Yoshinaga's Criterion and the topology of complexified complements

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IPAM Workshop
Computational Interactions between
Algebra, Combinatorics, and Discrete Geometry

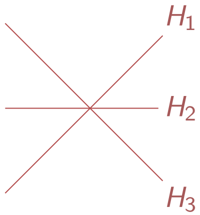
TEXAS A&M UNIVERSITY

Outline

Motivation

Arrangements of Hyperplanes in $\mathbb{R}^d$

- A **real hyperplane** is an affine linear subspace of codimension 1 in $V \cong \mathbb{R}^d$.
- A collection of finitely-many (distinct) hyperplanes is an **arrangement**.



Arrangements of Hyperplanes in $\mathbb{R}^d$

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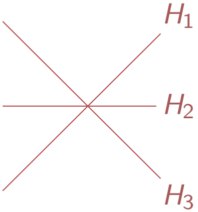


Provide interesting special cases of matroids and oriented matroids, useful tool in for linear optimization, polytopal combinatorics, reflection groups, and geometry.

Big Question: what is the topology of the "complexified complement"?

The Complexified Complement

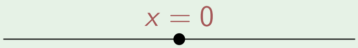
- The **complexification** of a real vector space V is $V^{\mathbb{C}} = V + iV$.
- If H is a hyerplane defined as the zero set of f, its **complexification** $H^{\mathbb{C}}$ is the zero set of

$$f^{\mathbb{C}} = f + i(f - f(0)) .$$

  If $H \subseteq \mathbb{R}^d$, then $H^{\mathbb{C}}$ is the zero set of f evaluated on $\mathbb{C}^d$.
- For a real arrangement $\mathcal{A}$, the **complexified complement** of $\mathcal{A}$ is $M(\mathcal{A}) = V^{\mathbb{C}} \setminus \bigcup_{H \in \mathcal{A}} H^{\mathbb{C}}$.

Example

Consider the following arrangement of one hyperplane in $\mathbb{R}$

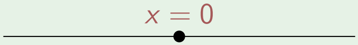


The complexified complement lives in $\mathbb{C}$ ...

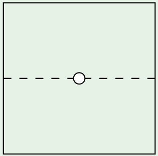
The Complexified Complement

Example

Consider the following arrangement of one hyperplane in $\mathbb{R}$



The complexified complement is $\mathbb{C} \setminus \{z = 0\}$ and looks like



The dashed line represents $\mathrm{im}(z) = 0$ the dot is the puncture where $\mathrm{im}(z) = 0$ and the real part lies on a hyperplane.

State of the Art

- Cohomology ring of $M(\mathcal{A})$: Orlik–Solomon algebra, depends only on the matroid
- Cohomology ring of a related space $M_3(\mathcal{A})$: Cordovil algebra, has simple presentation, depends on the oriented matroid (but *not* the specific orientation)

  (Cordovil Algebra = assoicated-graded of the Varchenko–Gelfand ring)
- The homotopy type is somewhat understood (there are combinatorial models like Salvetti's complex), but not completely.

Major Open Question: Is there a simple combinatorial criterion on $\mathcal{A}$ that determines when $\pi_k(M(\mathcal{A}))$ is trivial for $k \geq 2$?

Example: Graphical Arrangements

Theorem (Stanley, 1972)

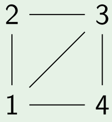
Let G be a finite graph and $\mathcal{A}_G$ its graphical arrangement. Then G is supersolvable if and only if every cycle with at least four edges is cut by a chord.

Theorem (Terao, 1983)

supersolvable $\Rightarrow$ the complexified complement is $K(\pi, 1)$

Example

A graph G and its graphical arrangement $\mathcal{A}_G$. Stanley and Terao tell us that $M(\mathcal{A}_G)$ is $K(\pi, 1)$.



$$x_1 - x_2 = 0, x_2 - x_3 = 0, x_3 - x_4$$
$$x_1 - x_4 = 0, x_1 - x_3 = 0$$

Example: Graphical Arrangements

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Theorem (Terao, 1983)

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Example

A graph G and its graphical arrangement $\mathcal{A}_G$.

Stanley tells us that $\mathcal{A}_G$ is *not* supersolvable, Terao is inconclusive

$$\begin{array}{ccc} 2 & \text{———} & 3 \\ | & & | \\ 1 & \text{———} & 4 \end{array}$$

$$x_1 - x_2 = 0, x_2 - x_3 = 0, x_3 - x_4$$
$$x_1 - x_4 = 0$$

From Falk–Randell, 1986:

	fibered	free	factor	fiber type	SS	LCS	rat'l $K(\pi, 1)$	formal	$K(\pi, 1)$	comb.
R-Refl.	??	T	T	F	F	F′)	F 1)	T	T	NA
Simp.	?	F	F	F	F	F	F	T	T	T 2)
C-Refl.	??	T	T	F	F	F′)	F	?	??	NA
fibered		F	F	F	F	F	F	T	T	?
free	?		T	F	F	F	F	??	??	??
factor	F	F		F	F	F	F	F	F	T
fiber type	T	T	T		T	T	T	T	T	T
super-solvable	T	T	T	T		T	T	T	T	T
lower central series	?	F *)	F *)	F 1)	F 1)		F 1)	??	??	T
Rat'l $K(\pi, 1)$	?	F *)	F *)	F *)	F *)	T 3)		T	??	T
Parallel	?	F	F	F	F	T *)	T *)	T *)	??	T
formal	F	F	F	F	F	F	F		F	?
simple	F	F	F	F	F	F	F	F	F	T
$K(\pi, 1)$	??	F	F	F	F	F	F	T		??

Key: rows are hypotheses
columns are conclusions
T = true
F = false
NA = not applicable
? = not known
?? = not known, of significant interest

*) Depends on the assertion that parallel implies rational $K(\pi, 1)$, proof of which has not appeared.
1) D_4 is not rational $K(\pi, 1)$ [8], but is LCS [T. Kohno, Poincaré series of the Malcev completion of generalized pure braid groups, preprint].
2) C. Toda, as communicated by H. Terao.
3) M. Falk [8] and T. Kohno [Rational $K(\pi, 1)$ arrangements satisfy the LCS formula, preprint].

They updated this in 1998, but this area is still very active (including last week!)

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	fibered	free	factor	fiber type	SS	LCS	rat'l $K(\pi, 1)$	formal	$K(\pi, 1)$	comb.
R-Refl.	??	T	T	F	F	F[1]	F[1]	T	T	NA
Simp.	?	F	F	F	F	F	F	T	T	T[2]
C-Refl.	??	T	T	F	F	F[1]	F	?	??	NA
fibered		F	F	F	F	F	F	T	T	?
free	?		T	F	F	F	F	??	??	??
factor	F	F		F	F	F	F	F	F	T
fiber type	T	T	T		T	T	T	T	T	T
super-solvable	T	T	T	T		T	T	T	T	T
lower central series	?	F[*]	F[*]	F[1]	F[1]		F[1]	??	??	T
Rat'l $K(\pi, 1)$	?	F[*]	F[*]	F[*]	F[*]	T[3]		T	??	T
Parallel	?	F	F	F	F	T[*]	T[*]	T[*]	??	T
formal	F	F	F	F	F	F	F		F	?
simple	F	F	F	F	F	F	F	F	F	T
$K(\pi, 1)$	??	F	F	F	F	F	F	T		??

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In This Talk: Give an easy-to-compute criterion to test if $M(\mathcal{A})$ *fails* to be $K(\pi, 1)$.

From Geometry to Algebra

Overview

- Typically, combinatorial properties are *stronger* than the $K(\pi,1)$ property

  ex. supersolvable $\Rightarrow$ $K(\pi,1)$
- Last year, Yoshinaga introduced a new one.
  It goes the other direction: $K(\pi,1)$ $\Rightarrow$ Yoshinaga, but not the other way around
- Unfortunately, this criterion is difficult to compute by hand.
- By translating Yoshinaga's criterion into an algebraic statement, we can use Gröbner bases to do calculations and formulate conjectues.

Prep for Yoshinaga's Criterion

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ a central, essential real arrangement with a fixed **orientation**, i.e. for each H_i we have two open halfspaces H_i^+ and H_i^-. Let $E = \{1, \dots, n\}$.

- A **signed subset** D of E is an ordered pair of disjoint subsets (D^+, D^-) of $\{1, \dots, n\}$.
- A signed subset $D = (D^+, D^-)$ of E with $D^+ \cup D^- = E$ is ***k*-consistent** if, for any subset $S \subset E$ with $\# S \leq k+1$ and any signed subset $D = (D^+, D^-)$ with $D^+ \cup D^- = S$, we have

$$\bigcap_{i \in D^+} H_i^+ \cap \bigcap_{i \in D^-} H_i^- \qquad \text{is nonempty}\,.$$

Yoshinaga's Criterion

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ a central, essential real arrangement with a fixed orientation.

- $\Sigma_k = \{S \mid S \text{ is } k\text{-consistent}\}$
- $\sigma_k = \#\Sigma_k$
- $\mathcal{A}$ is **Yoshinaga** of rank k if $\sigma_k = \sigma_r$, where r is the rank of $\mathcal{A}$.

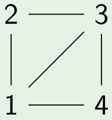
Theorem (Yoshinaga, 2024)

Let $\mathcal{A}$ be an arrangement in a real vector space V. If its complexified complement $M(\mathcal{A})$ is $K(\pi, 1)$, then $\mathcal{A}$ is Yoshinaga of rank 2.

Yoshinaga's Criterion

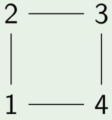
For graphical arrangements, need an orientation of the edges of the graph so that no $(k+1)$-subset forms an oriented cycle.

Example



Check: either way to have an oriented 4-cycle will force a "bad" orientation of one of the three cycles
$\Rightarrow$ Yoshinaga's criterion holds

Example



$$\sigma_2 = 2^4 = 16$$
$$\sigma_3 = 2^4 - 2 = 14$$

$\sigma_2 > \sigma_3 \;\Rightarrow\; \mathcal{A}_G$ is *not* $K(\pi, 1)$

Prep for Algebraic Yoshinaga's Criterion

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ a central, essential, oriented real arrangement.

- A **signed dependence** $D = (D^+, D^-)$ is a signed subset of $\{1, \ldots n\}$ such that

$$\bigcap_{i \in D^+} H_i^+ \cap \bigcap_{j \in D^-} H_j^- = \emptyset\,.$$

- The **Varchenko–Gelfand ideal** is an ideal of the polynomial ring $\mathbb{Q}[z_1, \ldots, z_n]$ generated by
  1. Heaviside Relations: $z_i\,(z_i - 1)$ for $i \in E$ and
  2. Dependence Relations:

$$\prod_{i \in D^+} z_i \prod_{j \in D^-} (z_j - 1) - \prod_{i \in D^+} (z_i - 1) \prod_{j \in D^-} z_j$$

  for each signed dependence (D^+, D^-).

Algebraic Yoshinaga's Criterion

Theorem (Denham-DB-Proudfoot, 2025+)

Yoshinaga's criterion holds for rank k if and only if the entire Varchenko–Gelfand ideal is generated by Heaviside relations *and* dependence relations *coming from dependencies of size $k + 1$ or less.*

Upshot: Instead exhaustive searches of signed sets, we can check containment of ideals.

For the experts: this is a statement about an *inhomogeneous* ideal. The corresponding statement fails for the Cordovil relations!

Algebraic Yoshinaga's Criterion

Example

$$\begin{tikzcd} 2 \arrow[r, dash] \arrow[d, dash] & 3 \arrow[d, dash] \\ 1 \arrow[r, dash] & 4 \end{tikzcd}$$

Check that Yoshinaga's criterion fails for rank 2:

- no dependence relations with 3 or fewer elements

$\Rightarrow$ $\sigma_2 = 2^4$

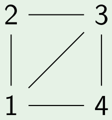
- two dependence relations with 4 or fewer elements

$\Rightarrow$ $\sigma_3 = 2^4 - 2$

$\sigma_2 \neq \sigma_3 \Rightarrow$ Yoshinaga fails at rank 2

Algebraic Yoshinaga's Criterion

Example



Check that Yoshinaga's criterion holds for rank 2:

1. Fix a reference orientation of G in order to define the Varchenko–Gelfand ideal (e.g. edges $i \to j$ if $i < j$).
2. Let g_{123} and g_{134} denote the dependence relations of 123 and 134 where we choose the orientations of 123 and 134 so that 13 has opposite orientations.
3. Easy to check that the $g_{1234} \in \langle g_{123}, g_{134} \rangle$.

Implementation (Sage vs Macaulay2)

Why does run-time matter to us?

- There are partial characterizations of the $K(\pi, 1)$ property for subarrangements of the type B reflection arrangement
  starts with Edelman–Reiner's counterexamples to Saito's conjecture, continued by Bailey, Proudfoot–Falk, and Suyama, Daisuke, Torielli, Tsujie
- This algebraic condition is the right thing to help us complete the characterization
- In order to figure out the right statement, we ran exhaustive searches (for small n) on the Whistler compute cluster at Texas A&M.
  - With the naïve implementation, this would have taken months or years.
  - With the algebraic version, it took us a few weeks.

Two Implementations

Naïve version (Sage):

naive_implementation.py 3, U

naive_implementation.py > ...

```python
import time

def is_flat_consistent(X,topes,eps):
    '''
    INPUT:
    A - a real arrangement
    X - list of the hyperplanes containing a flat of A
    e - a sign vector

    OUTPUT:
    A boolean, true if e is consistent with the flat X and false otherwi

    EXAMPLES:
    sage: A = hyperplane_arrangements.Coxeter("B2"); topes = [list(A.sig
    sage: X = {0,1,2}; eps = [+1,+1,-1,+1]; is_flat_consistent(X,topes,e
    False

    sage: X = {0,1,2}; eps = [+1,+1,+1,+1]; is_flat_consistent(X,topes,e
    True

    sage: X = {0,1}; eps = [+1,+1,-1,+1]; is_flat_consistent(X,topes,eps
    True
    '''
    new_eps = [eps[k] for k in list(X)]
    new_topes = []
    for T in topes:
        new_topes.append([T[k] for k in list(X)])
    return new_eps in new_topes
```

Using rings (Macaulay2):

.gitignore | sigma.tex | code.m2

code.m2

```
checkContainments = (A,R) -> (
    firstTwo := byDegree0 + ideal for c in circs list
    return all(circs, c -> ((# c == 3) or
      (relationFromCircuit(nrmls,c) % firstTwo) == 0))
    );

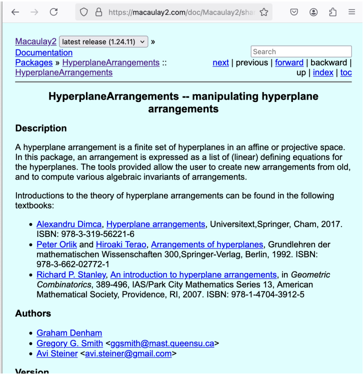
bigVGIdeal = (A,R) -> (
    byDegree0 := ideal for i from 0 to # hyperplanes A - 1 list (e_i - 1) * e_i ;
    nrmls := coefficients A;
    circs := circuits A;
    return byDegree0 + ideal for c in circs list relationFromCircuit(nrmls,c);
    );

shortVGIdeal = (A,R,k) -> (
    idempoties := ideal for i from 0 to # (hyperplanes A) - 1 list (e_i - 1) * e_i ;
    nrmls := coefficients A;
    circs := select(circuits A, c -> (#c <= k + 1));
    return idempoties + ideal for c in circs list(relationFromCircuit(nrmls,c));
    );

shortCircuits= (A,R,k) -> (
    shortCircs := select(circuits A, c -> (#c <= k + 1));
    checkDigit := true;
    while checkDigit == true do(
        checkDigit = false;
        for c in shortCircs do(
            for d in shortCircs do(
                if #(set(c) * set(d)) == 1 then(
                    newCirc := set(c) + set(d) - set(c)*set(d);
```

Two Implementations

Macaulay2 version uses the “HyperplaneArrangements” pacakge



https://macaulay2.com/doc/Macaulay2/sha

Macaulay2 latest release (1.24.11) »
Documentation
Packages » HyperplaneArrangements :: HyperplaneArrangements

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HyperplaneArrangements -- manipulating hyperplane arrangements

Description

A hyperplane arrangement is a finite set of hyperplanes in an affine or projective space. In this package, an arrangement is expressed as a list of (linear) defining equations for the hyperplanes. The tools provided allow the user to create new arrangements from old, and to compute various algebraic invariants of arrangements.

Introductions to the theory of hyperplane arrangements can be found in the following textbooks:

- Alexandru Dimca, Hyperplane arrangements, Universitext,Springer, Cham, 2017. ISBN: 978-3-319-56221-6
- Peter Orlik and Hiroaki Terao, Arrangements of hyperplanes, Grundlehren der mathematischen Wissenschaften 300,Springer-Verlag, Berlin, 1992. ISBN: 978-3-662-02772-1
- Richard P. Stanley, An introduction to hyperplane arrangements, in *Geometric Combinatorics*, 389-496, IAS/Park City Mathematics Series 13, American Mathematical Society, Providence, RI, 2007. ISBN: 978-1-4704-3912-5

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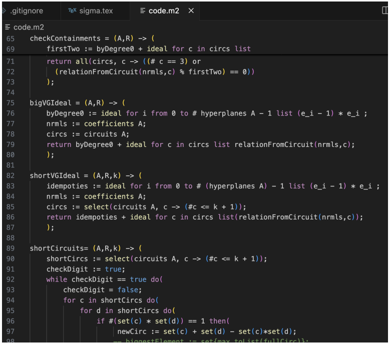
- Avi Steiner <avi.steiner@gmail.com>

Using rings (Macaulay2):



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    nrmls := coefficients A;
    circs := select(circuits A, c -> (#c <= k + 1));
    return idempoties + ideal for c in circs list(relationFromCircuit(nrmls,c));
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    while checkDigit == true do(
        checkDigit = false;
        for c in shortCircs do(
            for d in shortCircs do(
                if #(set(c) * set(d)) == 1 then(
                    newCirc := set(c) + set(d) - set(c)*set(d);
```

Comparing Run-Times of the Two Implementations

Naïve version (Sage) - several minutes

```
sage: A = hyperplane_arrangements.braid(6)
sage: %time yoshinagas_criterion(A)
CPU times: user 2min 29s, sys: 510 ms,
total: 2min 30s
Wall time: 2min 30s
True
```

Using rings (in Macaulay2) - a few seconds

```
i2 : A = typeA(5, QQ); R = setupRing(A);
time checkContainments(A,R)
 -- used 0.533831s (cpu); 0.266322s (thread);
  0s (gc)
o4 = true
```

Ring calculations are consistently fast

Using rings (in Macaulay2) - a few seconds

```
i2 : A = typeA(6, QQ); R = setupRing(A);
time checkContainments(A,R)
-- used 8.18428s (cpu); 4.16699s (thread); 0s (gc)
o4 = true
```

Using rings (in Macaulay2) - a few seconds

```
    i2 : A = typeA(7, QQ); R = setupRing(A); time
    checkContainments(A,R)
    -- used 470.341s (cpu); 210.63s (thread); 0s (gc)
   o4 = true
```

Where do we go from here?

Successes so Far

- Fast check to determine if an arrangement *could* be $K(\pi, 1)$
- Uses the Varchenko–Gelfand ring to gather information about the *homotopy groups* of $M(\mathcal{A})$

  The Varchenko–Gelfand ring and its associated graded (aka the Cordovil algebra) are interesting in their own right, and we'll see them again tomorrow morning in Ayah and Sarah's talks.
- For a natural notion of chordal oriented matroid, we can show that chordal implies Yoshinaga's criterion
- In the case of subarrangements of the Type B reflection arrangement (including graphical arrrangements, braid arrangements, threshold arrangements, etc), get more precise results because we can precisely characterize dependent sets

Loose Threads

- Are there other interesting families of arrangements for which we can say concretely whether Yoshinaga's criterion fails or not?
  This is interesting for ranks ≥ 2, and has a topological interpretation.
- Can we make the calculations more efficient by exploiting the symmetry of the arrangement?
  Interesting special case: graphical arrangements where the underlying graph is not chordal.
- Can this criterion be "upgraded" to get a combinatorial condition equvivalent to $K(\pi, 1)$-ness?

Thank you for your attention!