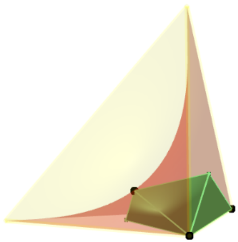

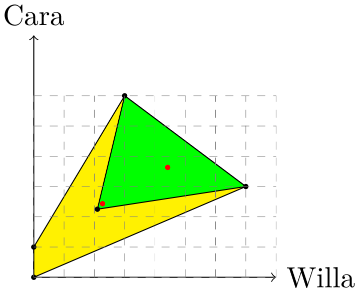

Combinatorics of equilibria in game theory

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Board games for date night*

An example of a 2×2 game. The pair has agreed that Willa will clear the table and set up a game, while Cara brings home a suitable dessert: Belgian waffle fixings for Wingspan, or delicate cookies from their favorite French bakery for Carcassonne.

		Player 2 = Cara	
		1 = Waffles	2 = Cookies
Player 1 = Willa	1 = Wingspan	$(7,3)$	$(0,1)$
	2 = Carcassonne	$(0,0)$	$(3,6)$

Willa and Cara's joint mixed strategies: $\big((p_1{}^{(1)}, p_2{}^{(1)}), (p_1{}^{(2)}, p_2{}^{(2)})\big) \in \Delta_1 \times \Delta_1$.

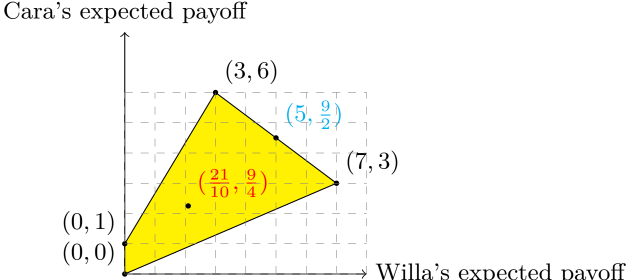
Happiness in the short term: Two *Nash equilibria*
Always waffles and Wingspan $= (1,0,1,0)$
Always cookies and Carcassonne $= (0,1,0,1)$

*Elliptic curves come to date night, U. Whitcher, Mathematical Reviews (AMS)

Board games for date night†

Uncoordinated mixed strategies: Willa reasons that 75% of her possible happiness comes from setting up Wingspan, so she should do so 75% of the time. Similarly, Cara decides to buy cookies 70% of the time.
Coordinated flip-coin: Choose Wingspan and waffles for heads but Carcassonne and cookies for tails.

$$\text{Willa's expected payoff} = \pi_W := W_{11}p_1^{(1)}p_1^{(2)} + W_{12}p_1^{(1)}p_2^{(2)} + W_{21}p_2^{(1)}p_1^{(2)} + W_{22}p_2^{(1)}p_2^{(2)}$$
$$\text{Cara's expected payoff} = \pi_C := C_{11}p_1^{(1)}p_1^{(2)} + C_{12}p_1^{(1)}p_2^{(2)} + C_{21}p_2^{(1)}p_1^{(2)} + C_{22}p_2^{(1)}p_2^{(2)}$$



†Elliptic curves come to date night, U. Whitcher, Mathematical Reviews (AMS)

Algebraic game theory

Let's set up a normal-form game for n players.

- $(d_1 \times \cdots \times d_n)$-game: an n-player game where player i has d_i pure strategies. Willa and Cara: (2×2)-game.
- The entry ${p_k}^{(i)}$ (mixed strategy) is the probability Player i chooses the pure strategy $k \in [d_i]$.
- Each Player i has a $d_1 \times \cdots \times d_n$ *payoff tensor* $X^{(i)}$. $X^{(1)} = W \in \mathbb{R}^{2\times 2}$ and $X^{(2)} = C \in \mathbb{R}^{2\times 2}$.
- The expected payoff for Player i

$$\pi_i := PX^{(i)} = \sum_{j_1=1}^{d_1} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1\cdots j_n} p^{(1)}_{j_1} \cdots p^{(n)}_{j_n}.$$

Definition

A point $P \in \Delta_{d_1-1} \times \cdots \times \Delta_{d_n-1}$ is called a *Nash equilibrium* for a n-player game X, if none of the players can increase their expected payoff by changing their strategy **while assuming the other players have fixed mixed strategies**.

Nash equilibria and real algebraic varieties

- The existence of equilibrium was first proven for any zero-sum games (Minimax theorem)*. It is considered the start point of game theory.
- By the result of Nash in 1950†, there exists a Nash equilibrium for any finite game. Proof: an application of the Kakutani fixed-point theorem.
- Study of Nash equilibria via systems of multilinear equations.‡ §
- In the general case, one solves $d_1 + \ldots + d_n$ multilinear equations:

$$p_k^{(i)} \left(\pi_i - \sum_{j_1=1}^{d_1} \cdots \sum_{j_i=1}^{\widehat{d_i}} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1 \ldots k \ldots j_n} p^{(1)}_{j_1} \cdots p^{(i-1)}_{j_{i-1}} p^{(i+1)}_{j_{i+1}} \cdots p^{(n)}_{j_n} \right) = 0$$

for all $k \in [d_i]$ and where each parenthesized expression is nonnegative.

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*Von Neumann, Zur Theorie der Gesellschaftsspiele, 1928.

†Nash. Equilibrium points in n-person games, 1950.

‡McKelvey and McLennan. Computation of equilibria in finite games, 1996.

§Sturmfels. Solving Systems of Polynomial Equations, 2002.

Nash equilibria and real algebraic varieties

Example: Willa and Cara

A point $\left((p_1^{(1)}, p_2^{(1)}), (p_1^{(2)}, p_2^{(2)})\right) \in \Delta_1 \times \Delta_1$ is a Nash equilibrium if and only if

$$
\begin{aligned}
p_1^{(1)}(7p_1^{(1)}p_1^{(2)} + 3p_2^{(1)}p_2^{(2)} - 7p_1^{(2)}) &= 0 \\
p_2^{(1)}(7p_1^{(1)}p_1^{(2)} + 3p_2^{(1)}p_2^{(2)} - 3p_2^{(2)}) &= 0 \\
p_1^{(2)}(3p_1^{(1)}p_1^{(2)} + p_1^{(1)}p_2^{(2)} + 6p_2^{(1)}p_2^{(2)} - 3p_1^{(1)}) &= 0 \\
p_2^{(2)}(3p_1^{(1)}p_1^{(2)} + p_1^{(1)}p_2^{(2)} + 6p_2^{(1)}p_2^{(2)} - p_1^{(1)} - 6p_2^{(1)}) &= 0
\end{aligned}
$$

where the parenthesized expressions are nonnegative.

- Computing Nash equilibria is PPAD-hard*, but we can still try: Check the example online of a $3 \times 3 \times 3$ game and its computation on `HomotopyContinuation.jl`†.

*Papadimitriou. On the complexity of the parity argument and other inefficient proofs of existence, 1994.

Computing Nash equilibria

- For totally mixed Nash equilibria (strictly positive probabilites), we consider the parenthesized expressions. Eliminate the variables π_i to obtain $d_1 + \cdots + d_n - n$ multilinear equations in $d_1 + \cdots + d_n$ variables.

$$\sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} \left(X^{(i)}_{j_1 \cdots k \cdots j_n} - X^{(i)}_{j_1 \cdots 1 \cdots j_n} \right) p^{(1)}_{j_1} \cdots p^{(i-1)}_{j_{i-1}} p^{(i+1)}_{j_{i+1}} \cdots p^{(n)}_{j_n} = 0 \tag{1}$$

for all $k \in 2, \cdots, d_i$ and for all $i \in [n]$.

- Set $p^{(i)}_{d_i} = 1 - \sum_{j=1}^{d_i - 1} p^{(i)}_j$ to apply Bernstein–Khovanskii–Kushnirenko (BKK) theorem for the maximal number of totally mixed Nash equilibria of a game.

- Newton polytopes of the multilinear equations are in the form of product of simplices:

$$\Delta^{(i)} := \Delta_{d_1 - 1} \times \ldots \times \Delta_{d_{i-1} - 1} \times \{0\} \times \Delta_{d_{i+1} - 1} \times \ldots \times \Delta_{d_n - 1}$$

Number of Nash equilibria

Theorem: ‡McKelvey and McLennan

The maximum number of (isolated) totally mixed Nash equilibria for any n-person game where the player i has d_i pure strategies equals the mixed volume of

$$\left(\Delta^{(1)}, \ldots, \Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(2)}, \ldots, \Delta^{(n)}, \ldots, \Delta^{(n)}\right)$$

where $\Delta^{(i)}$ appears $d_i - 1$ times. This mixed volume equals the number of partitions of

$$\{p_k^{(i)} \mid i = 1, \ldots, n,\ k = 1, \ldots, d_i - 1\} = \bigcup_{i=1}^{n} B_i$$

such that

1. $|B_i| = d_i - 1$ for each $i = 1, \ldots, n$, and
2. $p_k^{(i)} \notin B_i$ for any k.

‡McKelvey and McLennan, The maximal number of regular totally mixed Nash equilibria,1994

Number of Nash equilibria

$k_{1,2}$ many $p^{(2)}$
$\vdots$
$k_{1,n}$ many $p^{(n)}$

$|B_1| = d_1 - 1$

$\sum_{i \neq 1} k_{1,i} = d_1 - 1$

$\cdots$

$k_{n,1}$ many $p^{(1)}$
$\vdots$
$k_{n,n-1}$ many $p^{(n-1)}$

$|B_n| = d_n - 1$

$\sum_{i \neq n} k_{n,i} = d_n - 1$

Example: Linear algebra with mixed volumes

For a $d_1 \times d_2$ game, we have $\Delta^{(1)} = \{0\} \times \Delta_{d_2-1}$ and $\Delta^{(2)} = \Delta_{d_1-1} \times \{0\}$. If $d_1 = d_2$, then the mixed volume is one. Otherwise, it is zero.

Corollary: §

Let $d_1 \leqslant \cdots \leqslant d_n$. For a generic game, there exists no totally mixed Nash equilibrium if and only if $d_n - 1 > \sum_{i=1}^{n-1}(d_i - 1)$.

§Abo, 🍑, Sodomaco. A vector bundle approach to Nash equilibria, 2025+.

Correlated equilibrium

- For Nash equilibrium, there is a causal independence for the strategies of the players. Aumann* introduced a new concept of equilibria which allows dependency for the choices of strategies between players.
- Setup: The mixed strategy $P = (p_{j_1 j_2 \ldots j_n}) \in \Delta_{d_1 \cdots d_n - 1}$ is the (joint) probability Player 1 chooses the pure strategy $j_1 \in [d_1]$, Player 2 chooses the pure strategy $j_2 \in [d_2]$ etc.

  (2×2)-game: $p = (p_{11}, p_{12}, p_{21}, p_{22}) = (p_1^{(1)} p_1^{(2)}, p_1^{(1)} p_2^{(2)}, p_2^{(1)} p_1^{(2)}, p_2^{(1)} p_2^{(2)})$

Definition

A joint probability distribution $(p_{j_1 \cdots j_n}) \in \Delta_{d_1 \times d_n - 1}$ is called a *correlated equilibrium*, if no player can raise their expected payoff by breaking their part of the (agreed) joint distribution while assuming that the other players adhere to their own recommendations.

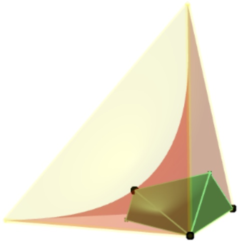
*Aumann. Subjectivity and correlation in randomized strategies, 1974

Correlated equilibrium polytope

- Aumann shows* that this definition is equivalent to the following: A point $P \in \Delta_{d_1 \cdots d_n - 1}$ is a *correlated equilibrium* for a game X if and only if

$$\sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} \left(X^{(i)}_{j_1 \cdots j_{i-1} k j_{i+1} \cdots j_n} - X^{(i)}_{j_1 \cdots j_{i-1} l j_{i+1} \cdots j_n} \right) p_{j_1 \cdots j_{i-1} k j_{i+1} \cdots j_n} \geqslant 0.$$

for all $k, l \in [d_i]$, and for all $i \in [n]$. The set of all such equilibria is the *correlated equilibrium polytope* C_X of the game X.



*Aumann. Correlated equilibrium as an expression of bayesian rationality, 1987.

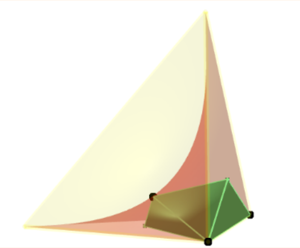
Correlated equilibrium for 2×2-games

Example

A point $(p_{11}, p_{12}, p_{21}, p_{22}) \in \Delta_3$ is a correlated equilibrium if and only if

$$\begin{aligned}
(W_{11} - W_{21})p_{11} + (W_{12} - W_{22})p_{12} &\geqslant 0 \\
(W_{21} - W_{11})p_{21} + (W_{22} - W_{12})p_{22} &\geqslant 0 \\
(C_{21} - C_{22})p_{21} + (C_{11} - C_{12})p_{11} &\geqslant 0 \\
(C_{22} - C_{21})p_{22} + (C_{12} - C_{11})p_{12} &\geqslant 0
\end{aligned}$$

$$\begin{aligned}
7p_{11} - 3p_{12} &\geqslant 0 \\
-7p_{21} + 3p_{22} &\geqslant 0 \\
-6p_{21} + 2p_{11} &\geqslant 0 \\
6p_{22} - 2p_{12} &\geqslant 0
\end{aligned}$$



The Nash equilibria $(1,0,0,0)$, $(0,0,0,1)$, $\left(\frac{9}{40}, \frac{21}{40}, \frac{3}{40}, \frac{7}{40}\right)$ are vertices of the correlated equilibrium polytope C_X. The polytope is a bipyramid over a triangle with 5 vertices and 6 facets.

Two universality results [¶][‖]

Theorem: Datta, 2003

Every real algebraic variety is isomorphic to the set of totally mixed Nash equilibria of a 3-player game, and also of an n-player game in which each player has two pure strategies.

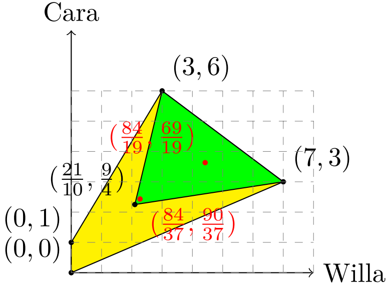
Theorem: Viossat, Solan, Lehrer, 2011

For any polytope $P \subseteq R^n$, there exists an n-player game X such that the projection of the correlated equilibrium polytope to the payoff region is equal to P.

[¶] Datta, Universality of Nash equilibria, 2003

[‖] Viossat, Solan, Lehrer, Equilibrium payoffs of finite games, 2011

Nash and correlated equilibria**



Proposition: Nau, Gomez Canovas, Hansen 2004

Let X be a non-trivial game. Then the Nash equilibria lie on a face of the correlated equilibrium polytope C_X of dimension at most $d_1 \cdots d_n - 2$.

The correlated equilibrium polytope C_X for 2×2-games is either a point or a 3-dimensional bipyramid with 5 vertices and 6 facets.*

*Calvo-Armengol. The Set of Correlated Equilibria of 2 x 2 games, 2003

**Nau, Gomez Canovas, Hansen, 2004

Combinatorics of correlated equilibrium polytope

- The oriented matroid stratification of (2×3)-games can be used to completely determine the possible combinatorial types of the polytope for payoffs Y which are generic with respect to the algebraic boundary.

Theorem: [††]

Let X be a (2×3)-game and C_X be the associated correlated equilibrium polytope. Then one of the following holds:

- C_X is a point,
- C_X is of maximal dimensional 5 and of a unique combinatorial type,
- There exists a (2×2)-game X' such that $C_{X'}$ has maximal dimensional 3 is and combinatorially equivalent to C_X.

[††] Brandenburg, Hollering, and . Combinatorics of correlated equilibria, 2022.

Combinatorics of correlated equilibrium polytope

Unique Combinatorial Types by Dimension					
Dimension	0	3	5	7	9
(2×2)	1	1	0	0	0
(2×3)	1	1	1	0	0
(2×4)	1	1	1	3	0
(2×5)	1	1	1	3	4

Table: The number of unique combinatorial types of C_X of each dimension for a $(2 \times n)$-game in a random sampling of size 100 000.

Check out the relevant code on Mathrepo!

Further outlook

Theorem: Draisma, unpublished, 2024

Let X be a *generic* $(2 \times n)$-game with generic payoff matrices and let C_X be its correlated equilibrium polytope. If C_X is not of maximal dimension, then there exists a $(2 \times k)$-game X' where $k < n$ such that $C_{X'}$ is has maximal dimension and C_X and $C_{X'}$ are combinatorially equivalent.

Draisma, Hoyer, 🍊: Generalization to $d_1 \times d_2$ games.
Deligeorgaki, Hill, Kagy, Sorea: CE polytope for zero-sum games.

- Another natural next step after handling (2×2)-games is $(2 \times 2 \times 2)$-games
- In a sample of $100,000$ random payoff matrices for $(2 \times 2 \times 2)$-games, we found $14,949$ distinct combinatorial types which are of maximal dimension.
- The number of faces can also range quite wildly

$$\begin{aligned} f_{P_{X_1}} &= (1, 8, 28, 56, 70, 56, 28, 8, 1) \\ f_{P_{X_2}} &= (1, 119, 458, 728, 616, 302, 87, 14, 1), \\ f_{P_{X_3}} &= (1, 119, 460, 733, 620, 303, 87, 14, 1). \end{aligned}$$