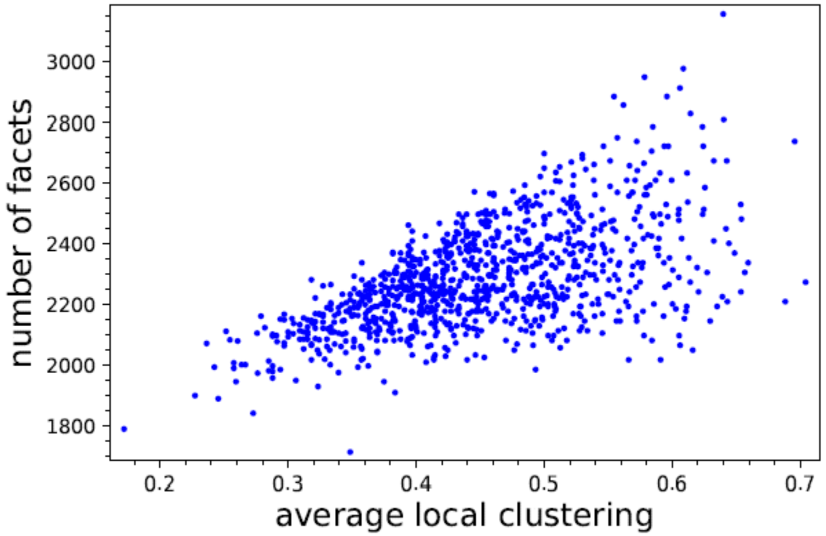
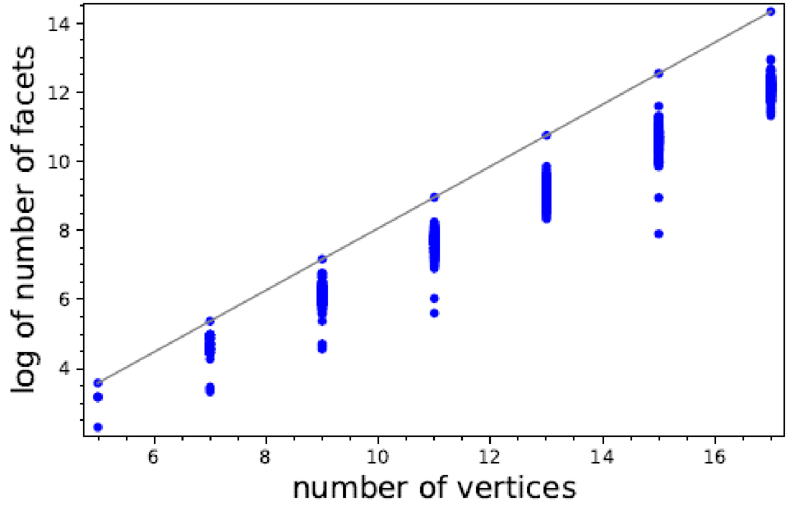
Facets of Symmetric Edge Polytopes: Clustering, Degree Sequences, and Graphs with Few Edges

- Benjamin Braun, University of Kentucky
- Kaitlin Bruegge, University of Cincinnati
- Matthew Kahle, The Ohio State University

- Partially supported by NSF awards DMS-1953785 and DMS-2005630.







Before we begin

Science

SCIENCEINSIDER

NIH slashes overhead payments for research, sparking outrage

Move to cut indirect cost rate to 15% could cost universities billions of dollars

7 FEB 2025 • 9:42 PM ET • BY DAVID MALAKOFF

Science

SCIENCEINSIDER | SCIENCE AND POLICY

'My boss was crying.' NSF confronts potentially massive layoffs and budget cuts

Trump could propose slashing agency's budget by two-thirds

7 FEB 2025 • 2:20 PM ET • BY JEFFREY MERVIS

OPINION

Scientific institutions have a long history of anticipatory obedience

BY PHILIP BALL | 7 FEBRUARY 2025

Racial Equity in STEM Education (EDU Racial Equity)

Home / Funding at NSF / Funding Search / Racial Equity in STEM Education (EDU Racial Equity) / NSF 22-634

Archived funding opportunity
This document has been archived.

Mathematical and Physical Sciences Ascending Postdoctoral Research Fellowships (MPS-Ascend)

Home / Funding at NSF / Funding Search / Mathematical and Physical Sciences Ascending Postdoctoral Research Fellowships (MPS-Ascend) / NSF 23-501

Archived funding opportunity
This document has been archived.

I believe that our collective goal should not be the "advancement of mathematics". Our goal should be the **advancement of our humanity.** Mathematics is a human endeavor. Our mathematical communities can be centered on care and compassion above all else, if we choose.

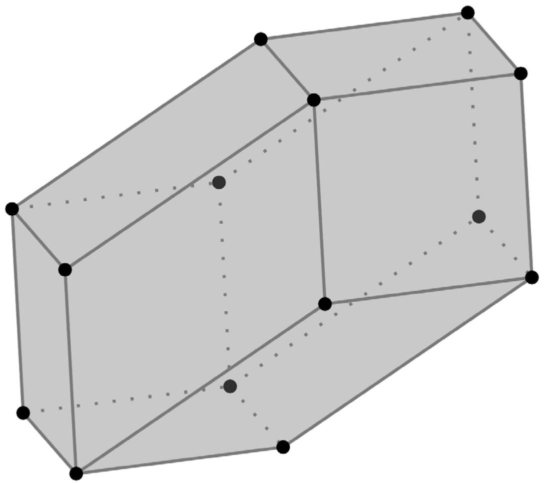
Structure of this talk

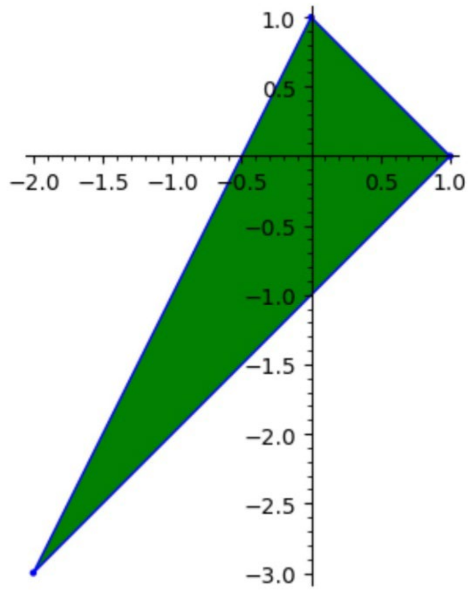
- **Part 1**: Survey talk about geometric and algebraic properties of lattice polytopes with focus on symmetric edge polytopes
- **Part 2**: Work by me, Bruegge, and Kahle regarding facet counting for symmetric edge polytopes

Part 1

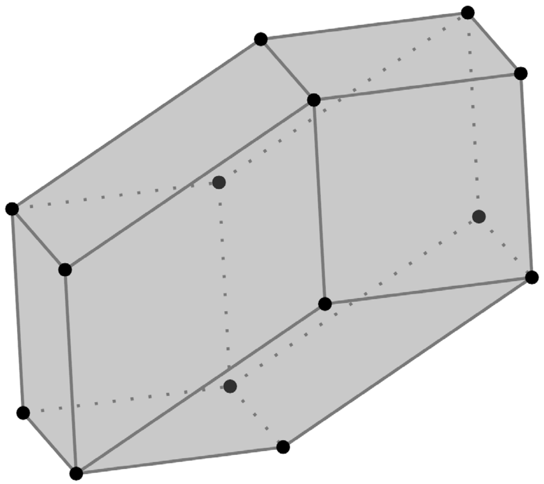
Lattice polytopes

A lattice polytope P is the convex hull of n lattice points in m-dimensional space. It is also a bounded intersection of a finite set of closed half-spaces. Assume that P has dimension d.

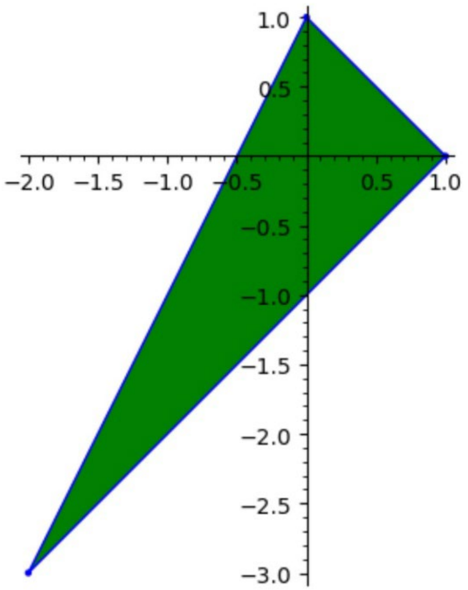




Lattice polytopes



A lattice polytope P is the convex hull of n lattice points in m-dimensional space. It is also a bounded intersection of a finite set of closed half-spaces. Assume that P has dimension d.



We typically replace Euclidean volume with *normalized volume*:

$$\mathrm{Vol}(P) = d!\mathrm{vol}(P)$$

Symmetric edge polytopes, i.e., SEPs

Given a finite, simple graph G, the *symmetric edge polytope* (SEP) defined by G is:

$$P_G := \operatorname{conv}\{\pm(e_i - e_j) : ij \in E(G)\}.$$

Note: This is the vertex description. Facet description, i.e., the maximal non-trivial faces, will show up later in the talk.

Symmetric edge polytopes, i.e., SEPs

Given a finite, simple graph G, the *symmetric edge polytope* (SEP) defined by G is:

$$P_G := \operatorname{conv}\{\pm(e_i - e_j) : ij \in E(G)\}.$$

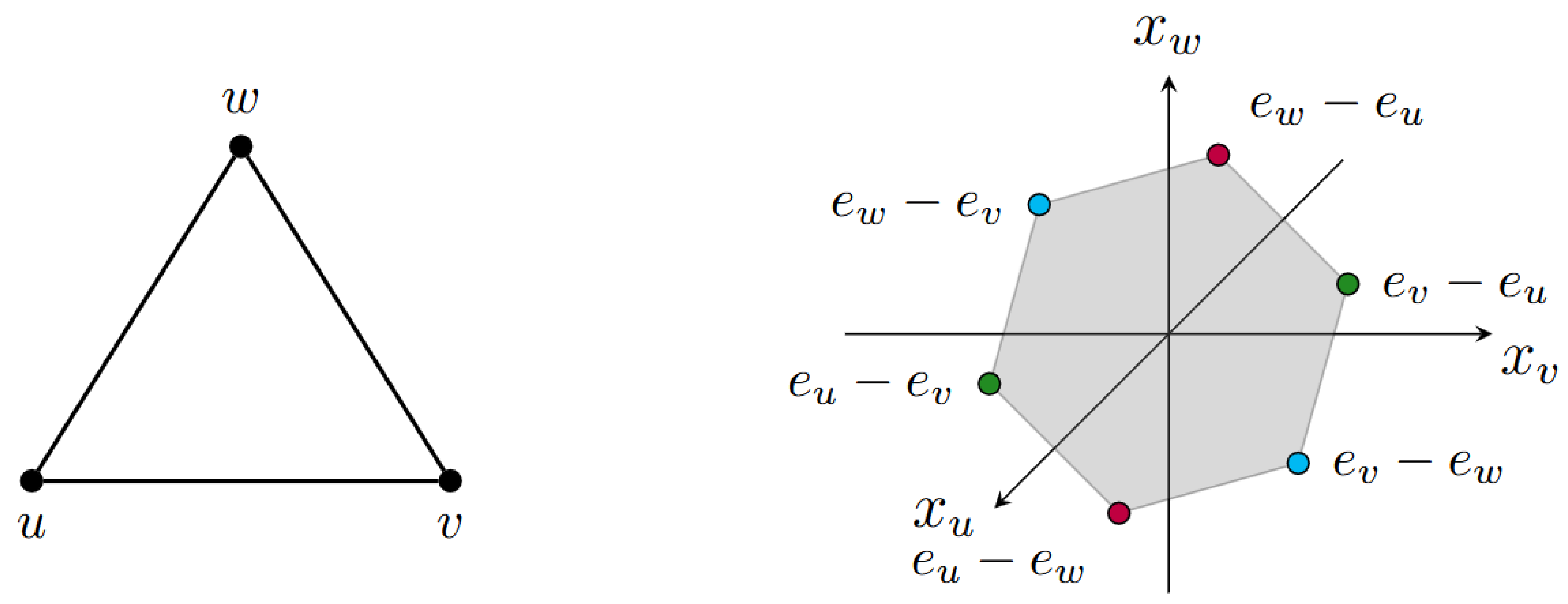


Image from Katie Bruegge's PhD thesis

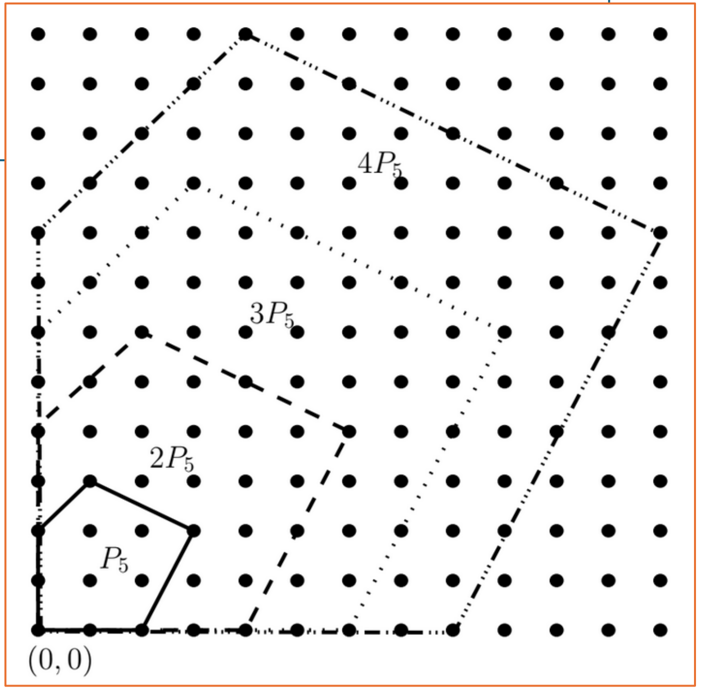
Ehrhart theory

Problem: Count the lattice points in integer dilates of a polytope.

For a lattice polytope P, the *Ehrhart series* of P is

$$\mathrm{Ehr}(P;z) = \sum_{t=0}^{\infty} |tP \cap \mathbb{Z}^d| z^t = \frac{h_0^* + h_1^* z + \cdots + h_d^* z^d}{(1-z)^{d+1}}$$

where $h_i^* \in \mathbb{Z}_{\geq 0}$ for all i.



Rationality is due to Ehrhart. Non-negativity is due to Stanley. The numerator is the h^*-*polynomial* and its coefficient vector is the h^*-*vector* of the polytope.

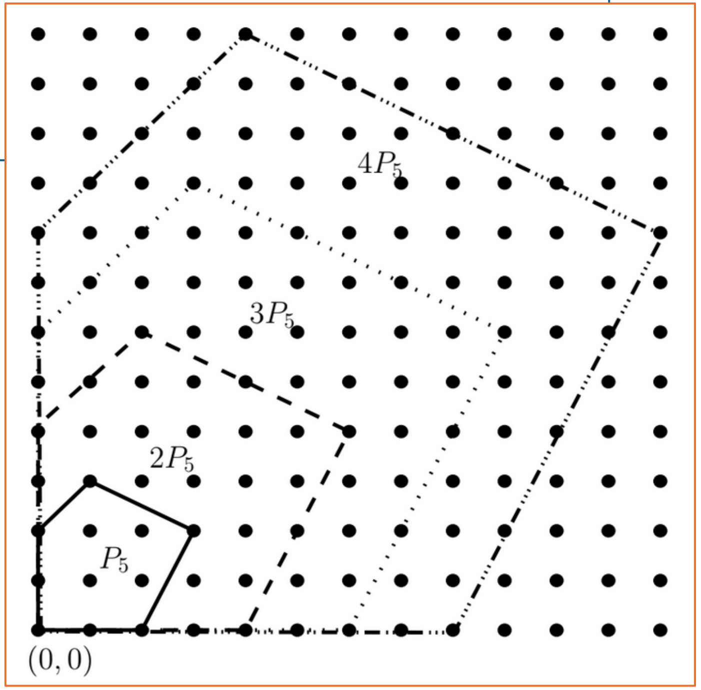
Ehrhart theory

Problem: Count the lattice points in integer dilates of a polytope.

For a lattice polytope P, the *Ehrhart series* of P is

$$\mathrm{Ehr}(P;z) = \sum_{t=0}^{\infty} |tP \cap \mathbb{Z}^d| z^t = \frac{h_0^* + h_1^* z + \cdots + h_d^* z^d}{(1-z)^{d+1}}$$

where $h_i^* \in \mathbb{Z}_{\geq 0}$ for all i.



Rationality is due to Ehrhart. Non-negativity is due to Stanley. The numerator is the h^*-*polynomial* and its coefficient vector is the h^*-*vector* of the polytope.

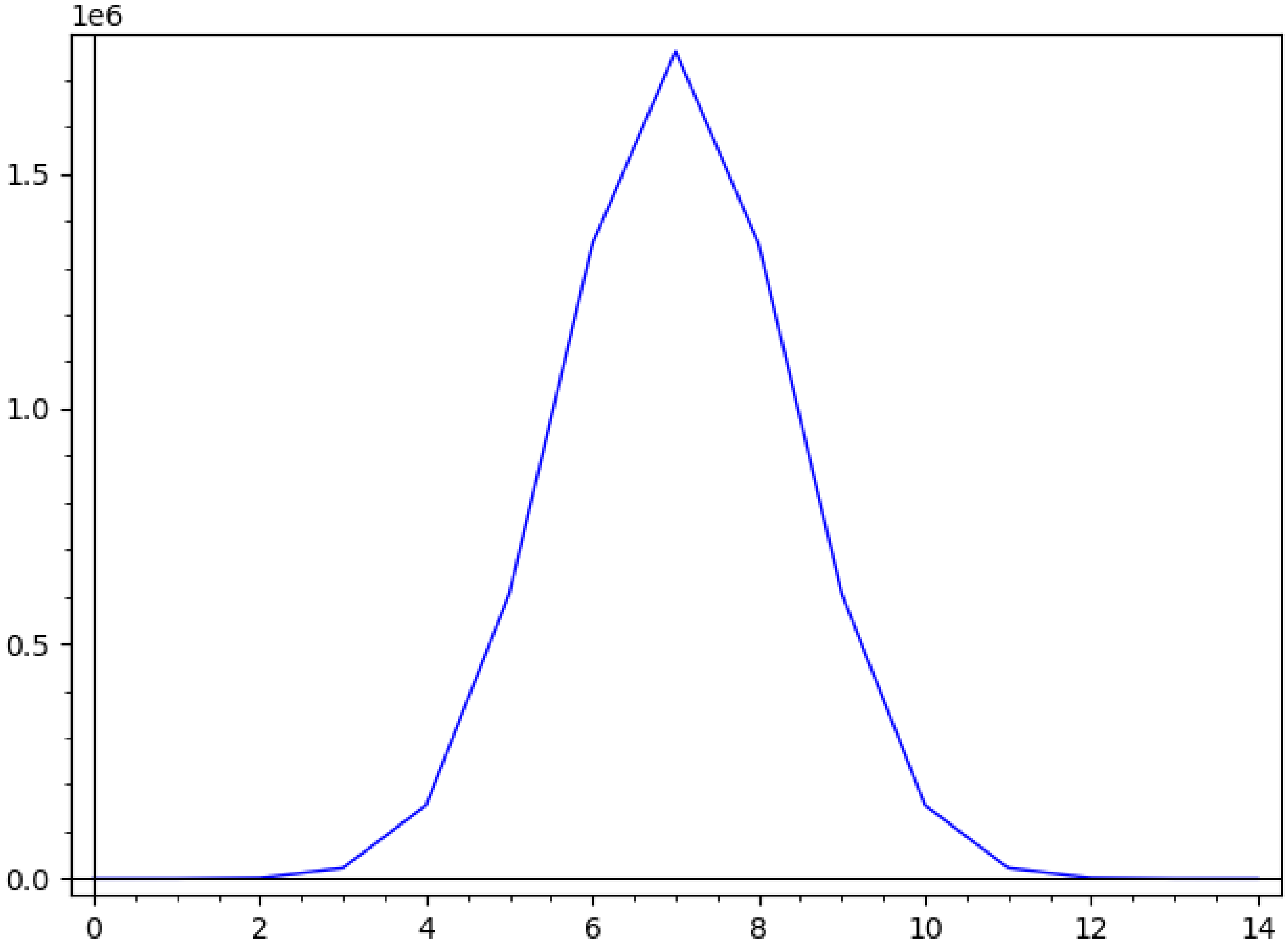
Theorem: $\sum_i h_i^* = \mathrm{Vol}(P)$

h^*-vectors of SEPs

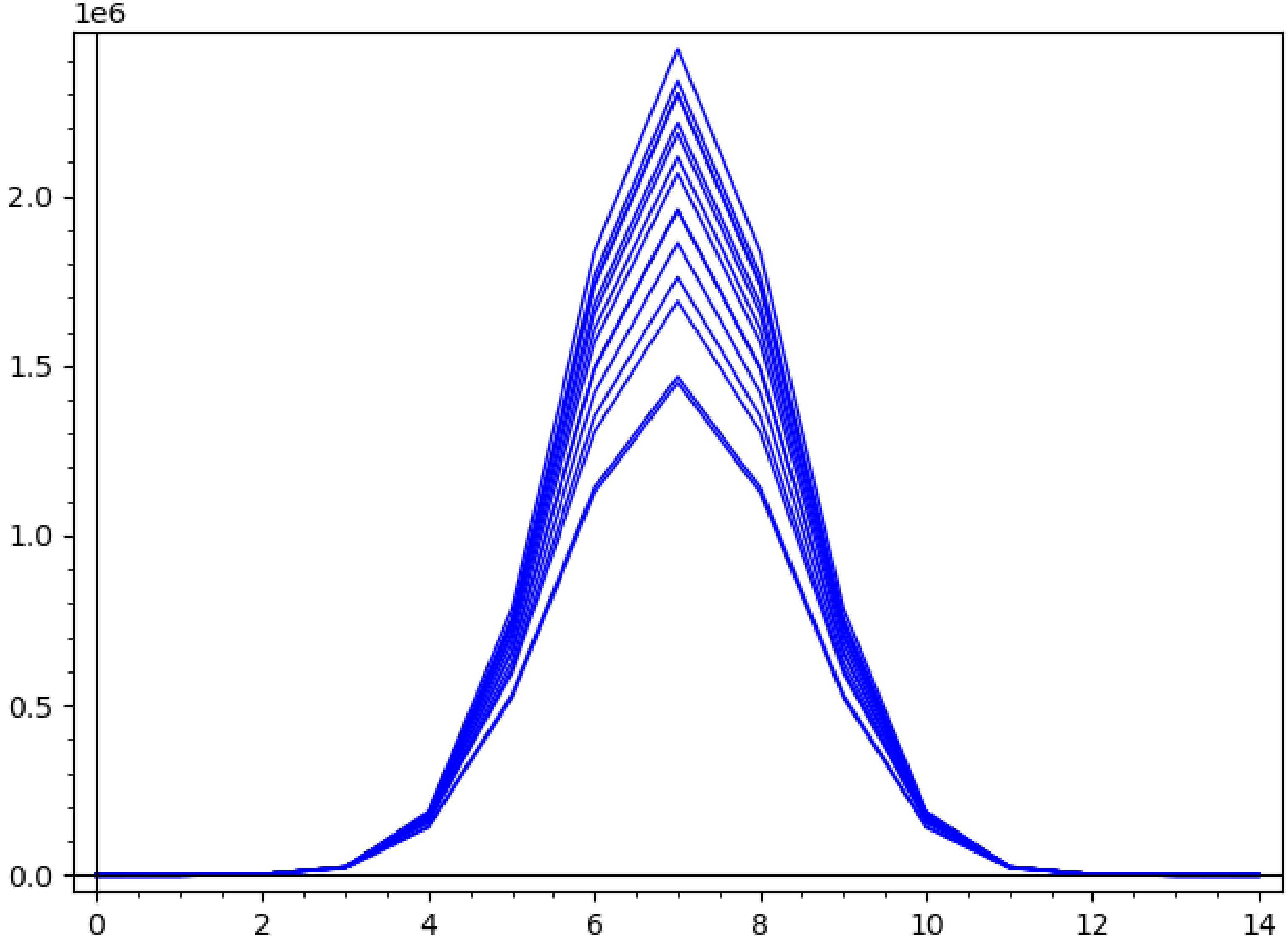
1,
66,
1735,
22534,
156343,
605970,
1350661,
1761000,
1350661,
605970,
156343,
22534,
1735,
66,
1

h^*-vectors of SEPs

1,
66,
1735,
22534,
156343,
605970,
1350661,
1761000,
1350661,
605970,
156343,
22534,
1735,
66,
1



h^*-vectors of SEPs



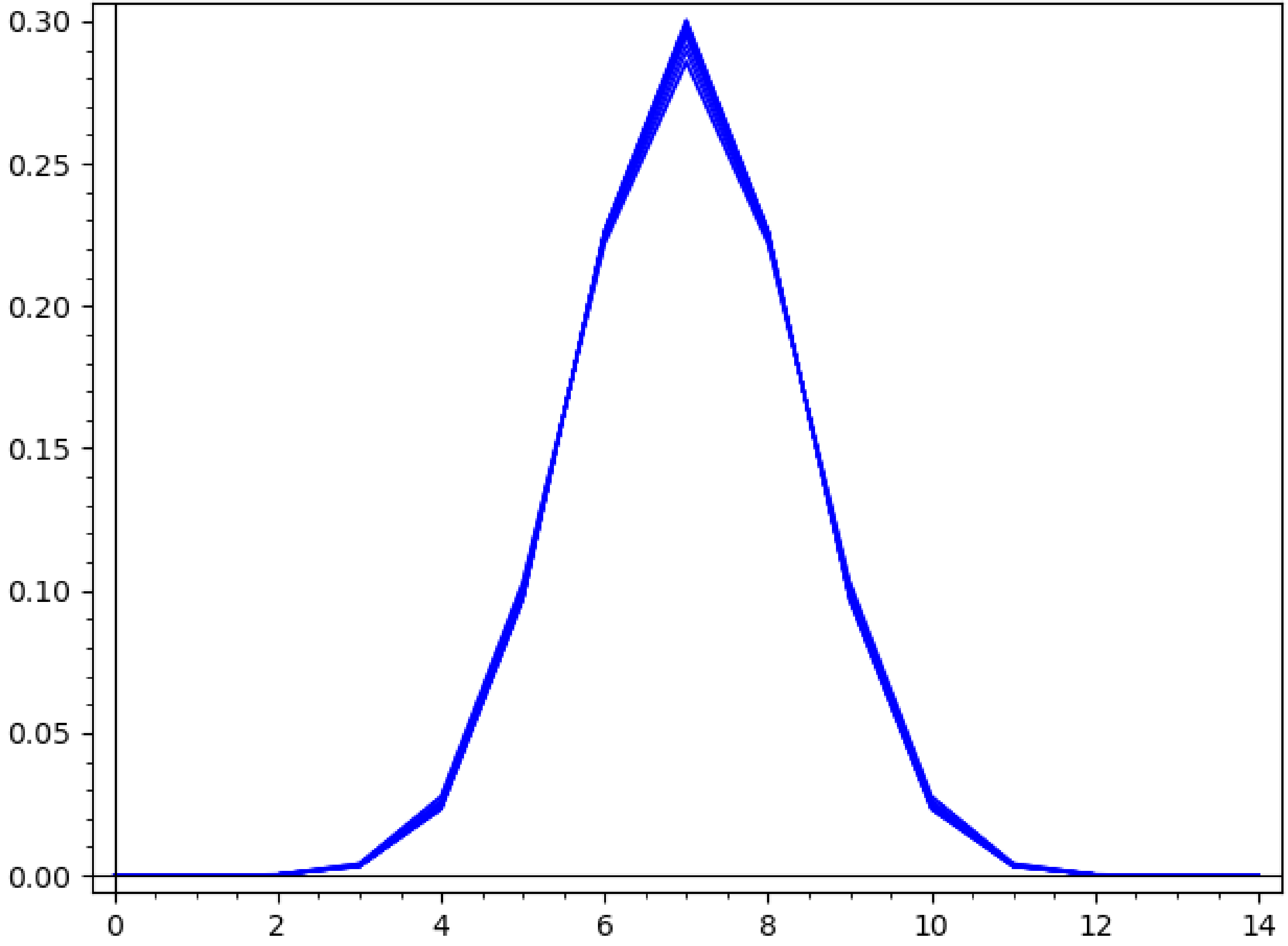
15 random connected graphs on 15 vertices with 40 edges. Computation with SageMath.

h^*-distributions of SEPs

$h^*(z)/h^*(1)$

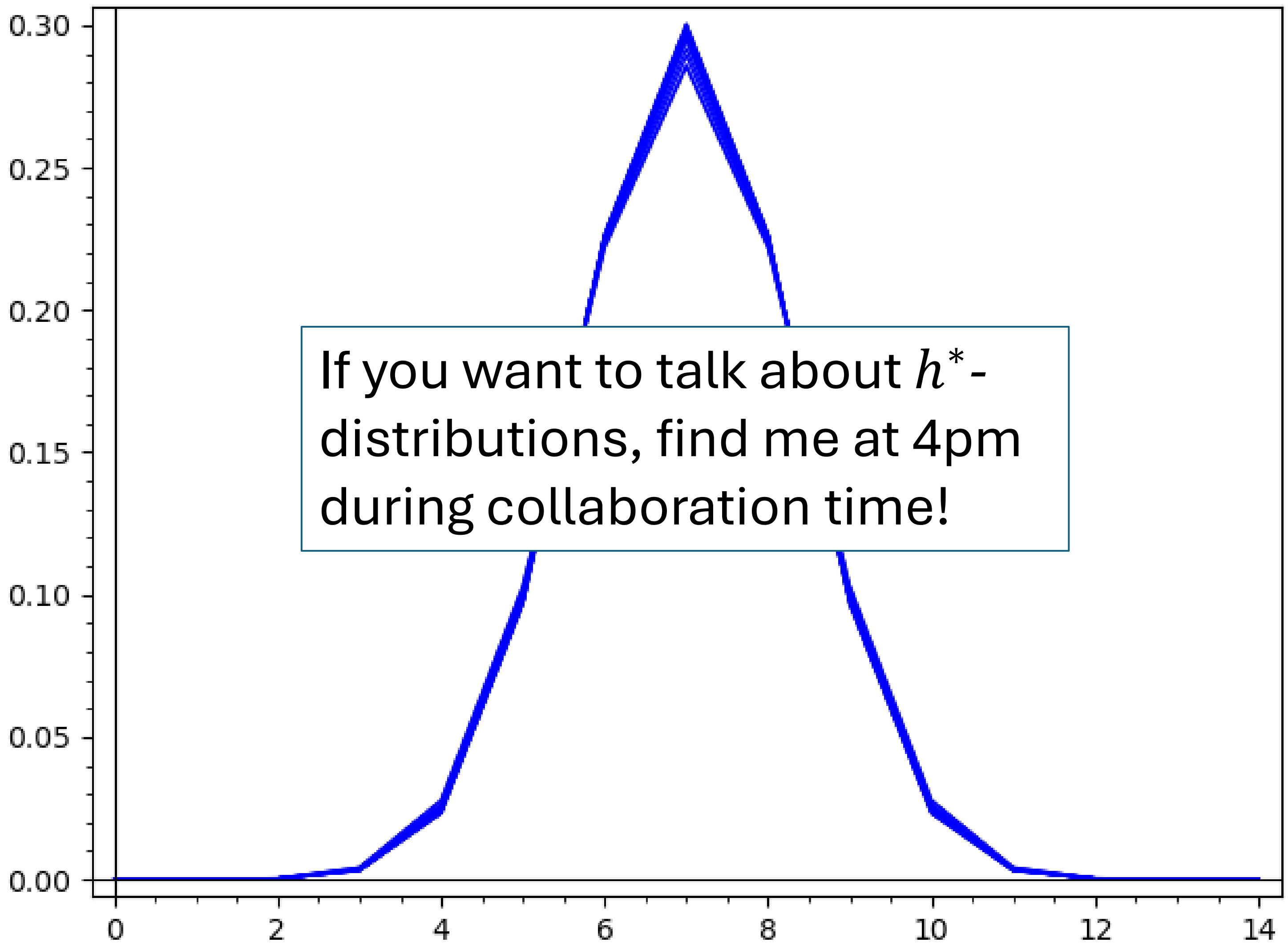
h^*-distributions of SEPs

$h^*(z)/h^*(1)$



h^*-distributions of SEPs

$h^*(z)/h^*(1)$



If you want to talk about h^*-distributions, find me at 4pm during collaboration time!

Gorenstein polytopes

A lattice polytope P is *reflexive* if P is unimodularly equivalent to a lattice polytope Q containing the origin in its interior such that the dual of Q is a lattice polytope.

Q is *Gorenstein* if rQ is reflexive for a positive integer r.

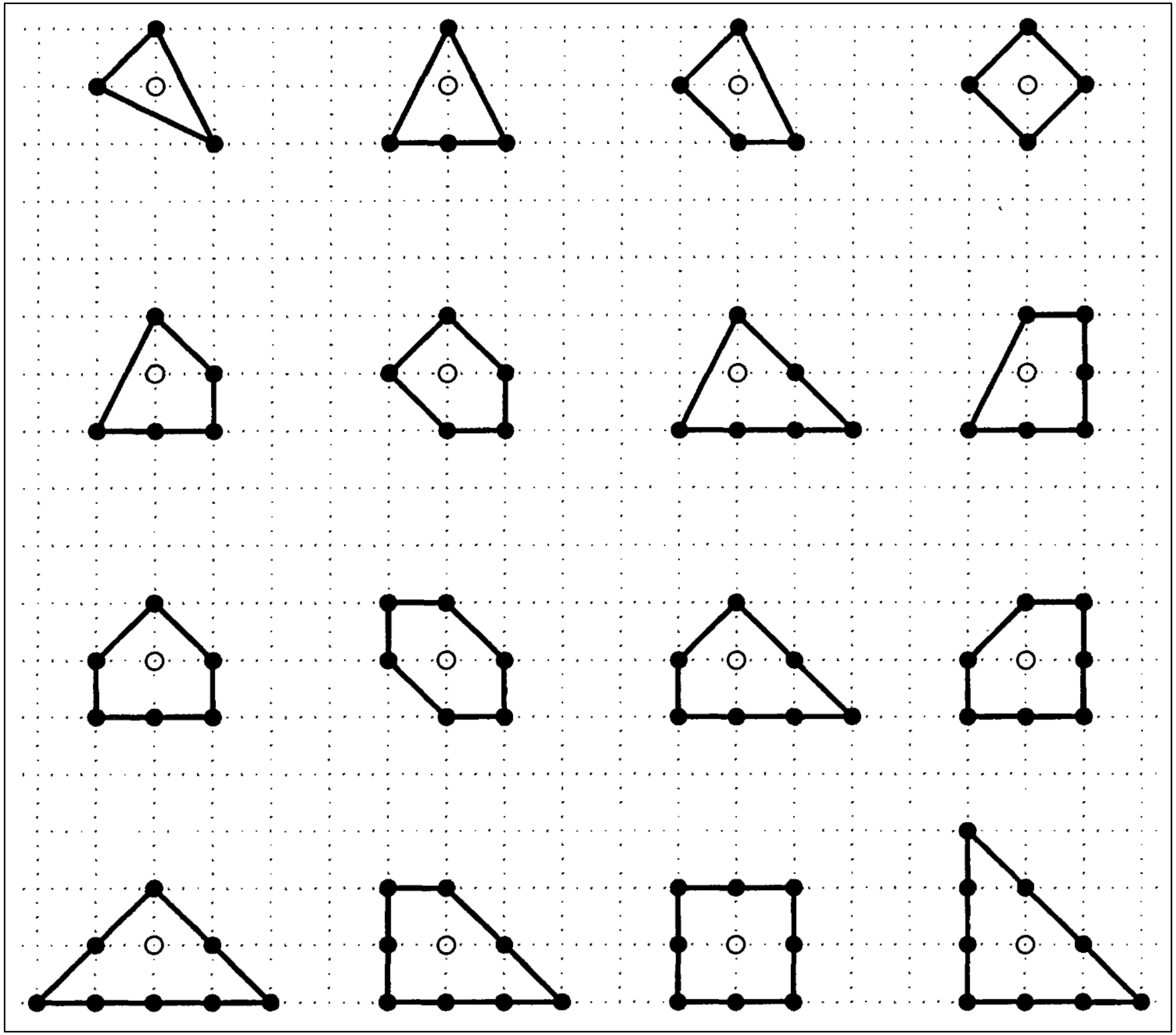


Image: The 16 reflexive polygons, from: Poonen, B., and Rodriguez-Villegas, F. “Lattice Polygons and the Number 12.” The American Mathematical Monthly 107, no. 3 (2000): 238–50.

Gorenstein polytopes and Ehrhart theory

A lattice polytope P is *reflexive* if P is unimodularly equivalent to a lattice polytope Q containing the origin in its interior such that the dual of Q is a lattice polytope.

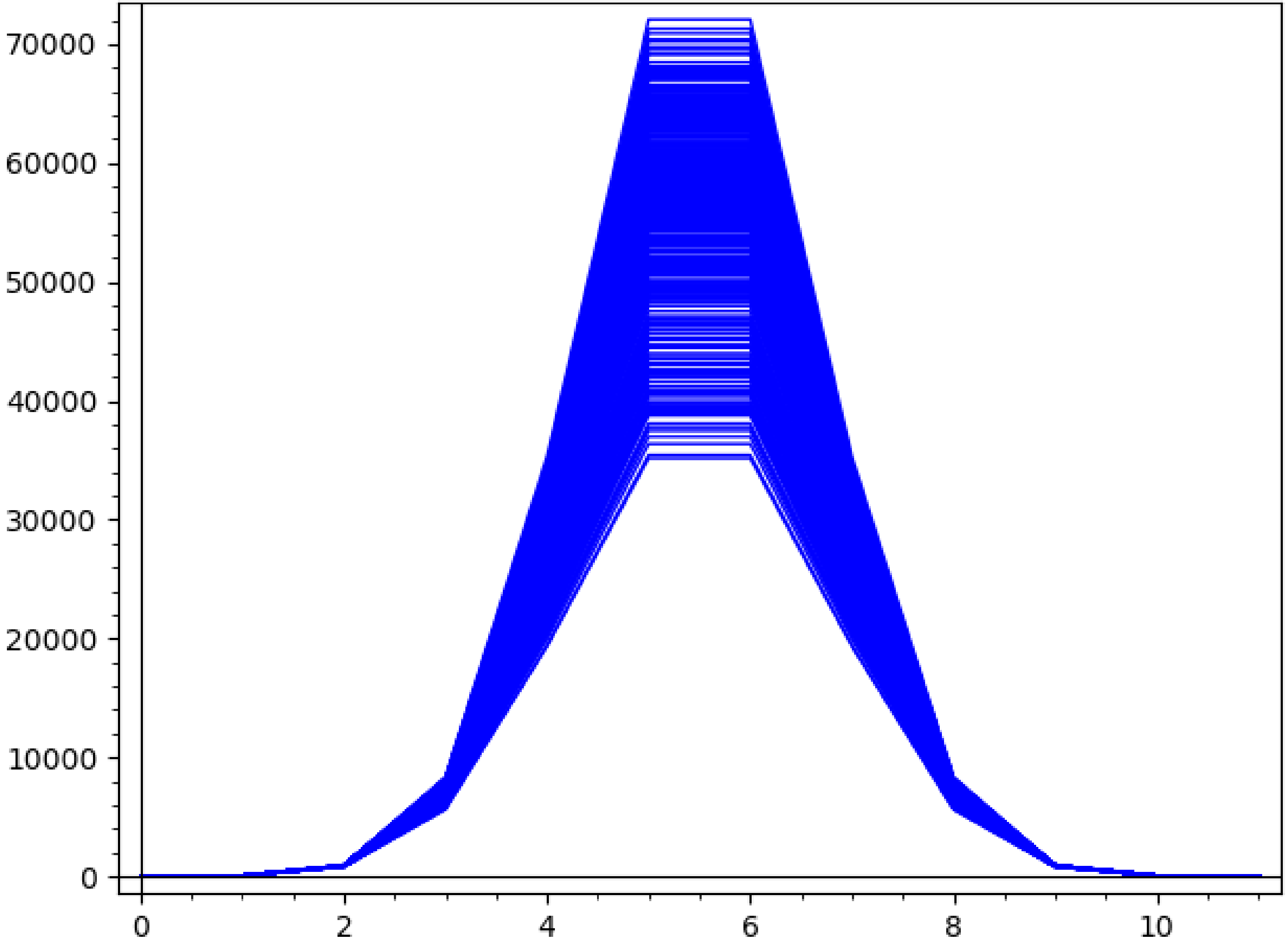
Q is *Gorenstein* if rQ is reflexive for a positive integer r.

Theorem: (Stanley; Hibi in the reflexive case) Suppose P has $h^*(P;z) = h_0^* + h_1^* z + h_2^* z^2 + \cdots + h_s^* z^s$ for some $s \leq d$. Then P is Gorenstein if and only if $h_i^* = h_{s-i}^*$ for all i.

SEPs are reflexive

Theorem: (Matsui et al. 2011 and Higashitani 2015) SEPs are reflexive.

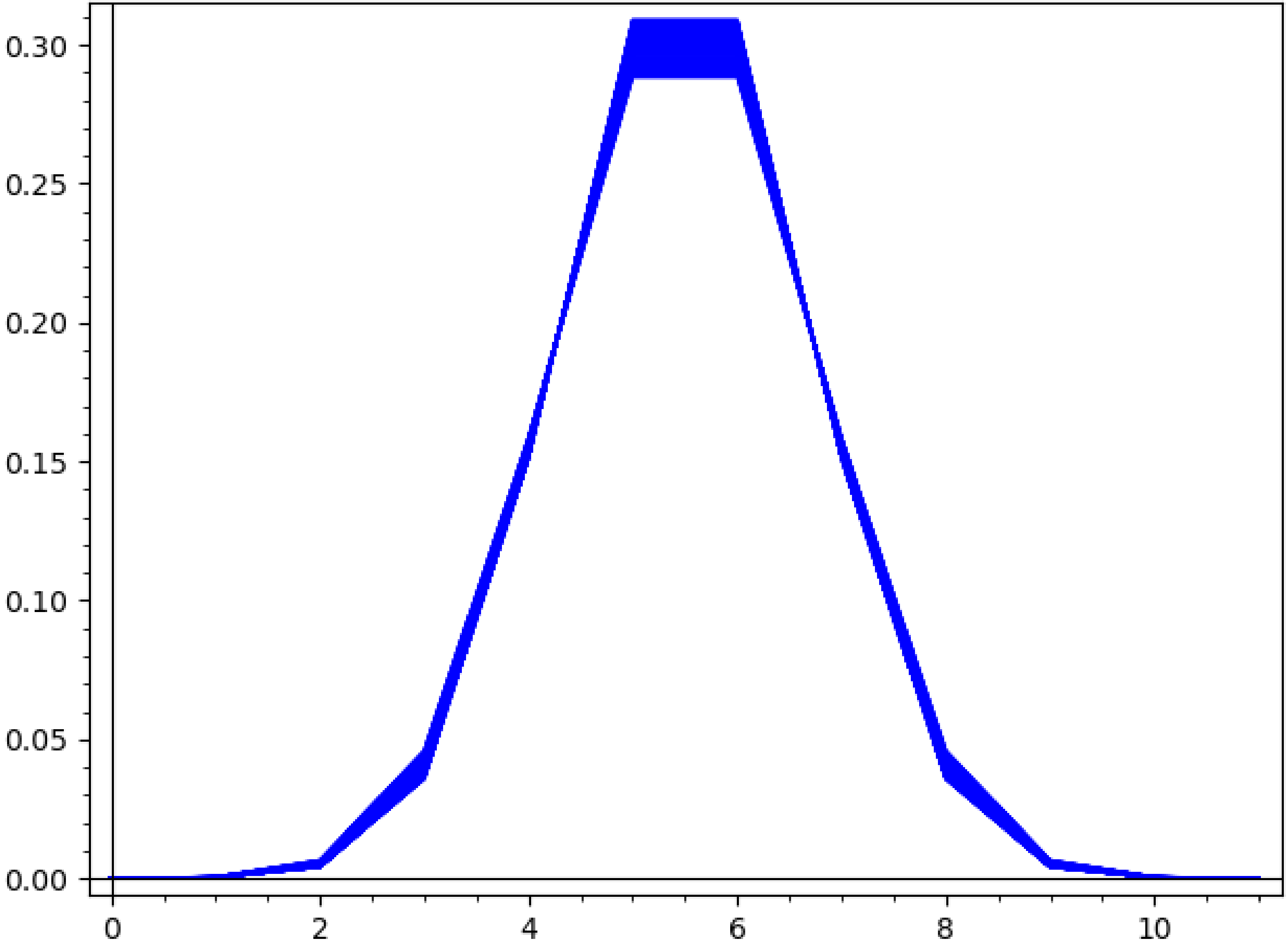
396 random connected graphs on 12 vertices with 30 edges. Note that there are 12,195,279,971 unlabeled simple connected graphs on 12 vertices with 30 edges. (per computations by Brendan McKay)



SEPs are reflexive

Same objects, but graphing distributions instead of vectors.

$$h^*(z)/h^*(1)$$



γ-vectors

$$(1+z)^3 + 8z(1+z) =$$
$$(1+3z+3z^2+z^3) + 8(z+z^2) =$$
$$1 + 11z + 11z^2 + z^3$$

If $p(z) = \sum_{i=0}^{d} a_i z^i$ satisfies $a_i = a_{d-i}$,

p is *γ-non-negative* if $p(z) = \sum_{j=0}^{\lfloor d/2 \rfloor} \gamma_j z^j (1+z)^{d-2j}$

with $\gamma_j \in \mathbb{Z}_{\geq 0}$ for all j.

γ-vectors

Conjecture (Gal's conjecture, in special case of reflexive polytopes): If a reflexive lattice polytope has a regular unimodular flag triangulation with 0 in every maximal simplex, then the h^*-vector is γ-nonnegative.

Motivating problems about lattice polytopes (including SEPs)

- Determine facet and/or vertex descriptions
- Find interesting/effective volume formulas
- Classify h^*-vectors for polytopes in a specified family
- Find combinatorial interpretations of h^*-vectors
- Investigate inequalities for h^*-coefficients, including unimodality, log-concavity, real-rootedness, γ-nonnegativity, etc.
- Investigate triangulations and their structure
- Classify special polytopes, e.g., Gorenstein, integer decomposition property, smooth, canonical line property, etc.

SEP fundamental results

- 2002, Ohsugi and Hibi
- 2011, Matsui et al.
- 2015, Higashitani
- 2018, Chen, Davis, Mehta
- 2019, Higashitani, Jochemko, Michalek
- 2020, Ohsugi, Tsuchiya
- 2023, Kara, Portakal, Tsuchiya
- 2024, D'Ali, Juhnke-Kubitzke, Koch

Some things we know:

- 2018: The volume of an SEP gives an upper bound on the number of solutions to associated algebraic Kuramoto equations.
- 2019: Facet description and explicit regular unimodular triangulation.
  - Corollary: SEPs are h^*-unimodal.
- 2024: SEPs are unimodularly equivalent if and only if they have isomorphic graphic matroids. Further, the definition of SEP can be extended to any regular matroid, giving a *generalized SEP*.

SEP Ehrhart theory results

- 2011, Matsui et al.
- 2012, Ohsugi, Shibata
- 2017, Higashitani, Kummer, Michalek
- 2019, Higashitani, Jochemko, Michalek
- 2021, Ohsugi, Tsuchiya
- 2022, Kálmán, Tóthmérész
- 2023, D'Ali, Juhnke-Kubitzke, Koehne, Venturello
- 2024, Davis, Higashitani, Ohsugi
- 2024, Kölbl

Some things we know:

- 2017: Some SEPs have Ehrhart polynomials with roots on the line $Re(z) = -1/2$, i.e., they have the *canonical line property*.
- 2021: All SEPs are conjectured to have γ-nonnegative h^*-vectors.
- 2022: h^* description via shellable simplicial dissections of facets.
- 2023: γ_2 nonnegative; asymptotically almost surely γ-nonnegativity results for Erdos-Renyi graphs.
- 2024 (Davis et al): *Generalized* SEPs are not all γ-nonnegative.

SEP facet and face structure results

- 2019, Higashitani, Jochemko, Michalek
- 2022, Kálmán, Tóthmérész
- 2022, D’Ali, Delucchi, Michalek
- 2023, Chen, Davis, Korchevskaia
- 2023, BB, Bruegge
- 2023, BB, Bruegge, Kahle
- 2024, Mori, Mori, Ohsugi

Part 2

Facet description of SEPs

Theorem: (Higashitani, Jochemko, Michałek) Let $G = (V, E)$ be a finite simple connected graph and

$$\sum_{v \in V} f(v) x_v \leq 1$$

a hyperplane in $\mathbb{R}^V$. Then $f : V \to \mathbb{Z}$ is facet-defining if and only if both of the following hold.

(i) For every edge $e = uv$ we have $|f(u) - f(v)| \leq 1$.

(ii) The subset of edges

$$E_f = \{e = uv \in E \ : \ |f(u) - f(v)| = 1\}$$

forms a spanning connected subgraph of G.

Facet description of SEPs

Theorem: (Higashitani, Jochemko, Michałek) Let $G = (V, E)$ be a finite simple connected graph and

$$\sum_{v \in V} f(v) x_v \leq 1$$

a hyperplane in $\mathbb{R}^V$. Then $f : V \to \mathbb{Z}$ is facet-defining if and only if both of the following hold.

(i) For every edge $e = uv$ we have $|f(u) - f(v)| \leq 1$.

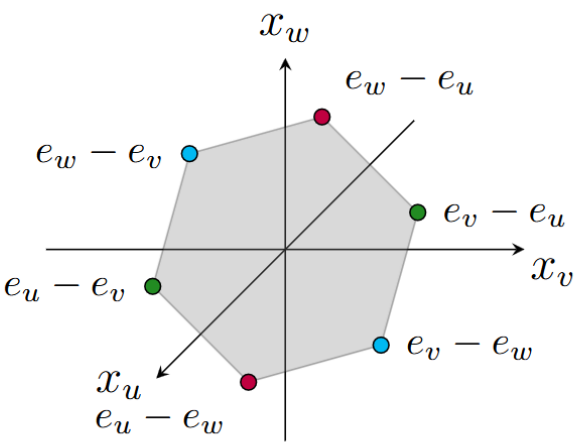
(ii) The subset of edges

$$E_f = \{e = uv \in E \ : \ |f(u) - f(v)| = 1\}$$

forms a spanning connected subgraph of G.

Example:
K_n has $2^n - 2$ facets.

Any non-trivial subset A of the vertices induces 0/1-labeling, 0 if in A, 1 if not. This is the only type of labeling that can arise for a facet normal.



How many facets do we expect?

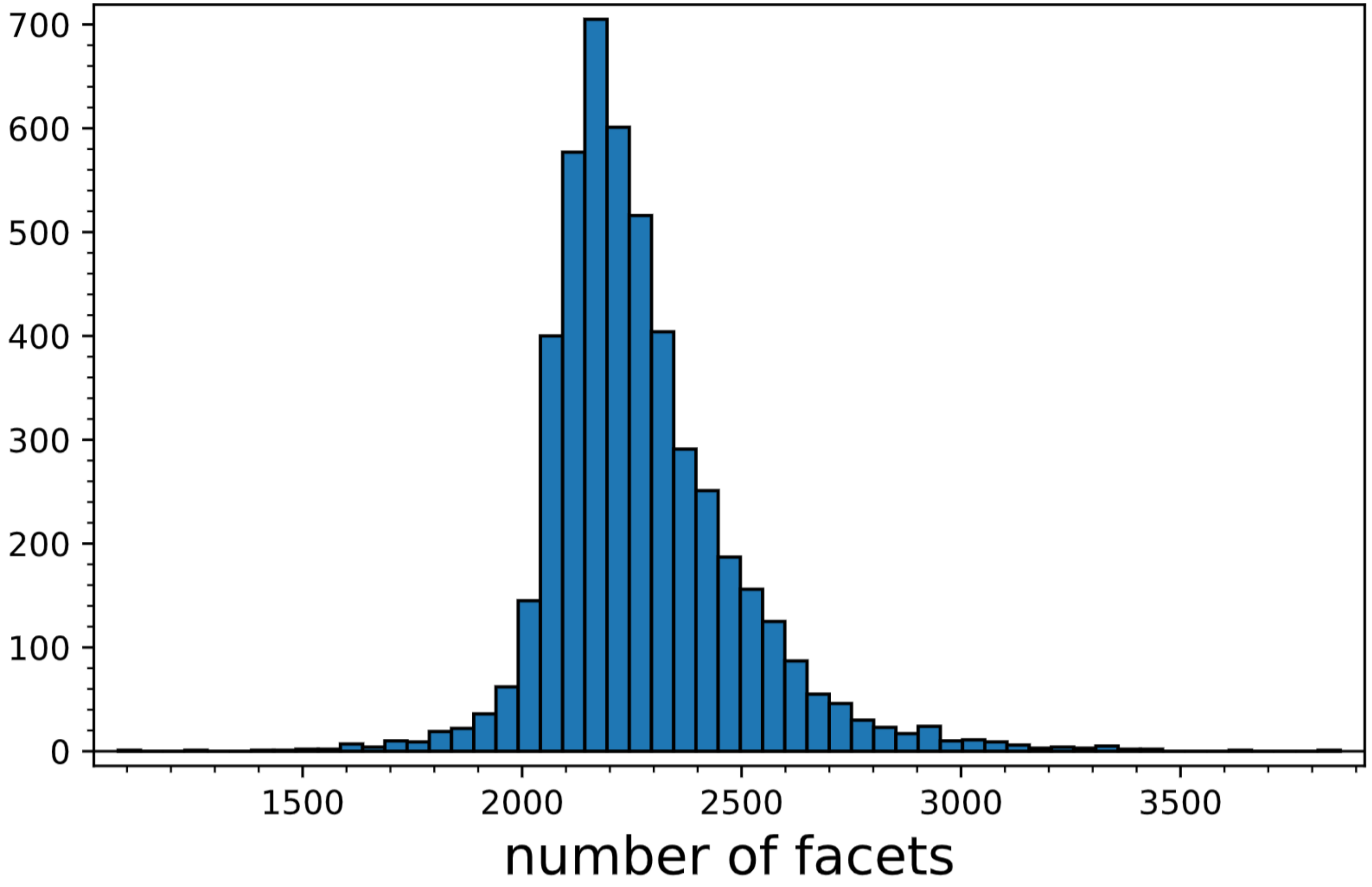


FIGURE 2. Histogram of $N(G)$ for 4874 connected graphs sampled from $G(11, 0.45)$.

How many facets do we expect?

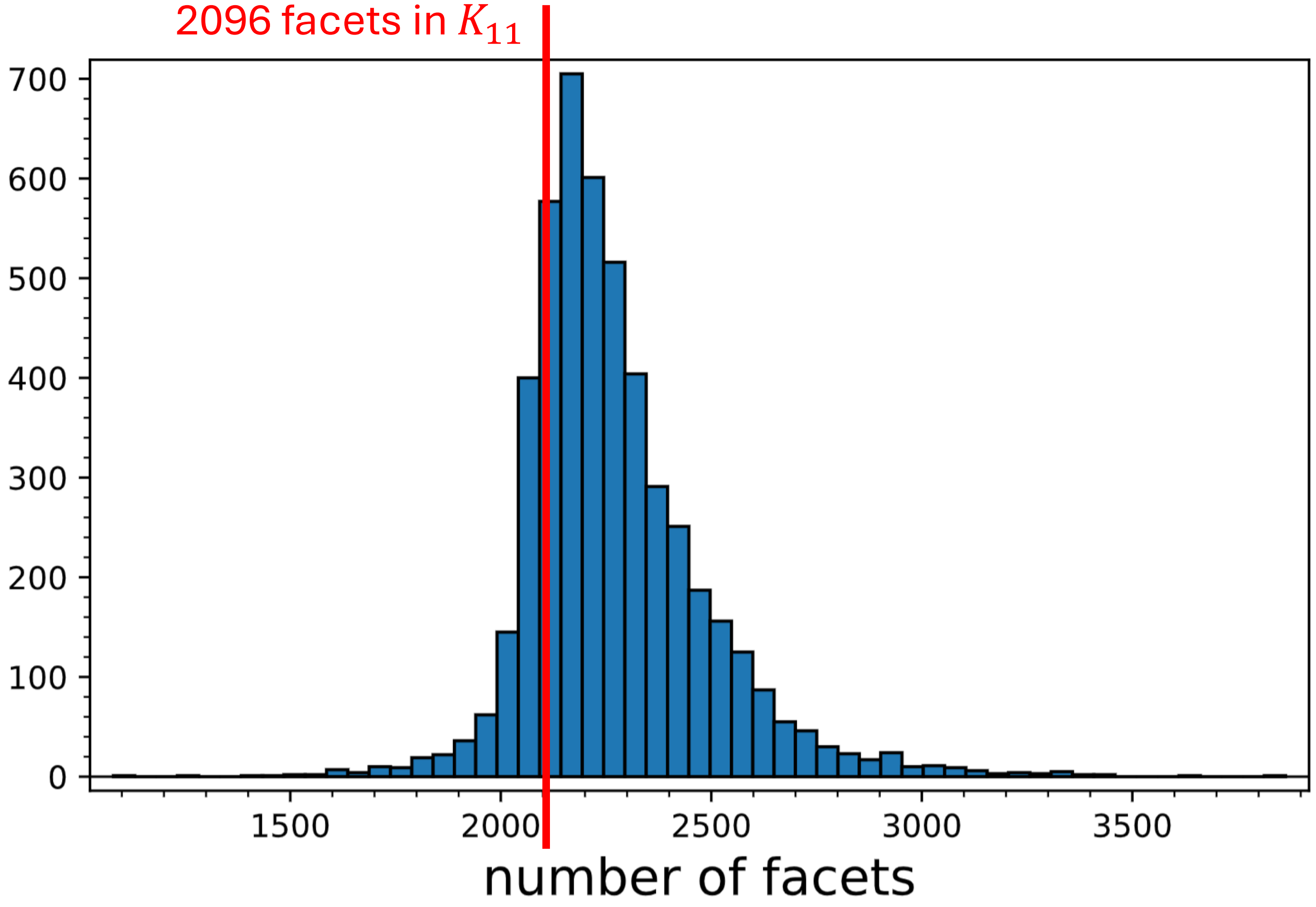


FIGURE 2. Histogram of $N(G)$ for 4874 connected graphs sampled from $G(11, 0.45)$.

How many facets do we expect?

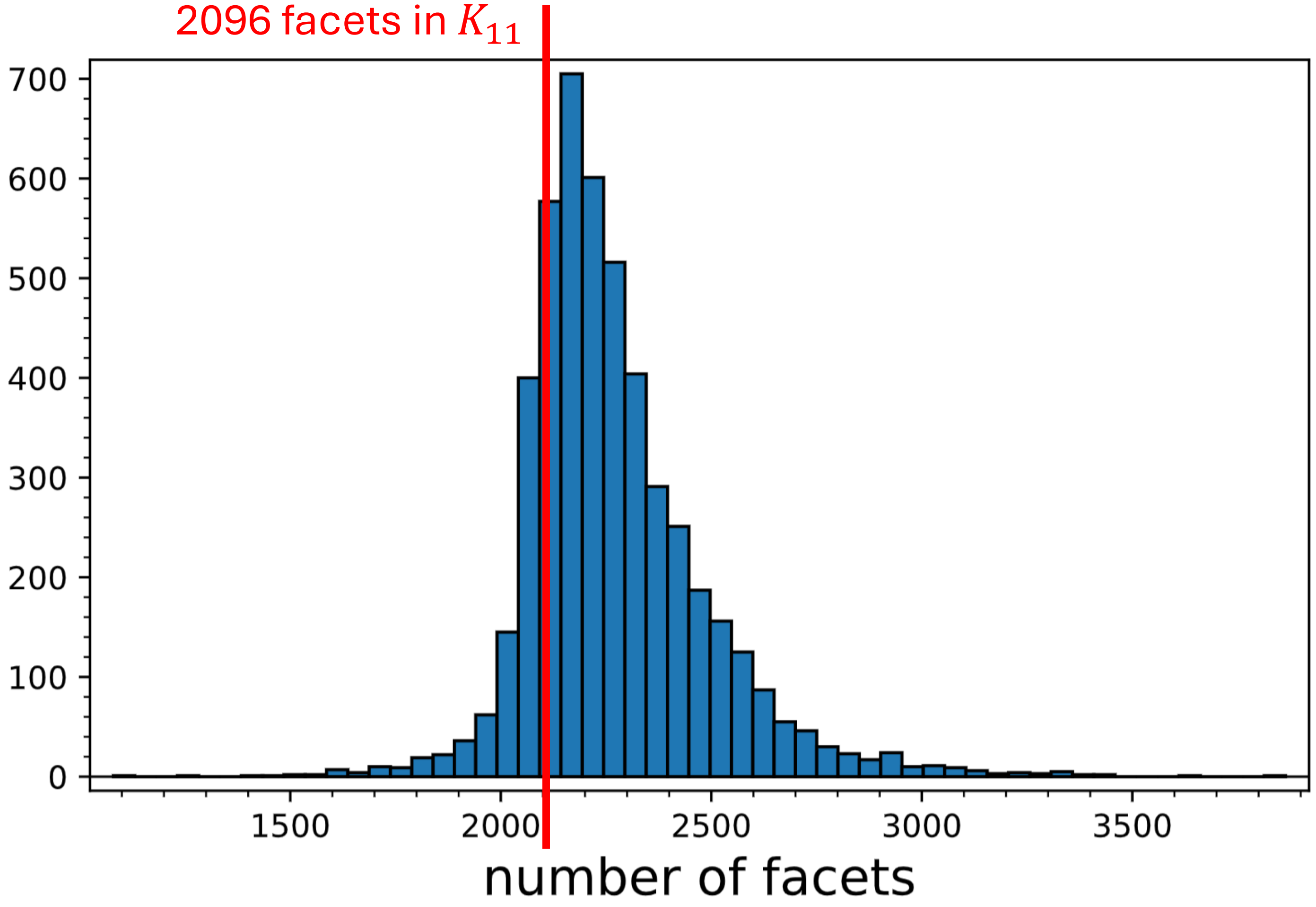


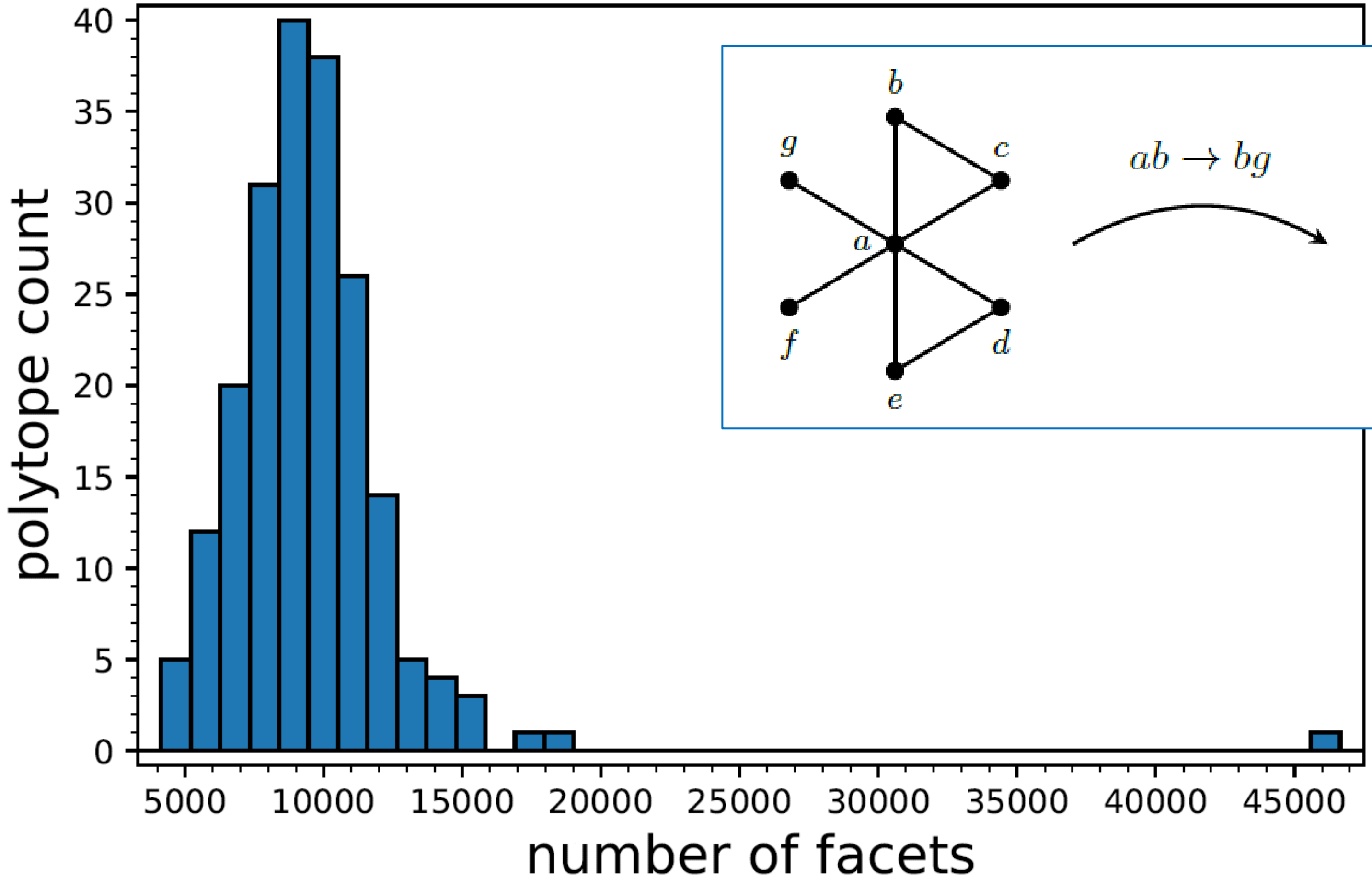
FIGURE 2. Histogram of $N(G)$ for 4874 connected graphs sampled from $G(11, 0.45)$.

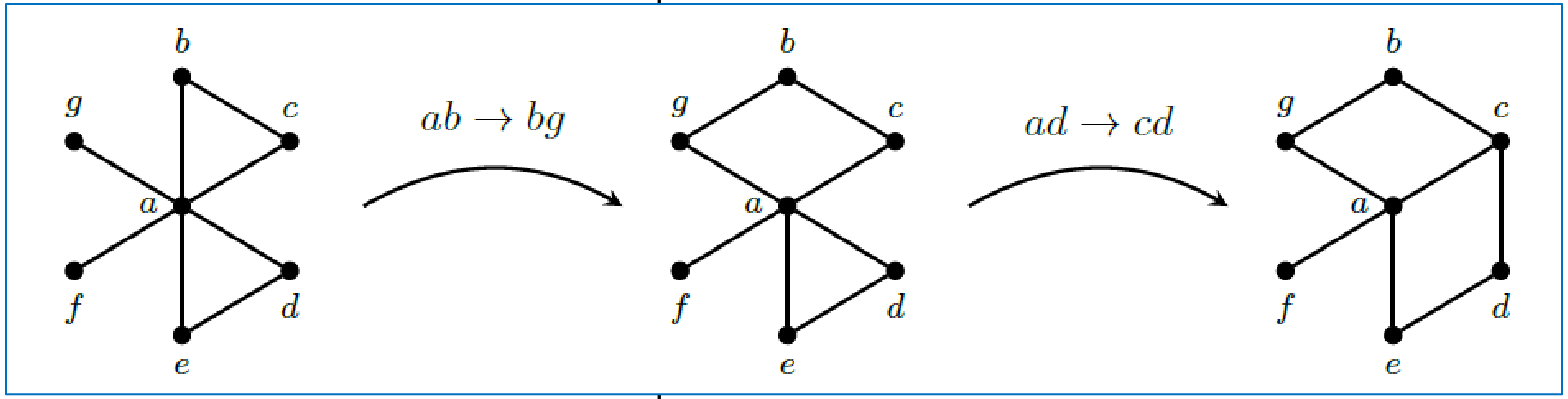
Question:

What if we sample from restricted families of graphs?

Earlier, we fixed the number of edges.

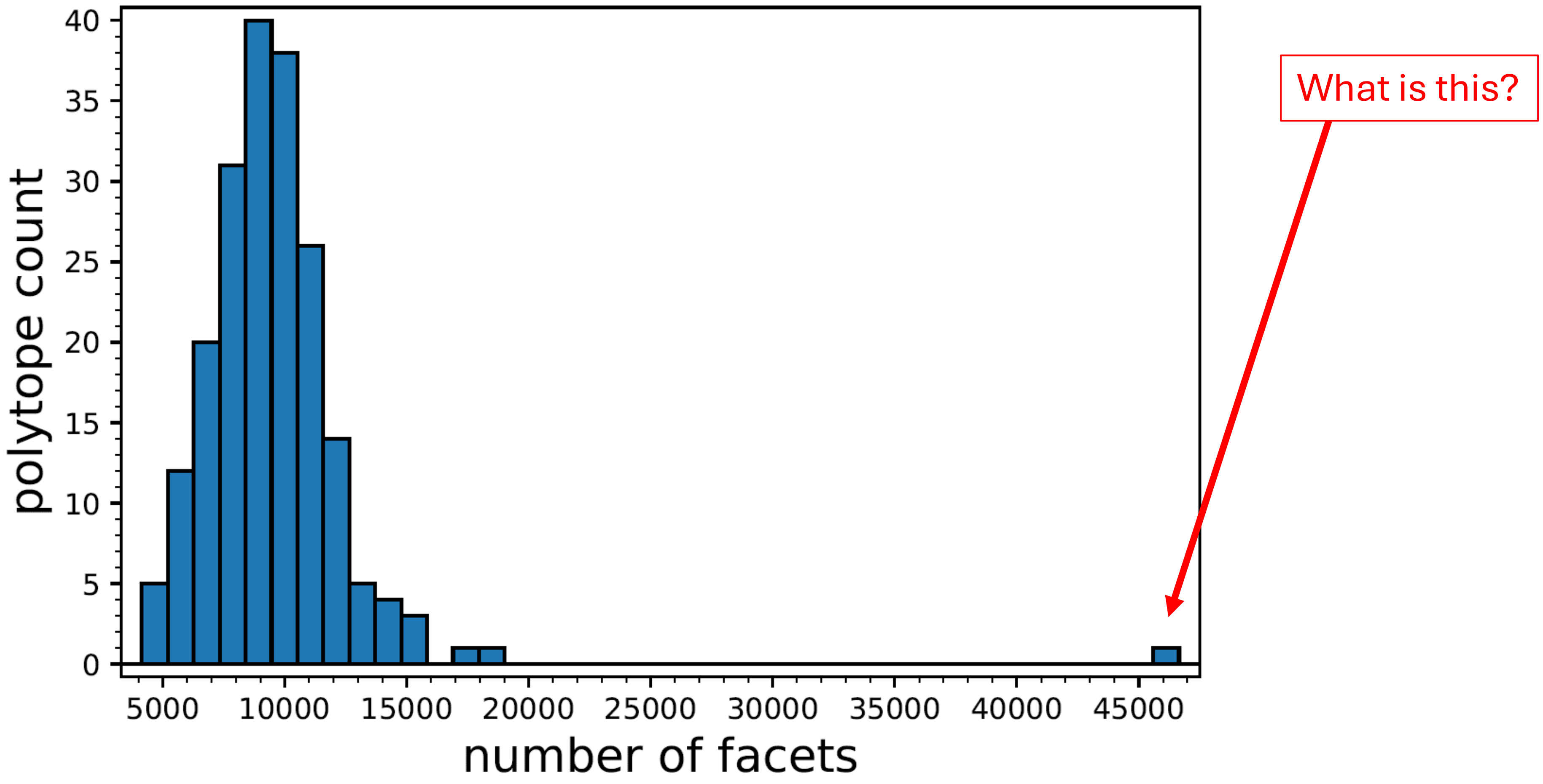
Fixed number of edges





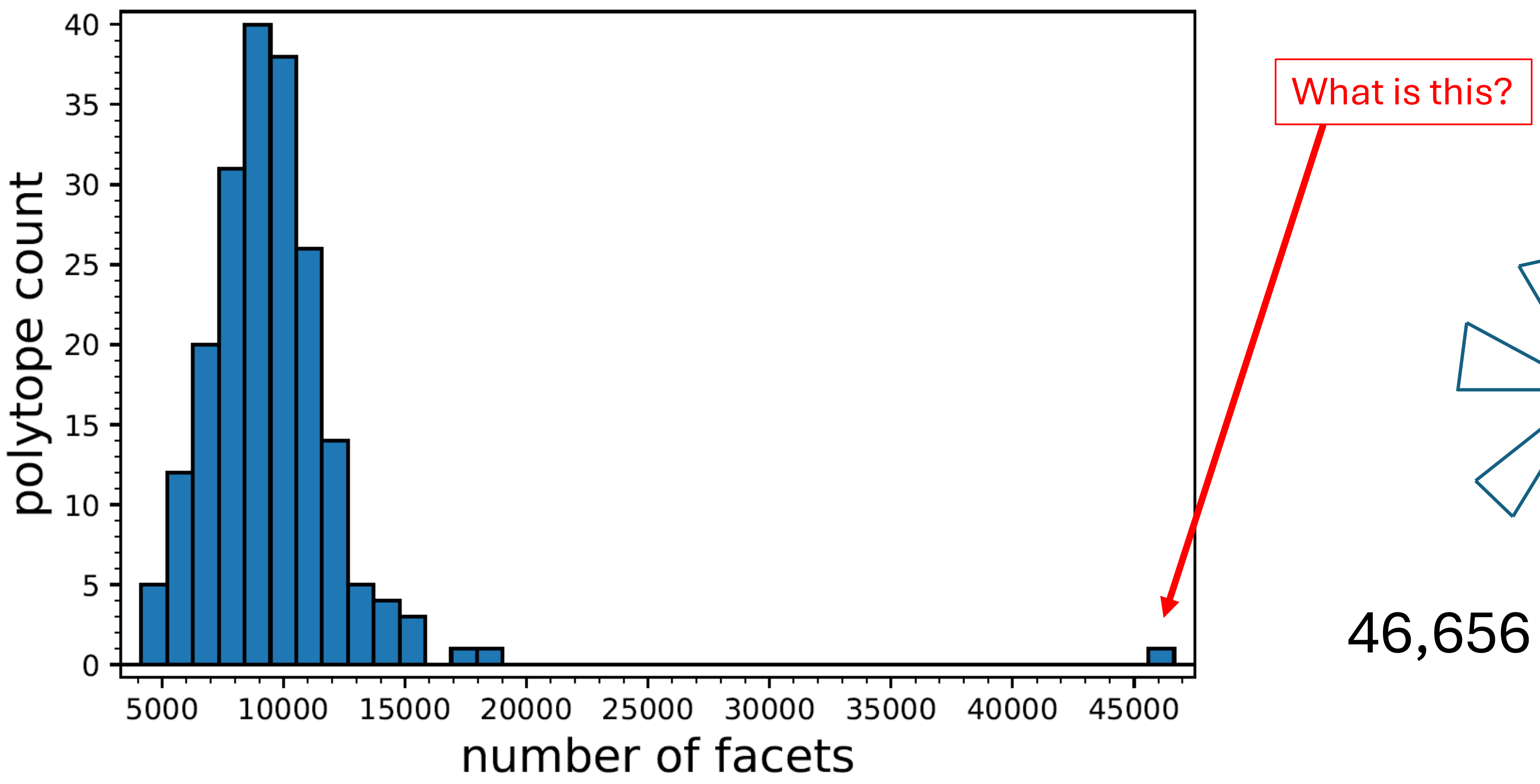
13 vertices, 18 edges, ~200 graphs sampled via single-edge-swap MCMC.

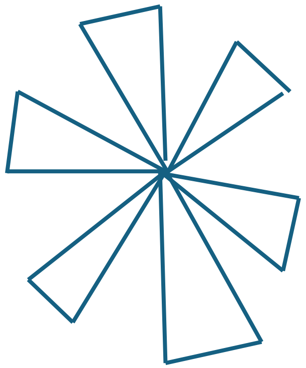
Fixed number of edges



13 vertices, 18 edges, ~200 graphs sampled via single-edge-swap MCMC.

Fixed number of edges





46,656 SEP facets

13 vertices, 18 edges, ~200 graphs sampled via single-edge-swap MCMC.

Conjecture (BB and Bruegge, 2023):

1. *For $n = 2k+1$, the maximum number of facets for P_G for a connected graph G on n facets is 6^k, which is attained by a wedge of k cycles of length three.*

2. *For $n = 2k$, the maximum number of facets for P_G for a connected graph G on n facets is $14 \cdot 6^{k-2}$, which is attained by a wedge of K_4 with $k-2$ cycles of length three.*

3. *For $n = 2k+1$, the minimum number of facets for P_G for a connected graph G on n facets is $3 \cdot 2^k - 2$, which is attained by $K_{k,k+1}$.*

4. *For $n = 2k$, the minimum number of facets for P_G for a connected graph G on n facets is $2^{k+1} - 2$, which is attained by $K_{k,k}$.*

"Take home" version: the most facets from wedges of triangles;
the least facets from complete bipartite graphs on "equal" partitions.

Clustering metrics

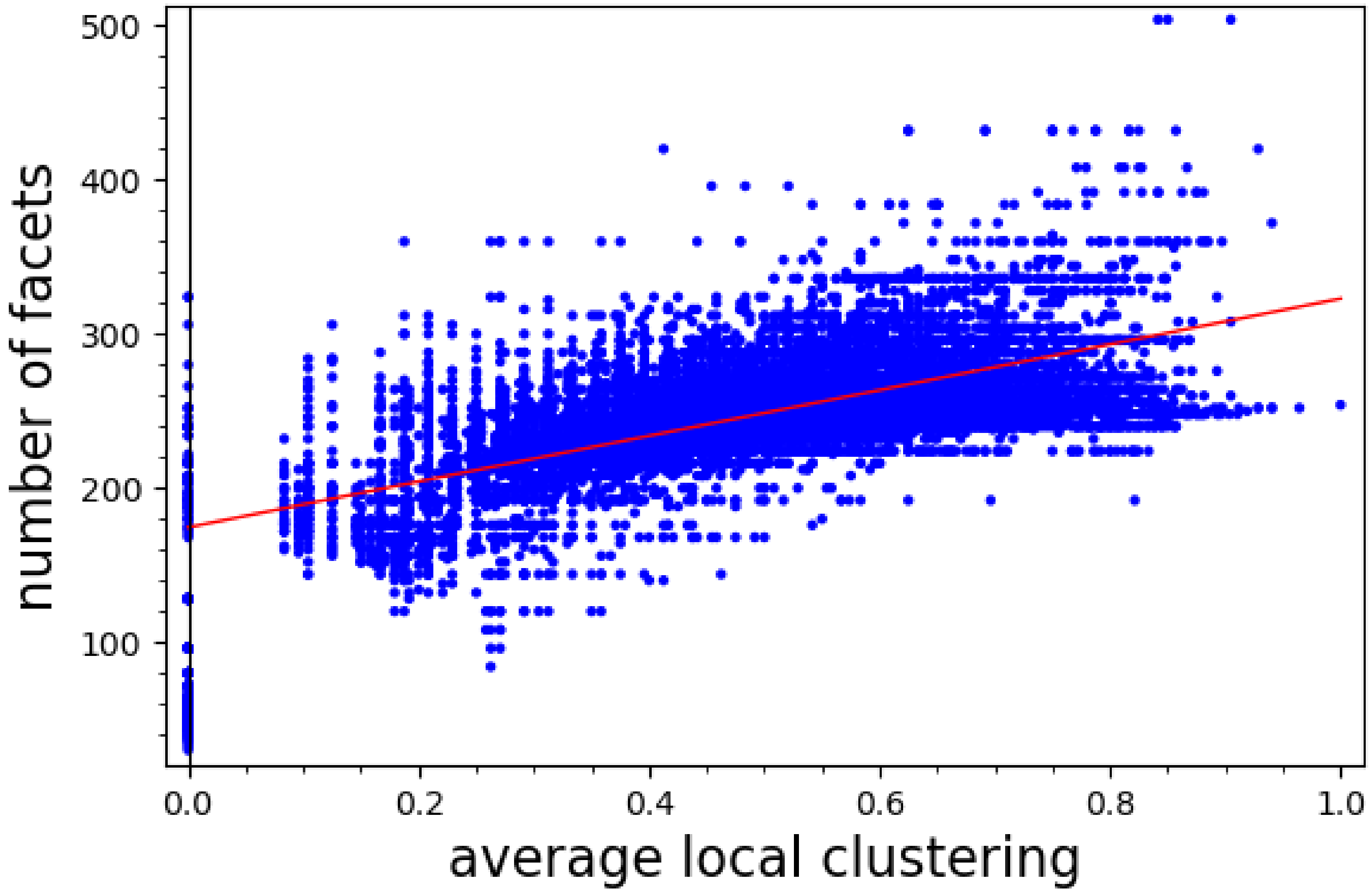
Definition: For a vertex v, let $C_{WS}(v)$ denote the number of edges connecting two neighbors of v divided by the number of possible edges between neighbors of v.

The *average local clustering* of a graph is

$$C_{WS} = \frac{1}{|V(G)|} \sum_{v \in V(G)} C_{WS}(v)$$

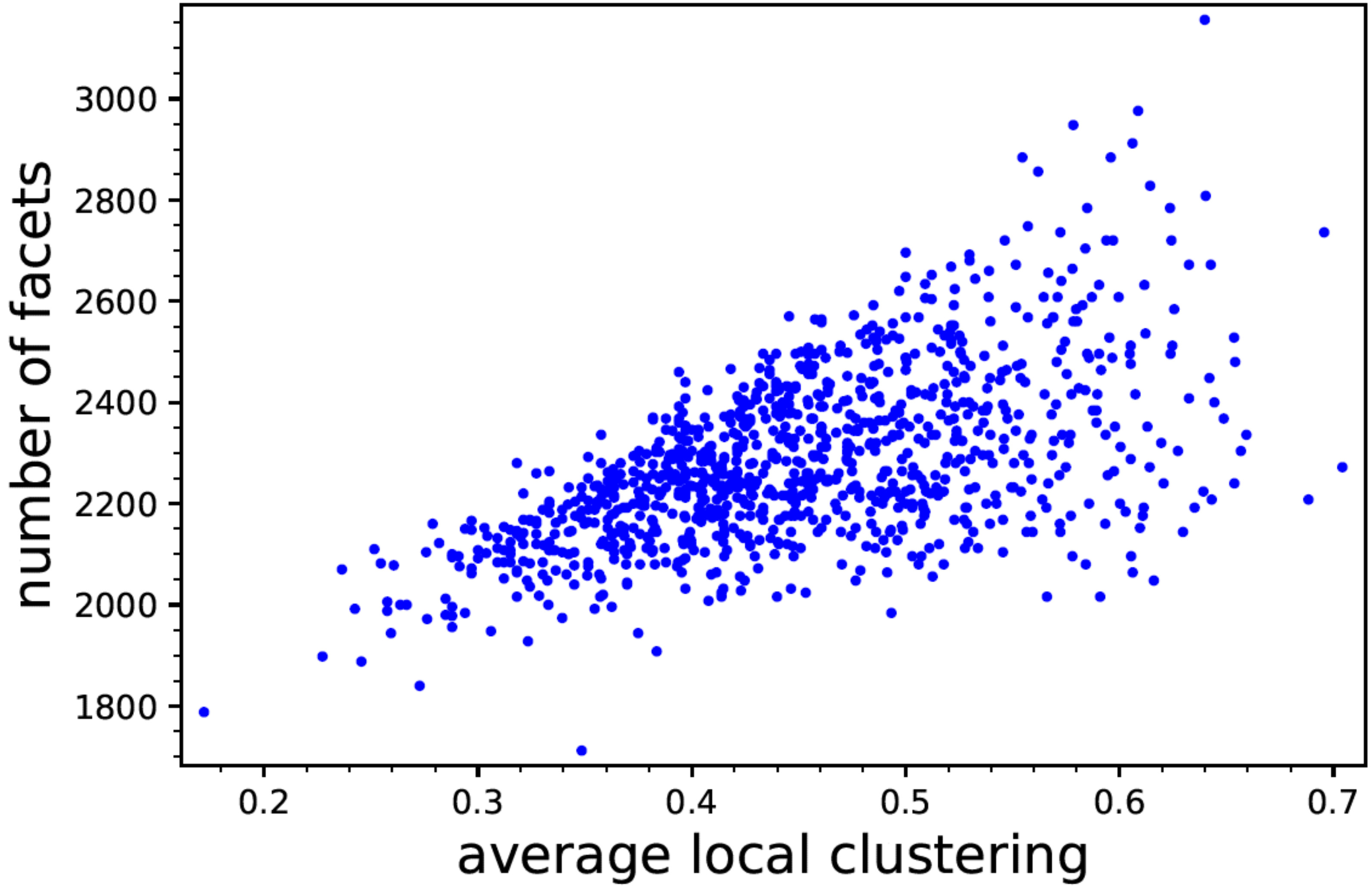
Experiments

All connected graphs on 8 vertices.

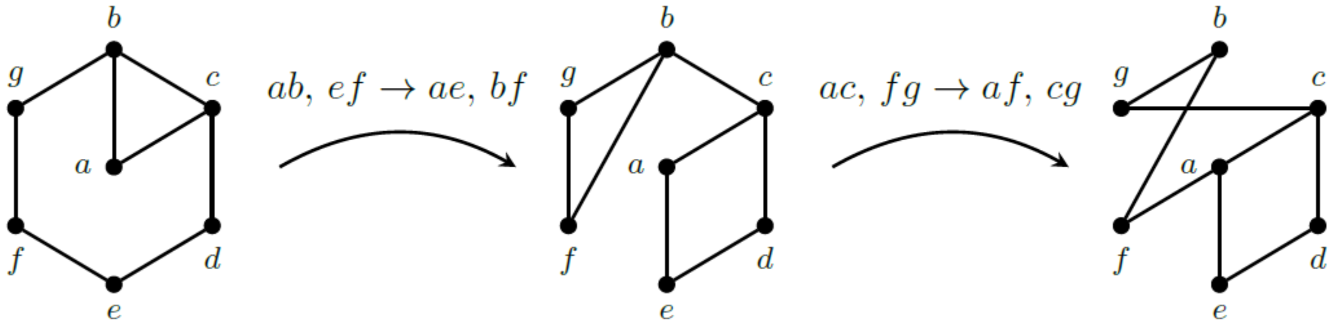


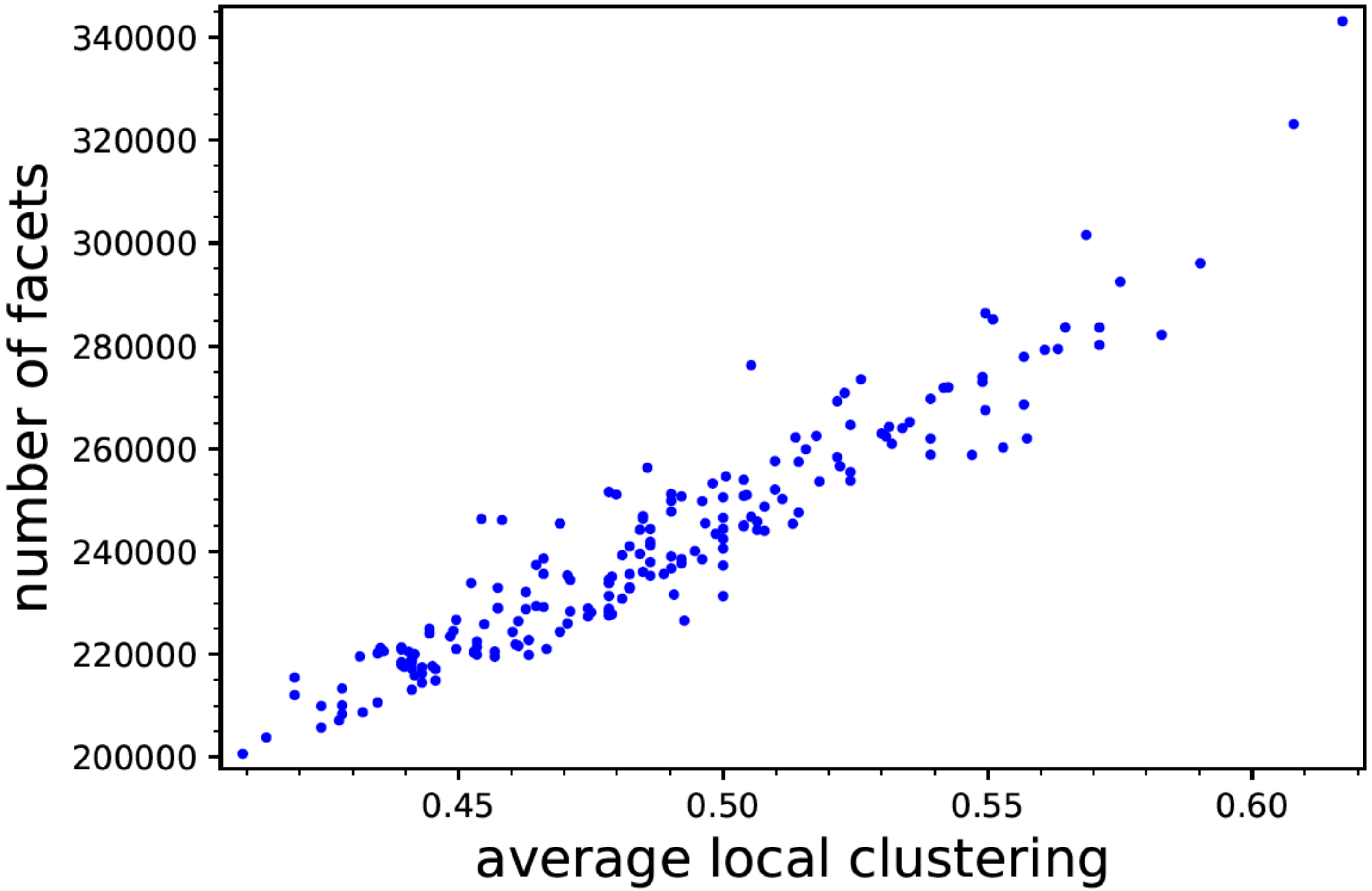
Experiments

1001 graphs with 11 vertices and 25 edges, obtained via single-edge swap MCMC.



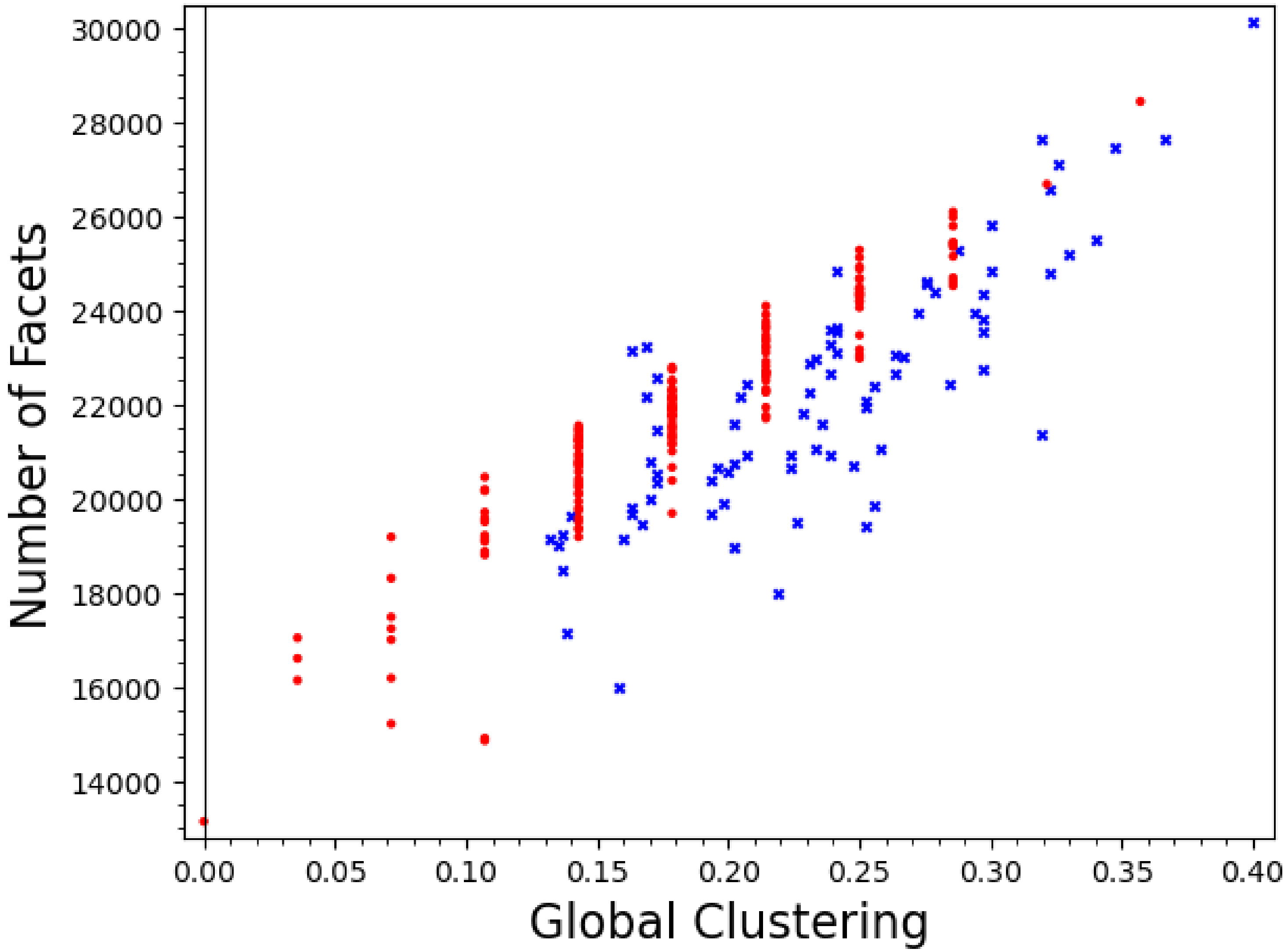
Experiments





192 connected graphs with 17 vertices and degree sequence [3,3,3,4,4,...,4,4,5,5,5,5,15], obtained via double-edge swap MCMC.

Global clustering



154 4-regular graphs on 14 vertices (shown as red dots) and
82 8-regular graphs on 14 vertices (shown as blue x's).

Facet subgraphs

(i) For every edge $e = uv$ we have $|f(u) - f(v)| \leq 1$.

(ii) The subset of edges

$$E_f = \{e = uv \in E \ : \ |f(u) - f(v)| = 1\}$$

forms a spanning connected subgraph of G.

Given a facet-defining function f, we write G_f for the corresponding *facet subgraph*, i.e., the subgraph with vertex set V and edge set E_f.

Facet subgraphs

(i) For every edge $e = uv$ we have $|f(u) - f(v)| \leq 1$.

(ii) The subset of edges

$$E_f = \{e = uv \in E \ : \ |f(u) - f(v)| = 1\}$$

forms a spanning connected subgraph of G.

Given a facet-defining function f, we write G_f for the corresponding *facet subgraph*, i.e., the subgraph with vertex set V and edge set E_f.

Theorem (Chen, Davis, Korchevskaia, 2023)
Every facet subgraph is a spanning connected induced bipartite graph.

Let $B(A, G)$ denote the induced bipartite graph for the bipartition $(A, V \setminus \mathrm{A})$ of G.

Theorem (BB, Bruegge, Kahle, 2023):

Let $G = (V, E) \sim G(n, p)$.

1. If $p < 1/2$ is fixed, then w.h.p. there exists an $\lfloor n/2 \rfloor$-subset A of V such that $B(A, G)$ is not connected.
2. If $p > 1/2$ is fixed, then w.h.p. for every subset $A \subset V$, we have $B(A, G)$ consists of a single connected component unioned with isolated vertices.
3. Further, if $p = 1/2 + \epsilon$ is fixed for some $\epsilon > 0$, then w.h.p. for every subset $A \subset V$ with

$$||A| - n/2| < \epsilon(1/2 - \epsilon)n$$

we have that $B(A, G)$ is connected and spans V.

Theorem (BB, Bruegge, Kahle, 2023):

Let $G = (V, E) \sim G(n, p)$.

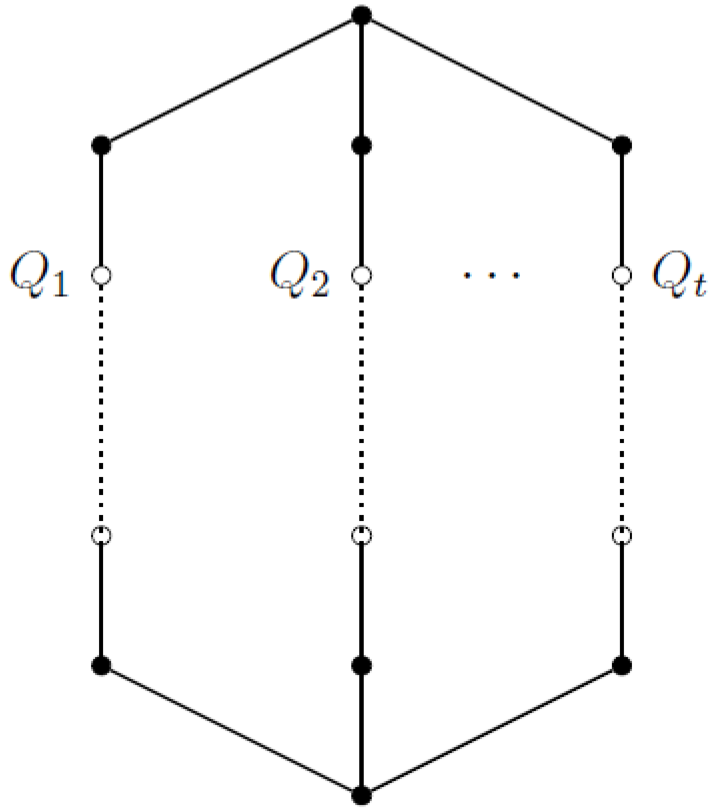
1. If $p < 1/2$ is fixed, then w.h.p. there exists an $\lfloor n/2 \rfloor$-subset A of V such that $B(A, G)$ is not connected.
2. If $p > 1/2$ is fixed, then w.h.p. for every subset $A \subset V$, we have $B(A, G)$ consists of a single connected component unioned with isolated vertices.
3. Further, if $p = 1/2 + \epsilon$ is fixed for some $\epsilon > 0$, then w.h.p. for every subset $A \subset V$ with

$$||A| - n/2| < \epsilon(1/2 - \epsilon)n$$

we have that $B(A, G)$ is connected and spans V.

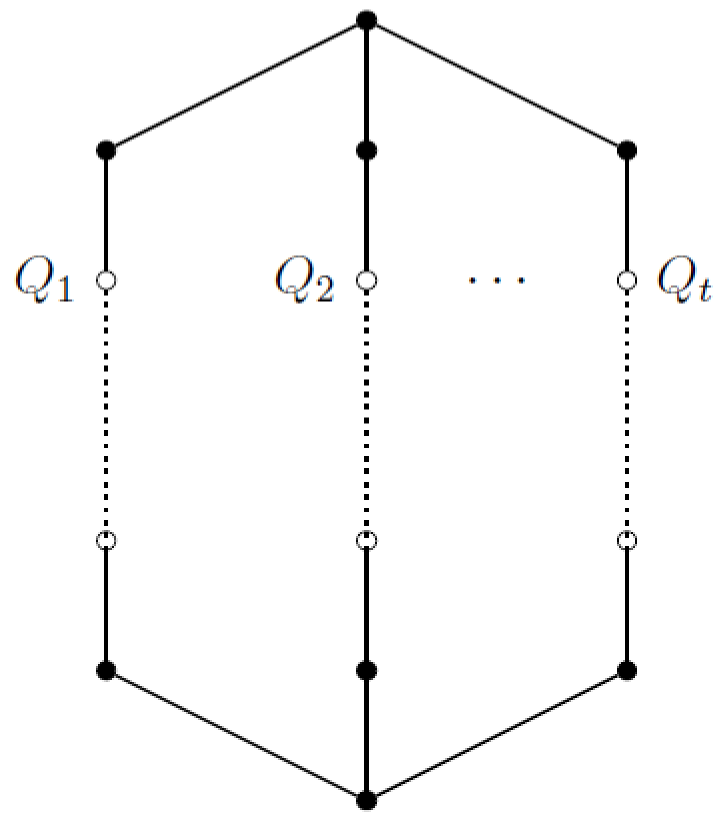
Note that every such connected spanning subgraph supports at least two facets where the values of the function are 0 and 1.

Generalized theta graphs



$$m_1 \geq m_2 \geq \cdots \geq m_t$$

Generalized theta graphs



$$m_1 \geq m_2 \geq \cdots \geq m_t$$

Theorem (BB, Bruegge, 2023): If all the path lengths have the same parity, then the number of facets of the SEP for this graph is given by the function:

$$F(\mathbf{m}) = \sum_{j=0}^{m_t} \binom{m_t}{j} \left[\prod_{k=1}^{t-1} \binom{m_k}{\frac{1}{2}(m_k - m_t) + j} \right]$$

Generalized theta graphs

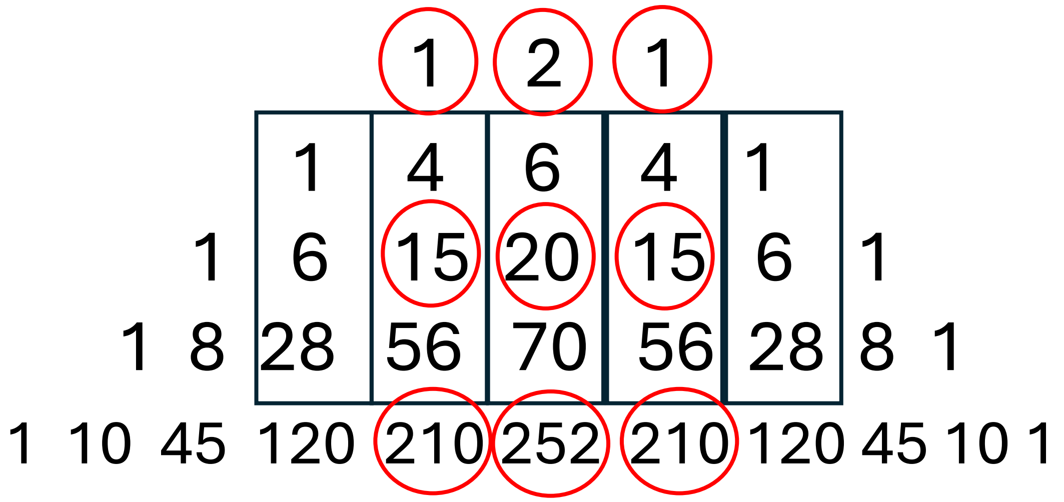
Conjecture (BB, Bruegge, 2023): if all x_i's have the same parity, then

$$F(x_1, x_2, x_3) \le F(x_1 + 2, x_2, x_3 - 2)$$

$$\sum_{j=0}^{x_3} \binom{x_3}{j} \binom{x_2}{\frac{1}{2}(x_2 - x_3) + j} \binom{x_1}{\frac{1}{2}(x_1 - x_3) + j} \le \sum_{j=0}^{x_3 - 2} \binom{x_3 - 2}{j} \binom{x_2}{\frac{1}{2}(x_2 - x_3 + 2) + j} \binom{x_1 + 2}{\frac{1}{2}(x_1 - x_3) + j}$$

and

$$F(x_1, x_2, x_3) \le F(x_1 + 2, x_2 - 2, x_3)$$



Thank you!