Geometric families of degenerations from mutations of polytopes

Laura Escobar

University of California, Santa Cruz

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Toric degenerations

Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope with vertices in $\mathbb{Z}^d$ and let $P \cap \mathbb{Z}^d = \{\alpha_0, \dots, \alpha_n\}$.

The **toric variety** X_P is the closure of the image of the map $(\mathbb{C}^*)^d \to \mathbb{P}^n$ given by $x \mapsto (x^{\alpha_0}, \dots, x^{\alpha_n})$.

The dimension of X_P is d and the degree of X_P is $d! \operatorname{Vol}_d(P)$.

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Definition. A **toric degeneration** of a variety $X \subseteq \mathbb{P}^n$ is a flat family $\pi : \mathfrak{X} \to \mathbb{C}$, where the general fiber is X and the special fiber is a toric variety X_P.

In this setting, the degree of X equals $d! \operatorname{Vol}_d(P)$.

Newton–Okounkov bodies

Let A be a finitely generated $\mathbb{C}$-algebra that is positively graded. Equip $\mathbb{Z}^n$ with a total order $\succ$. A function $\nu : A \setminus \{0\} \to \mathbb{Z}^n$ is a **valuation** if

- $\nu(f+g) \succeq \min\{\nu(f), \nu(g)\}$,
- $\nu(fg) = \nu(f) + \nu(g)$, and
- $\nu(c) = 0$ for all $c \in \mathbb{C}^*$.

Definition. (Okounkov, Lazarsfeld–Mustață, Kaveh–Khovanskii) Let X be a projective variety of dimension d and $\nu : \mathbb{C}[X] \setminus \{0\} \to \mathbb{Z}^{d+1}$ a valuation. The **Newton–Okounkov body** for (X, ν) is $\Delta(X, \nu) := \overline{\text{cone}(\text{im}(\nu))} \cap (\{1\} \times \mathbb{R}^d)$.

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When the tropicalization of X is *well-behaved* [Kaveh–Manon, 2016] construct valuations ν such that $\Delta(X, \nu)$ is a lattice polytope.

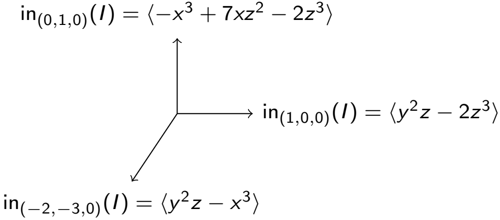
Tropical geometry and Newton–Okounkov bodies

Choose a presentation $\mathbb{C}[x_1, \ldots, x_n]/I$ for $\mathbb{C}[X]$, I homogeneous.

$\mathrm{Trop}(I) := \{w \in \mathbb{R}^n \mid \mathrm{in}_w(I) \text{ contains no monomials}\}$.

$\mathrm{Trop}(I)$ is a fan in $\mathbb{R}^n$ with cones $C_w = \overline{\{x \in \mathbb{R}^n \mid \mathrm{in}_x(I) = \mathrm{in}_w(I)\}}$ for $w \in \mathbb{R}^n$.

The tropicalization of $I = \langle y^2z - x^3 + 7xz^2 - 2z^3 \rangle$ is the product of $\mathbb{R}(1,1,1)$ with the union of the rays



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A cone C of $\mathrm{Trop}(I)$ is **prime** if $\mathrm{in}_w(I)$ is prime for some/all $w \in C^\circ$.

Theorem. (Kaveh-Manon, 2016) Let C be a prime cone of $\mathrm{Trop}(I)$ and $\{u_1, \dots, u_r\} \subset C$ be maximally linearly independent. There is a valuation ν_C of $\mathbb{C}[X]$ such that its Newton-Okounkov body $\Delta(\mathbb{C}[X], \nu_C) \subset \mathbb{R}^d$ is the convex hull of the columns of the matrix with rows $u_1, \dots, u_r$.

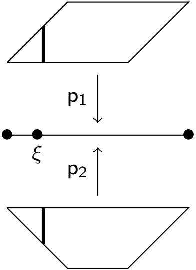
Mutations of Newton-Okounkov bodies

Theorem. (E-Harada, 2019) Let C_1 and C_2 be two prime cones of $\mathrm{Trop}(I)$ of maximal dimension sharing a codimension-1 face. There exist natural projections $\mathsf{p}_1, \mathsf{p}_2 : \mathbb{R}^d \to \mathbb{R}^{d-1}$ such that

$$\Delta(X, \nu_{C_1}) \xrightarrow{\mathsf{p}_1} \Delta_{C_1 \cap C_2} \xleftarrow{\mathsf{p}_2} \Delta(X, \nu_{C_2})$$

and the fibers are intervals of the same length (up to a global constant).

We obtain two piecewise-linear bijections $\Delta(X, \nu_{C_1}) \to \Delta(X, \nu_{C_2})$.



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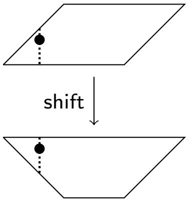
The first one **shifts** the intervals.



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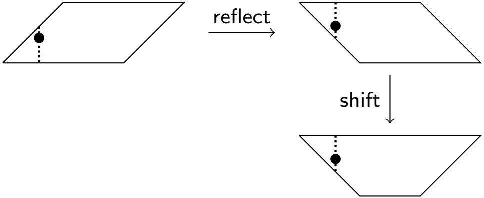
The second one **vertically reflects** $\Delta(X, \nu_{C_1})$ and then **shifts** the intervals.



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Remark 1. Ilten interprets this piecewise-linear bijection as a generalization of the combinatorial mutations of Akhtar-Coates-Galkin-Kasprzyk used to study mirror symmetry for Fano manifolds.

Remark 2. In the case of the Grassmannian of 2-planes in $\mathbb{C}^m$ the second bijection is connected to cluster mutations.

Families of degenerations from mutations of polytopes

$X \rightsquigarrow \mathrm{Trop}(X) \rightsquigarrow$ a collection of Newton–Okounkov polytopes and piecewise-linear bijections between them.

Mutations of polytopes also appear in the theory of cluster algebras/varieties, mirror symmetry, and the study of Fano manifolds/varieties.

Million-dollar question. Is there a systematic theory that can unify these?

Families of degenerations from mutations of polytopes

In [E–Harada–Manon, 2024] we have proposed a theory which generalizes the theory of toric varieties by

- replacing the classical lattice $M \cong \mathbb{Z}^r$ with a collection of lattices which are related by piecewise-linear bijections ("mutations"), and
- replacing the Laurent polynomial ring $\mathbb{C}[x_1^{\pm}, \cdots, x_r^{\pm}]$, together with its usual valuation with a more general $\mathbb{C}$-algebra $\mathcal{A}$ equipped with a valuation.

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By doing the above, we gain multiple benefits:

- We systematize and generalize the phenomenon in [E–Harada, 2022].
- We exhibit a family $\{\mathfrak{X}_\alpha\}$ of toric degenerations of a single variety X, where each of the resulting toric varieties are associated to a polytope which is mutation-related to the others in the family.
- We develop a generalization of the classical theory of polytopes together with a combinatorics-geometry dictionary.

Polyptych lattices

A **polyptych lattice** of rank r is $\mathcal{M} = (\{M_i\}_{i\in I}, \{\mu_{ij}\}_{i,j\in I})$ such that

- $M_i \simeq \mathbb{Z}^r$.
- $\mu_{ij} : M_i \to M_j$ is a piecewise linear bijection.
- $\mu_{ii} = \mathrm{id}$ and $\mu_{jk} \circ \mu_{ij} = \mu_{ik}$.

Example. The trivial polyptych lattice of rank r is $\mathcal{M}^r_\circ := (\{\mathbb{Z}^r\}, \{\mathrm{id}\})$.

Example. Let $\mathcal{M}_2 = (\{M_1, M_2\}, \{\mu_{12}\})$, where $M_1 \simeq M_2 \simeq \mathbb{Z}^2$ and $\mu_{12}(x, y) = (\min\{0, y\} - x, y)$.

An **element** of $\mathcal{M}$ is $\mathrm{m} = (m_i)_{i\in I}$ such that for all $i \in I$, $m_i \in M_i$ and for all $i, j \in I$, $\mu_{ij}(m_i) = m_j$.

Given $\mathcal{S} \subseteq \mathcal{M}$, the *i*-**th chart** of $\mathcal{S}$ is the set $S_i := \{s \mid \exists m \in \mathcal{S}, m_i = s\}$.

Polyptych lattice halfspaces

Roughly a **point** of $\mathcal{M}$ is a collection $p = \{p_i : M_i \to \mathbb{Z} \mid i \in I\}$ such that

- $\forall i, j \in I$, $p_j \circ \mu_{ij} = p_i$
- $\exists i \in I$ such that p_i is linear and
- $\forall i \in I$, p_i is a convex piecewise-linear function.

We denote by $\mathrm{Sp}(\mathcal{M})$ the collection of points of $\mathcal{M}$.

Example. For $\mathcal{M}_\circ^r := (\{\mathbb{Z}^r\}, \{\mathrm{id}\})$ we have that $\mathrm{Sp}(\mathcal{M}_\circ^r) = \mathrm{Hom}(\mathbb{Z}^r, \mathbb{Z})$.
$\mathrm{Sp}(\mathcal{M}_2) \cong \{(a, a', b) \in \mathbb{Z}^3 \mid a + a' = \min(0, b)\}$.

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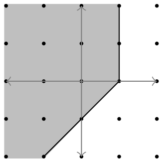
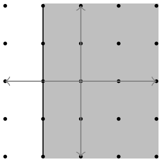
The **PL-halfspace** associated to $p \in \mathrm{Sp}(\mathcal{M})$ and $a \in \mathbb{Z}$ is $\mathcal{H}_{p,a} := \{\mathrm{m} \in \mathcal{M} \mid p(\mathrm{m}) \geq a\}$.

Example. An PL halfspace in $\mathcal{M}_2$ is:

M_1 

M_2 

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A **PL-polytope** is a bounded finite intersection of PL-halfspaces.

Example. A PL-polytope in $\mathcal{M}_2$:

An PL-polytope is **integral** if for all $i \in I$ its chart in M_i is an integral/lattice polytope.

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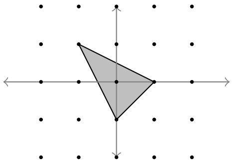
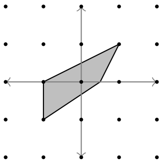
Example. A PL-polytope in $\mathcal{M}_2$ that is not integral:





An PL-polytope is **integral** if for all $i \in I$ its chart in M_i is an integral/lattice polytope.

Compactifications via polytopes

1. Toric case.

Let P be a lattice polytope in $\mathbb{R}^n$. The toric variety X_P is a compactification of the torus $\mathrm{Spec}(\mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}])$.

The homogeneous coordinate ring of the toric variety X_P is given by $\mathbb{C}[X_P] = \bigoplus_{k=0}^{\infty} \mathbb{C}[x^m \mid m \in \mathbb{Z}^r \cap kP]$.

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We have a valuation $\nu : \mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}] \to \{\text{PWL functions } \mathbb{Z}^r \to \mathbb{Z}\}$ given by $\nu(\sum c_\alpha x^\alpha)(-) := \min_{c_\alpha \neq 0} \langle \alpha, - \rangle$.

The **support function** $\psi_P : \mathbb{Z}^r \to \mathbb{R}$ of a polytope P is defined by $\psi_P := \min\{\langle m, - \rangle \mid m \in P\}$.

We have that $\mathbb{C}[X_P] = \bigoplus_{k=0}^{\infty} \{f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \mid \nu(f) \geq \psi_{kP}\}$.

Detropicalization of a polyptych lattice

Assume that $\mathcal{M}$ is **dualizable**. Roughly, this means there is a polyptych lattice $\mathcal{N}$ and a pair of bijections $\mathfrak{N} : \mathcal{N} \to \mathrm{Sp}(\mathcal{M})$ and $\mathfrak{M} : \mathcal{M} \to \mathrm{Sp}(\mathcal{N})$.

Let $\mathcal{P}_{\mathcal{N}}$ be the semialgebra generated by $\mathrm{Sp}(\mathcal{N})$ with respect to the operations $\oplus := \min$ and $\odot := +$.

Definition. Given a domain $\mathcal{A}$, a function $\nu : \mathcal{A} \to \mathcal{P}_{\mathcal{N}}$ is a **valuation** if

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A **detropicalization** of $\mathcal{M}$ is a domain $\mathcal{A}$ together with a valuation $\nu : \mathcal{A} \to \mathcal{P}_{\mathcal{N}}$ such that $\mathrm{Sp}(\mathcal{N}) \subseteq \mathrm{im}(\nu)$.

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Example. For the trivial polyptych lattice of rank r, $\mathcal{M}_\circ^r := (\{\mathbb{Z}^r\}, \{\mathrm{id}\})$, recall that $\mathrm{Sp}(\mathcal{M}_\circ^r) = \mathrm{Hom}(\mathbb{Z}^r, \mathbb{Z})$. The dual is $\mathcal{M}_\circ^r$ and $\mathcal{P}_{\mathcal{M}_\circ^r}$ is the set of piecewise-linear convex functions on $\mathbb{Z}^r$.

The ring $\mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$ is a detropicalization of $\mathcal{M}_\circ^r$ with valuation $\nu(\sum c_\alpha x^\alpha) := \bigoplus_{c_\alpha \neq 0} \langle \alpha, - \rangle$.

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Example. Let $\mathcal{A} = \mathbb{C}[x_1, x_2, t^{\pm 1}]/\langle x_1 x_2 - 1 - t\rangle$. There exists a valuation ν such that $(\mathcal{A}, \nu)$ is a detropicalization of $\mathcal{M}_2$.

Remark. For each $d, r \in \mathbb{N}$ we give a polyptych lattice $\mathcal{M}_{d,r}$ together with detropicalization $(\mathcal{A}_{d,r}, \nu_{d,r})$ where $\mathcal{A}_{d,r} = \mathbb{C}[x_1, \ldots, x_d, t_1^{\pm 1}, \ldots, t_r^{\pm 1}]/\langle x_1 \cdots x_d - t_1 - \cdots - t_r\rangle$.

Compactifications via polyptych lattice polytopes

2. Polyptych lattice case.

Let $\mathcal{A}$ be a detropicalization of $\mathcal{M}$ with valuation ν and Δ an integral PL-polytope.

Let $\mathcal{N}$ be the dual of $\mathcal{M}$ with bijection $\mathfrak{M} : \mathcal{M} \to \mathrm{Sp}(\mathcal{N})$.

The **support function** $\psi_\Delta : \mathcal{N} \to \mathbb{R}$ of Δ is defined by $\psi_\Delta := \min\{\mathfrak{M}(\mathrm{m}) \mid \mathrm{m} \in \Delta \cap \mathcal{M}\}$.

Define the graded algebra $\mathcal{A}_\Delta := \bigoplus_{k=0}^{\infty}\{f \in \mathcal{A} \mid \nu(f) \geq \psi_{k\Delta}\}$.

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Theorem. (E–Harada–Manon) $X_\Delta := \mathrm{Proj}(\mathcal{A}_\Delta)$ is a compactification of $\mathrm{Spec}(\mathcal{A})$. Moreover, for each $i \in I$, the chart image of Δ in M_i is a Newton–Okounkov body of X_Δ and these polytopes are connected by the PWL bijections μ_{ij}.

Geometric properties

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Moreover,

- $\mathcal{A}_\Delta$ is finitely generated.
- X_Δ is arithmetically Cohen-Macaullay.
- If $\mathcal{A}$ is normal, then X_Δ is also normal.
- If $\mathcal{A}$ is a UFD, then X_Δ has a finitely generated class group and a finitely generated Cox ring.

We give a family of rank-2, two-chart examples in (Cook–E–Harada–Manon), and also give lots of sample computations for this family, e.g. the PL analogue of Gorenstein-Fano polytopes.

Thank you!