Symmetries of Rings from Hyperplane Arrangements

Sarah Brauner
Brown University

Computational Interactions Between Algebra, Combinatorics and Discrete Geometry

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Joint work with: Megan Chang-Lee (Brown)
Trevor Karn (Minnesota)

BIG IDEA: SEQUENCES OF INTEGERS → DIMENSIONS OF SPACES → DIMENSIONS OF REPRESENTATIONS

OUTLINE:

I. COHOMOLOGY RINGS FROM HYPERPLANE COMPLEMENTS

II. SYMMETRY

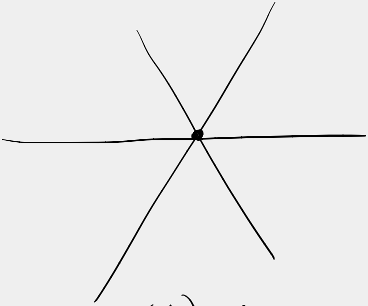
III. BRAID ARRANGEMENT

IV. THRESHOLD ARRANGEMENTS

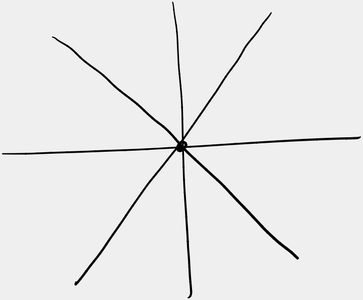
I. COHOMOLOGY RINGS FROM HYPERPLANE COMPLEMENTS

Set up: $\mathcal{A} \subset \mathbb{R}^k$ is a real, central hyperplane arrangement

e.g.



$c(\mathcal{A}) = 6$

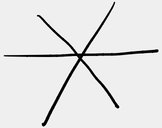


$c(\mathcal{A}) = 8$

$c(\mathcal{A})$ = # CHAMBERS ← full dimensional cones

Consider the complement of $\mathcal{A}$ in two different ways:

(i) **REAL COMPLEMENT:**



* $M_{\mathbb{R}}(\mathcal{A}) := \mathbb{R}^k \setminus \mathcal{A}$

* disjoint union of $c(\mathcal{A})$ contractible pieces

* **COHOMOLOGY RING** $H^* M_{\mathbb{R}}(\mathcal{A}) \cong \mathbb{Z}^{c(\mathcal{A})}$

has an INTERESTING PRESENTATION:

VARCHENKO-GELFAND RING $VG(\mathcal{A})$ ← more on this soon!

↳ presentation via Heaviside functions

↳ associated graded ring $grVG(\mathcal{A})$ with cohomological interpretation.

CORDIVAL ALGEBRA from Galen's talk

(2) **COMPLEX COMPLEMENT**

$$M_{\mathbb{C}}(\mathcal{A}) := \mathbb{C}^k \setminus \mathcal{A}$$

COHOMOLOGY RING

$$H^*(M_{\mathbb{C}}(\mathcal{A})) \cong OS(\mathcal{A}),$$

the **ORLIK SOLOMON ALGEBRA** of $\mathcal{A}$.

↳ **GRADED RING:** in cohomological degrees $0, 2, 4, \ldots, 2k$

↳ **TOTAL DIMENSION:** $c(\mathcal{A})$.

WHY STUDY THESE RINGS?

IMPORTANT and INTERESTING SEQUENCES OF INTEGERS

DIMENSIONS OF IMPORTANT and INTERESTING SPACES

FOR US:

CHAMBERS $c(\mathcal{A})$

DIMENSIONS of $OS(\mathcal{A})$ and $VG(\mathcal{A})$.

CHARACTERISTIC POLYNOMIAL

$$\chi(\mathcal{A},t) := \sum_{F \in \mathcal{L}(\mathcal{A})} \mu(F)\, t^{k-\mathrm{codim}(x)}$$

where $\mathcal{L}(\mathcal{A})$ is the lattice of intersection subspaces

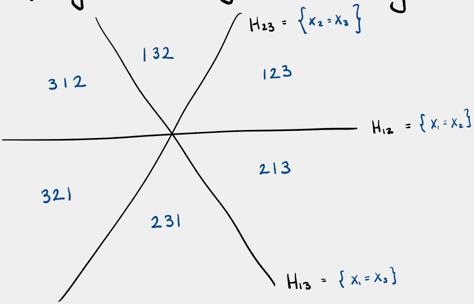
DIMENSIONS OF GRADED PIECES OF $OS(\mathcal{A})$, $gr(VG(\mathcal{A}))$

(up to signs).

EXAMPLE The BRAID ARRANGEMENT

$$\mathcal{A}_{n-1} := \left\{ H_{ij} := \{x_i - x_j = 0\} \right\}_{1 \le i < j \le n}$$

$\mathcal{A}_2 =$



* $c(\mathcal{A}) = n! =$ dimension of $VG(\mathcal{A})$ and $OS(\mathcal{A})$

$$\chi(\mathcal{A}_{n-1}, t) = (t-1)(t-2)\cdots(t-n+1)$$

$$= \sum_{j=1}^{n} s(n,j)\, t^{n-j}$$

STIRLING NUMBERS OF FIRST KIND

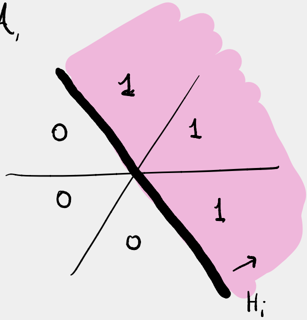
(IGNORING SIGN): $= \#\left\{ \sigma \in S_n : \sigma \text{ has } j \text{ cycles} \right\}$

* Every hyperplane $H_i \in \mathcal{A}$ divides $M_{\mathbb{R}}(\mathcal{A})$ in two pieces.
  ↳ choose a positive part: H_i^+
  and a negative part: H_i^-

* **VARCHENKO-GELFAND RING** is generated by linear functionals on $M_{\mathbb{R}}(\mathcal{A})$ called

HEAVISIDE FUNCTIONS: For $H_i \in \mathcal{A}$,

$$h_i : M_{\mathbb{R}}(\mathcal{A}) \longrightarrow \mathbb{Z}$$
$$p \longmapsto \begin{cases} 1 & p \in H_i^+ \\ 0 & p \in H_i^- \end{cases}$$

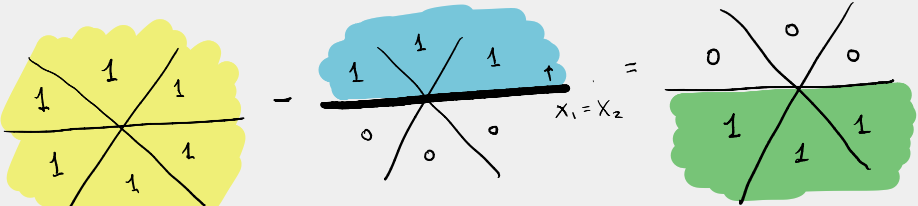


NOTE: $h_i \in H^0 M_{\mathbb{R}}(\mathcal{A})$

SURPRISING FACT: h_i form an algebra by pointwise **ADDITION** and **MULTIPLICATION**

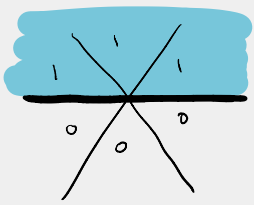
EXAMPLE: BRAID ARRANGEMENT has Heaviside functions h_{ij}

ADDITION: e.g. What is $1-h_{12}$ in $H^* M_{\mathbb{R}}(\mathcal{A})$?

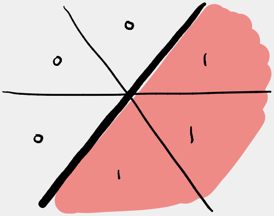


$1 \quad - \quad h_{12} \qquad\qquad 1-h_{12}$

EXAMPLE (MULTIPLICATION):

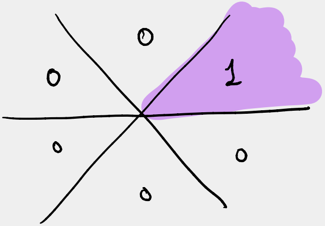


h_{12}



h_{23}

$=$



THEOREM (VARCHENKO - GELFAND)

There is a surjection

$$\mathbb{Z}[z_i]_{H_i \in \mathcal{A}} \longrightarrow H^* M_{\mathbb{R}}(\mathcal{A})$$

$$z_i \longmapsto h_i$$

with kernel $J(\mathcal{A})$ generated by "**CIRCUIT RELATIONS**"

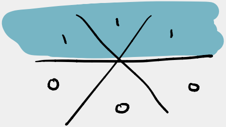
minimal linear dependency

DEFINE:

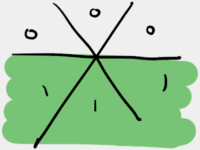
$$VG(\mathcal{A}) := \mathbb{Z}[z_i]_{H_i \in \mathcal{A}} / J(\mathcal{A})$$

EXAMPLE: BRAID ARRANGEMENT

$z_{12}(1 - z_{12}) \longmapsto$



$\cdot$



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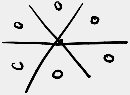
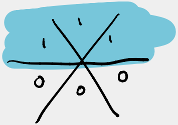
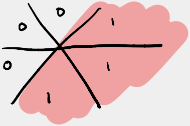


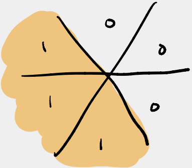
$z_{12} \quad z_{23} \quad (1 - z_{13}) \longmapsto$



h_{12}

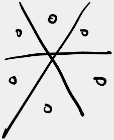


h_{23}



$1 - h_{13}$

$=$



BRAID ARRANGEMENT : $VG(\mathcal{A}_{n-1}) \cong \mathbb{Z}[z_{ij}] / J(\mathcal{A}_{n-1})$

where $J(\mathcal{A}_{n-1})$ is generated by.

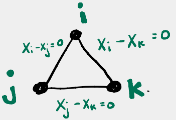
(1) $\quad z_{ij}(1-z_{ij}) = z_{ij}^2 - z_{ij}$

(2) $\quad z_{ij} z_{jk}(1-z_{ik}) + (1-z_{ij})(1-z_{jk}) z_{ik}$

$$= z_{ij} z_{jk} - z_{ij} \cancel{z_{jk}} z_{ik} + z_{ik} - z_{jk} z_{ik} - z_{ij} z_{ik} + z_{ij} \cancel{z_{jk}} z_{ik}$$

NOTE: The simplicity of this presentation is an artifact of $\mathcal{A}_{n-1}$ being very nice...

* only need circuits



$\longleftrightarrow \{H_{ij}, H_{ik}, H_{jk}\}$.

This makes it more tractable **COMPUTATIONALLY**

ASSOCIATED GRADED

* There is a filtration by degree on $VG(A)$

  F_i = monomials of degree i or less

  $$F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_k$$

* **ASSOCIATED GRADED RING**

  $$\mathrm{gr}\,VG(A) := \bigoplus F_{i+1}/F_i = \mathbb{Z}[z_i]/\mathrm{gr}(J).$$

INTUITION: relations in $\mathrm{gr}(J)$ $\longleftrightarrow$ take highest degree terms in relations in J.

INTUITION: relations in $gr(J) \longleftrightarrow$ take highest degree terms in relations in J.

EXAMPLE: BRAID ARRANGEMENT

Generating relations for $J(A_{n-1})$:

(1) $z_{ij}(1-z_{ij}) = z_{ij}^2 \cancel{- z_{ij}}$

(2) $z_{ij}z_{jk}(1-z_{ik}) + (1-z_{ij})(1-z_{jk})z_{ik}$

$= z_{ij}z_{jk} + \cancel{z_{ik}} - z_{jk}z_{ik} - z_{ij}z_{ik}$

$$gr\, VG(A_{n-1}) \cong \mathbb{Z}[z_{ij}] / \langle z_{ij}^2,\ z_{ij}z_{jk} - z_{jk}z_{ik} - z_{ij}z_{ik} \rangle$$

DIMENSION of $gr\, VG(A_{n-1})_{n-j}$ is :

$s(n,j)$ = STIRLING NUMBERS OF FIRST KIND = # $\sigma \in S_n$ with j cycles

EXAMPLE: $n=3$

	DIMENSION	PERMUTATIONS
$gr\, VG(A_2)_0$	$1 = s(3,3)$	$(1)(2)(3)$
$gr\, VG(A_2)_1$	$3 = s(3,2)$	$(12)(3)$, $(13)(2)$, $(23)(1)$
$gr\, VG(A_2)_2$	$2 = s(3,1)$	(123), (132),
$VG(A_2)$	$6 = 3!$	$(1)(2)(3)$ $(12)(3)$, $(13)(2)$, $(23)(1)$ (123), (132),

GENERAL FACTS ABOUT $gr(VG(\mathcal{A}))$:

Recall $\mathcal{A} \subset \mathbb{R}^k$

* $k+1$ graded pieces $gr(VG(\mathcal{A}))_j$ = j^{th} graded piece.

* COMMUTATIVE: $\mathbb{Z}[z_i]_{H_i \in \mathcal{A}} \,/\, gr(\mathcal{J}(\mathcal{A}))$ homogeneous ideal

* HILBERT SERIES:

  $\sum_j \dim\left(gr\,VG(\mathcal{A})_j\right) t^j$ matches $OS(\mathcal{A})$

* THM (Moseley): $gr\,VG(\mathcal{A})$ has a cohomological interpretation

$$gr\,VG(\mathcal{A}) \cong H^*\Big(\underbrace{\mathbb{R}^k \otimes \mathbb{R}^3}_{\mathbb{R}^{3k}} \setminus \bigcup_{H_i \in \mathcal{A}} H_i \otimes \mathbb{R}^3\Big).$$

OS($\mathcal{A}$) by EXAMPLE: BRAID ARRANGEMENT

$$OS(\mathcal{A}_{n-1}) \cong \Lambda\{y_{ij}\}_{1\le i<j\le n} \Big/ \langle y_{jk}y_{ik} - y_{ij}y_{jk} + y_{ij}y_{ik}\rangle$$

UPSHOT:

* gr $VG(\mathcal{A})$ can be thought of as a COMMUTATIVE analog of $OS(\mathcal{A})$
  - ↳ Presentations are very similar:

    generators indexed by hyperplanes modulo **CIRCUIT RELATIONS**

    minimal linear dependencies

- ↳ dimensions of graded pieces give coefficients of characteristic polynomial $\chi(\mathcal{A},t)$
- ↳ Both have Grobner basis given by NBC sets

  non-broken circuits

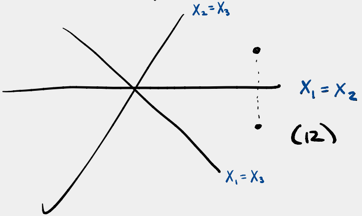
II SYMMETRY

Many of the most interesting arrangements have

SYMMETRY $\leftarrow$ (FINITE) group of automorphisms G acting on $\mathcal{A}$

This induces an action on $\mathcal{M}_{\mathbb{R}}(\mathcal{A})$, $\mathcal{M}_{\mathbb{C}}(\mathcal{A})$ ...

EXAMPLE: Symmetric group S_n acts on $\mathcal{A}_{n-1}$:



(12) acts by reflecting over $x_1 = x_2$

This induces a **G-REPRESENTATION** on

$$VG(\mathcal{A}), \quad gr\,VG(\mathcal{A}), \quad OS(\mathcal{A}).$$

G-REPRESENTATION: G action on a vector space..

e.g. S_3 acts on $VG(\mathcal{A}_2) = \mathbb{Z}[z_{ij}]_{ij} / J(\mathcal{A}_2)$

$$(12) \cdot z_{13} = z_{23}$$

$$(12) \cdot z_{13} z_{23} - z_{12} = z_{23} z_{13} - (-z_{12}) = z_{23} z_{13} + z_{12}.$$

WHY STUDY SYMMETRIES OF THESE RINGS?

IMPORTANT and
INTERESTING
SEQUENCES
OF INTEGERS

DIMENSIONS
OF IMPORTANT and
INTERESTING
SPACES

DIMENSIONS
OF IMPORTANT and
INTERESTING
REPRESENTATIONS

Q: How do we describe this representation?

DREAM SCENARIO:

Describe a **G-REPRESENTATION** in terms of its

IRREDUCIBLES = smallest building blocks of a finite group representation

COMPLICATIONS:

* this is often infeasible, even for **BRAID ARRANGEMENT!**
  ↳ hard computationally AND theoretically...
* next best thing: Formula for representations....

COMPLICATIONS, continued...

* G-REPRESENTATIONS of

VG(A)
gr VG(A)

are NOT THE SAME as OS(A)

same total rep...

can only relate them in a technical way (i.e. as an induced representation)

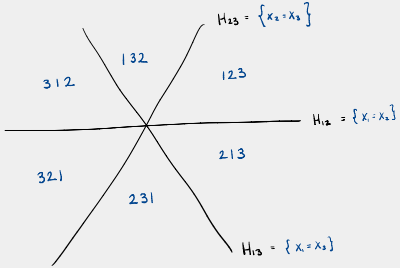
TODAY: VG(A) and gr VG(A)

NOTE: gr VG(A) decomposes VG(A)

III Braid Arrangement

Recall: $\mathcal{A}_{n-1} = \{\{x_i = x_j\}\}$ for $1 \le i < j \le n$

$\mathcal{A}_2$:



Symmetry: The symmetric group acts on $\mathcal{A}_{n-1}$

Representation Theory of Symmetric Group:

$\hookrightarrow$ parametrized by integer partitions: $\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \longleftrightarrow (2,2,1)$

We can encode representations of S_n as **SYMMETRIC FUNCTIONS:**

↳ this is called **FROBENIUS CHARACTERISTIC**

KEY PLAYERS:

SCHUR FUNCTION

$$s_\lambda = \sum_T x^T \longleftrightarrow \text{irreducible representations}$$

HOMOGENEOUS SYMMETRIC FUNCTION

$$h_n = \sum_{i_1 \le i_2 \le \cdots \le i_n} x_{i_1} \cdots x_{i_n} \longleftrightarrow \text{permutation representations}$$

UPSHOT:

* Dream scenario ⟷ Schur expansion
* Next best thing ⟷ symmetric function formula

What is **FROBENIUS CHARACTERISTIC** of $VG(A_{n-1})$?

ANSWER: $$(h_1)^n = \sum_{\lambda \vdash n} f^\lambda \cdot S_\lambda$$

f^λ = # Standard Young Tableaux of shape λ

↳ Also known as **REGULAR REPRESENTATION** of S_n

* Basis indexed by permutations
* very natural equivariant lift of $n!$

EXAMPLE: $$h_1^3 = 1 \cdot S_{(3)} + 2\, S_{(2,1)} + 1 \cdot S_{(1,1,1)}$$

$f^{(3)} = 1$: $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$

$f^{(2,1)} = 2$: $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$

$f^{(1,1,1)} = 1$: $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$

UPSHOT:

$$n! \longrightarrow VG(A_{n-1}) \longrightarrow \mathrm{ch}\, VG(A_{n-1}) = h_1^n$$

WHAT ABOUT GRADED PIECES? → Much harder!

Thm (Sundaram - Welker)

$\mathrm{gr}\, VG(A_{n-1})_j$ has **FROBENIUS CHARACTERISTIC**:

$$\sum_{\ell(\lambda) = n-j} \mathrm{Lie}_\lambda$$

THRALL'S HIGHER LIE CHARACTERS

number of parts of λ.

EXAMPLE: $n = 5$, $\mathrm{gr}\, VG(A_4)_2$:

$$\mathrm{Lie}_{(3,1,1)} + \mathrm{Lie}_{(2,2,1)}$$

$3 = 5-2$ parts

EXAMPLE: $n=3$

	DIMENSION	FROBENIUS CHARACTERISTIC
$grVG(A_2)_0$	$1 = S(3,3)$	$\mathrm{Lie}_{\ydiagram{1,1,1}} = S_{\ydiagram{3}}$ 3-0 parts
$grVG(A_2)_1$	$3 = S(3,2)$	$\mathrm{Lie}_{\ydiagram{2,1}} = S_{\ydiagram{2,1}} + S_{\ydiagram{1,1,1}}$ 3-1 parts
$grVG(A_2)_2$	$2 = S(3,1)$	$\mathrm{Lie}_{\ydiagram{3}} = S_{\ydiagram{2,1}}$ 3-2 parts
$VG(A_2)$	$6 = 3!$	$1 \cdot S_{\ydiagram{3}} + 2\, S_{\ydiagram{2,1}} + 1 \cdot S_{\ydiagram{1,1,1}}$ $f^{\ydiagram{3}} = 1 \quad f^{\ydiagram{2,1}} = 2 \quad f^{\ydiagram{1,1,1}} = 1$

THRALL'S HIGHER LIE CHARACTERS

* Indexed by partitions λ of n

* Decompose REGULAR REPRESENTATION h_1^n

* Can be expressed as

$$h_{m_1}[L_1]\, h_{m_2}[L_2] \cdots h_{m_n}[L_n]$$

Lie representation

$$\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$$

↳ **PLETHYSM**: Major open problem to expand plethysm into Schur functions...

(but COMPUTATIONALLY FEASIBLE in Sage)...

V. THE THRESHOLD ARRANGEMENT

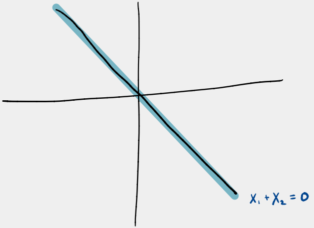
There are many other interesting arrangements with symmetry

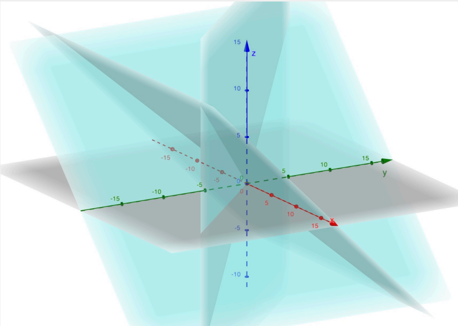
TODAY: THRESHOLD ARRANGEMENT.

$$T_n = \{x_i + x_j = 0\}_{1 \le i < j \le n}$$

$n=3$:

$n=2$:





WHY STUDY T_n?

(1) Appears in connection to other interesting math, :

+ neural networks
  (e.g "On encoding and enumerating threshold functions")
  Zunic in IEEE Trans. Neural Netw., 2004.

* algebraic statistics
  ↳ Geometry of Restricted Boltzmann Machine" Cueto-Morton-Sturmels
  in Algebraic Methods in Statistics and Probability, 2010

(2) $c(T_n)$ = # **LABELLED THRESHOLD GRAPHS** on n vertices

THRESHOLD GRAPHS: Well studied in enumerative combinatorics

See: "Threshold graphs and related topics" by Mahadev and Peled

↳ applications to linear programming and beyond!

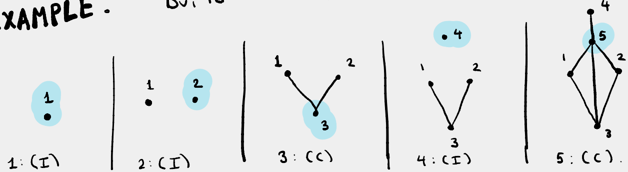
IDEA:

Build a graph with n vertices with two operations

(1) **ISOLATED VERTEX** (I)

(2) **CONNECTED VERTEX** (C)

EXAMPLE: Build $\overset{1}{(I)}\ \overset{2}{(I)}\ \overset{3}{(C)}\ \overset{4}{(I)}\ \overset{5}{(C)}$



BIJECTION: chambers of T_n $\longleftrightarrow$ labelled threshold graphs on n vertices

COUNTING # of THRESHOLD GRAPHS :

(1) EXPONENTIAL GENERATING FUNCTION

t_n = # of labeled threshold graphs with n vertices

THM (Beissinger - Peled)

$$\sum_{n \geq 0} t_n \frac{x^n}{n!} = \frac{e^x (1-x)}{2-e^x}$$

(2) RECURSIONS : for $n \geq 3$

* $$t_n = 2 + \sum_{k=2}^{n-1} \binom{n}{k} \cdot t_k$$

* $$t_n = n \cdot t_{n-1} + \sum_{k=2}^{n-1} \binom{n-1}{k} \cdot t_k \cdot t_{n-1-k}$$

WITH MEGAN CHANG-LEE AND TREVOR KARN:

Can we make these statements **EQUIVARIANT?**

$$t_n \longrightarrow \dim VG(T_n) \longrightarrow \boxed{?}$$

* The symmetric group acts on T_n.

EXAMPLE:

$$(23) \cdot \{x_1 + x_3\} = \{x_1 + x_2\}.$$

* This induces a **S_n - REPRESENTATION** on **$VG(T_n)$**

↳ What does this representation look like?

RECALL:

THEOREM (Beissinger-Peled)

$$\sum_{n \geq 0} t_n \frac{x^n}{n!} = \frac{e^x (1-x)}{2-e^x}$$

Let $\mathcal{T}_n$ be the Frobenius characteristic of $VG(T_n)$

THEOREM: GENERATING FUNCTION (B, Chang-Lee, Karn 25+)

Let $\mathcal{E}(x) := \sum_{i \geq 0} h_i x^i$. Then the **THRESHOLD REPRESENTATION**

is $\sum_{n \geq 0} ch(VG(T_n)) x^n = \sum_{n \geq 0} \mathcal{T}_n x^n = \dfrac{\mathcal{E}(x) \cdot (1 - h_1 x)}{2 - \mathcal{E}(x)}$

Note: does not seem like an equivalent statement holds for $OS(T_n)$...

RECALL RECURSION 1: for $n \geq 3$

$$* \; t_n = 2 + \sum_{k=2}^{n-1} \binom{n}{k} \cdot t_k$$

THEOREM: **RECURSION 1** (B, Chang-Lee, Karn 25+)

For $n \geq 3$,
$$\sigma J_n = 2h_n + \sum_{k=2}^{n-1} h_{n-k} \cdot \sigma J_k$$

EXAMPLE:

$$\sigma J_2 = 2h_2$$

Thus
$$\sigma J_3 = 2h_3 + h_1 \cdot (2h_2)$$
$$= 2h_3 + 2h_{21}.$$

UPSHOT: Rules for moving from σJ_n to σJ_{n+1} ....

RECALL RECURSION 2: for $n \geq 3$

$$* \; t_n = n \cdot t_{n-1} + \sum_{k=2}^{n-1} \binom{n-1}{k} \cdot t_k \cdot t_{n-1-k}$$

We show that this can be upgraded to

"**BRANCHING RULES**":

Rules for moving from $\mathcal{T}_{n+1} \longrightarrow \mathcal{T}_n \ldots$

THEOREM: RECURSION 2 (B, Chang-Lee, Karn 25+)

For $\mathcal{T}_n$, $n \geq 3$

$$\frac{\partial}{\partial p_1}\left(\mathcal{T}_n\right) = \frac{\partial}{\partial p_1}\left(p_1 \mathcal{T}_{n-1}\right) + \sum_{k=2}^{n-1} \mathcal{T}_k \, \mathcal{T}_{n-1-k}$$

"restriction" of symmetric functions

WHAT ABOUT GRADED PIECES?

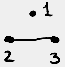
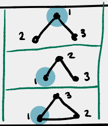
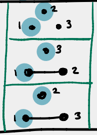
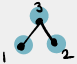
HARD!

* **OPEN PROBLEM**, Stanley in EC2 (Exercise 5.2):

Combinatorially interpret coefficients of **CHARACTERISTIC POLYNOMIAL** of T_n via **THRESHOLD GRAPHS**

* **ANSWERED** very recently by Deshpande - Menon - Singh, 21
  ↳ answer in terms of "odd anchors" of threshold graphs

EXAMPLE: $n=3$

	DIMENSION	FROBENIUS CHARACTERISTIC
$grVG(T_3)_0$	1 	S_3
$grVG(T_3)_1$	3 	$S_{21} + S_3$
$grVG(T_3)_2$	3 	$S_{21} + S_3$
$grVG(T_3)_3$	1 	S_3
$VG(T_3)$	$8 = t_3$	$4S_3 + 2S_{21} = 2h_{21} + 2h_3$

EVEN HARDER PROBLEMS:

(1) Give a BASIS for $\operatorname{gr} VG(T_n)$ in terms of threshold graphs

(2) Decompose T_n by graded pieces of $\operatorname{gr} VG(T_n)$

A FIRST STEP: Simplified presentation for $\operatorname{gr} VG(T_n)$

Recall: $VG(A) \cong \mathbb{Z}[z_H] / J(A)$ ⟵ ALL CIRCUIT RELATIONS

$\operatorname{gr} VG(A) \cong \mathbb{Z}[z_H] \; \operatorname{gr} J(A)$

We show only two circuits are necessary...

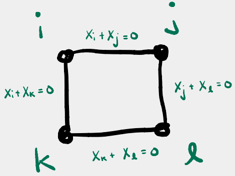
THEOREM (B, Chang-Lee, Karn 25+)

As a ring, $\operatorname{gr} VG(T_n) \cong \mathbb{Z}[z_{ij}]/I$

where I is generated by **CIRCUIT RELATIONS** from circuits

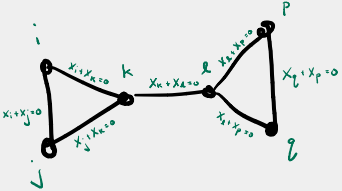
(1)

$(n \geq 4)$



(2)

$(n \geq 6)$



THE HOPE:

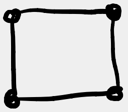
* Makes computations more feasible!
* Combine with representation stability results to give **PRECISE DESCRIPTIONS** of reps.

We think only two families of circuits are needed to obtain a Gröbner basis...

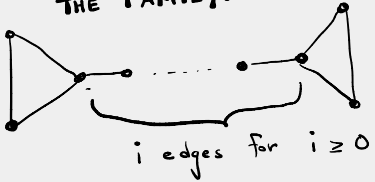
CONJECTURE (B, Chang-Lee, Karn 25+)

With $\mathrm{gr}\, VG(T_n) \cong \mathbb{Z}[z_{ij}] / \mathrm{gr}\, J(T_n)$, circuit relations from the following circuits form a **GRÖBNER BASIS** for $\mathrm{gr}\, J(T_n)$:

THE FAMILY:

(1) 

$(n \geq 4)$

(2) 

$(n \geq 5)$

SUMMARY:	BRAID ARRANGEMENT	THRESHOLD ARRANGEMENT
CHAMBERS:	$n! \longleftrightarrow$ biject with permutations	$t_n \longleftrightarrow$ biject with threshold graphs
FROBENIUS CHARACTERISTIC OF $VG(\mathcal{A})$	$h_1^n \longleftrightarrow$ REGULAR REPRESENTATION	$\sum_{n\geq 0} T_n x^n = \dfrac{\mathcal{E}(x)\cdot(1-h_1 x)}{2-\mathcal{E}(x)}$
CHARACTERISTIC POLYNOMIAL COEFFICIENTS	Stirling numbers of the first kind i.e. permutations with j cycles	Labelled threshold graphs with j odd anchors
FROBENIUS CHARACTERISTIC of $\mathrm{gr}\, VG(\mathcal{A})$	THRALL'S HIGHER LIE CHARACTERS	?!

THANK YOU!

CONTACT ME!

EMAIL: sarahbrauner@gmail.com

WEBSITE: sarahbrauner.com

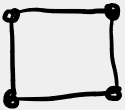
CONJECTURE (B, Chang-Lee, Karn 25$^+$)

Let $\operatorname{gr} J(T_n)$ be the defining ideal for $\operatorname{gr} VG(T_n)$.

Then (with appropriate term order), the following circuit relations form a **GRÖBNER BASIS** for $J(T_n)$:

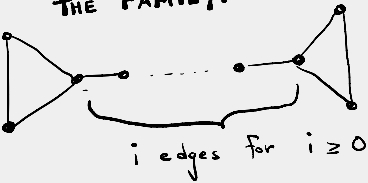
THE FAMILY:

(1)



$(n \geq 4)$

(2)



$(n \geq 5)$

THE HOPE:

* This would make computations more feasible
* Combine with representation stability results to give **PRECISE DESCRIPTIONS** of reps.

A REMARK ABOUT REFLECTION ARRANGEMENTS

* Let $\mathcal{A}_W$ be a reflection arrangement corr. to W.

↳ My past work dealt with describing the graded pieces of $\mathrm{gr}\, VG(\mathcal{A}_W)$

AGAIN: no known irreducible decomposition

INSTEAD: I describe the graded pieces in terms of other representations occurring in the literature, called

EULERIAN REPRESENTATIONS

↳ come from certain IDEMPOTENTS in DESCENT ALGEBRA

REPRESENTATION THEORY OF SYMMETRIC GROUP:

* IRREDUCIBLES $\longleftrightarrow$ INTEGER PARTITIONS

e.g. $\chi^{\ydiagram{2,2,1}}$ $\qquad$ $\ydiagram{2,2,1} \longleftrightarrow (2,2,1)$

We can encode representations of S_n as **SYMMETRIC FUNCTIONS** via **FROBENIUS CHARACTERISTIC MAP:**

ch: IRREDUCIBLE REP χ^{λ} $\longmapsto$ SCHUR FUNCTION $s_\lambda = \sum_T x^T$

SPECIAL CASE: ch: TRIVIAL REP $\chi^{\square\cdots\square}$ $\longmapsto$ HOMOGENEOUS SYMMETRIC FUNCTION $h_n = \sum_{i_1 \le i_2 \le \cdots \le i_n} x_{i_1} \cdots x_{i_n}$

UPSHOT: (1) This is an important **COMPUTATIONAL TOOL**

(2) New **DREAM SCENARIO:**

decompose symmetric functions into **SCHUR FUNCTIONS**

COMPLICATIONS:

this is often infeasible, even for **BRAID ARRANGEMENT!**

↳ hard computationally AND theoretically...

NOTE: Long term stability properties are better understood!

↳ e.g. Gadish, Proudfoot - Ramos