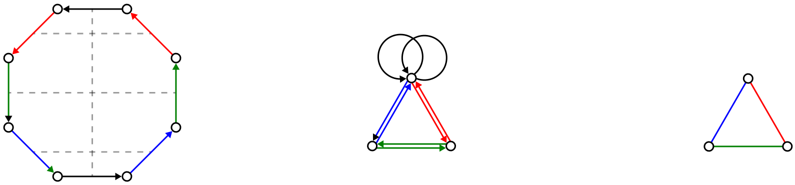
Finite-rank perturbations of random band matrices via infinitesimal free probability

Benson Au

arXiv:1906.10268



February 6th, 2020

IPAM: Asymptotic Algebraic Combinatorics

Think in threes

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The number three surrounds us: three is the first recognizable and easy to remember sequence of numbers; the three-act structure is the predominant model used in screenwriting; and a triangle is the strongest physical shape. That is why we rely on three security triads to lay the foundation of what it means to be secure!

-UC Cyber Security Awareness Fundamentals

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Figure: © The Security Awareness Company, LLC

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Combinatorics

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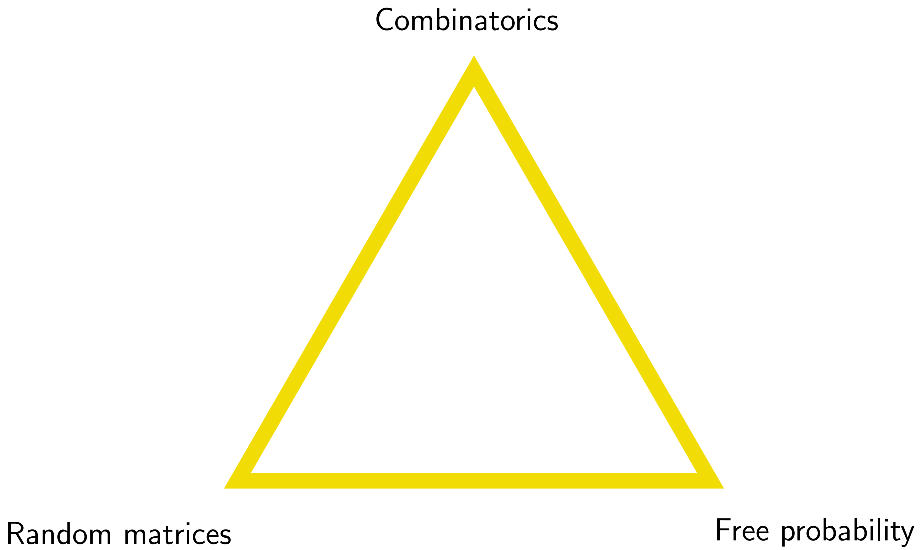
Random matrices

Think in threes



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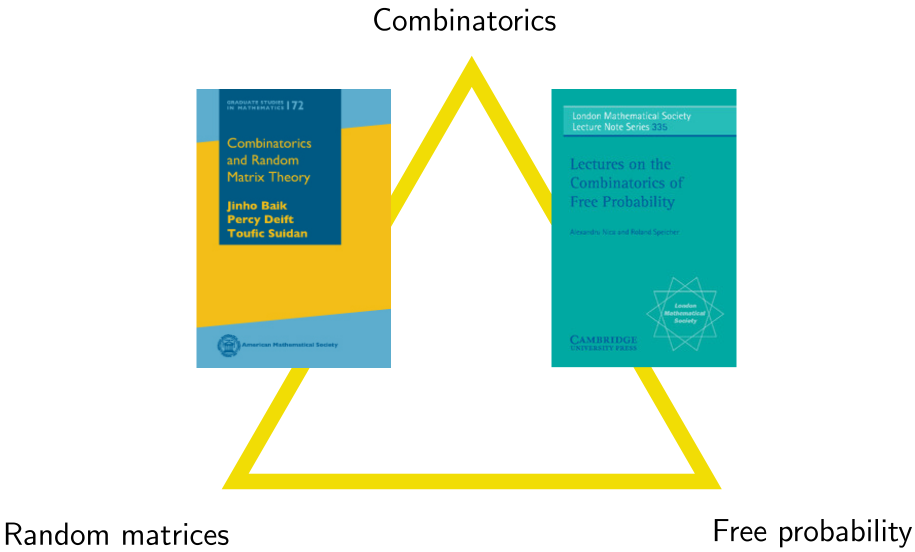

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- For a self-adjoint $N \times N$ matrix $\mathbf{A}_N \in \mathrm{Mat}_N(\mathbb{C})$, let

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  denote the eigenvalues of $\mathbf{A}_N$, counting multiplicity, arranged in a non-increasing order.
- We write $\mu(\mathbf{A}_N)$ for the *empirical spectral distribution* (or *ESD* for short) of $\mathbf{A}_N$, i.e.,

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- For a random matrix $\mathbf{A}_N$, the ESD $\mu(\mathbf{A}_N)$ becomes a random probability measure on the real line $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Wigner matrices

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- Let $(X_{i,j})_{1\leq i<j<\infty}$ and $(X_{i,i})_{1\leq i<\infty}$ be independent families of i.i.d. random variables: the former, complex-valued, centered, and of unit variance; the latter, real-valued and of finite variance.

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- We call $\mathbf{W}_N$ a *Wigner matrix* with $\beta = 1$ if $X_{i,j}$ is real-valued and $\beta = 2$ otherwise.

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Theorem (Wigner's semicircle law)

Let $\mathbf{W}_N$ *be a Wigner matrix. Then the ESD* $\mu(\mathbf{W}_N)$ *converges weakly almost surely to the semicircle distribution* $\mu_{\mathcal{SC}}(dx) = \frac{1}{2\pi}(4 - x^2)_+^{1/2}\, dx$*.*

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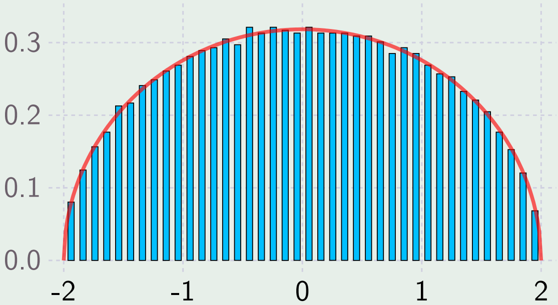

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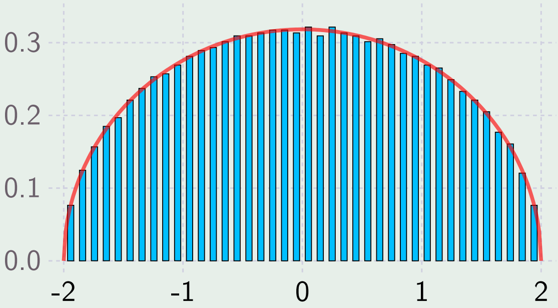

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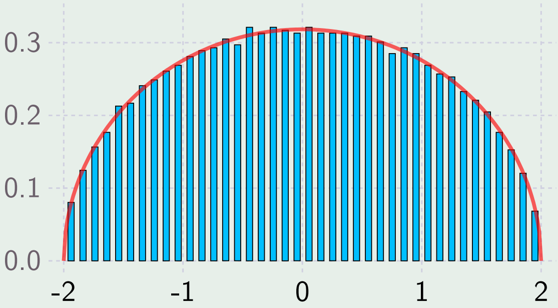

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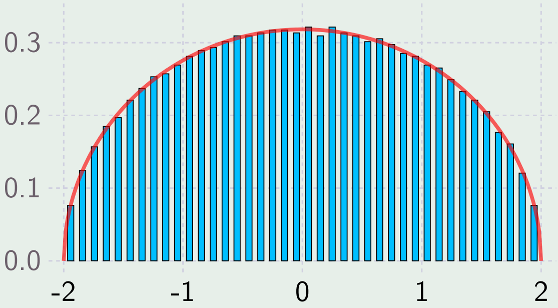

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Theorem (BDJ99)

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{\ell(\pi_N) - 2\sqrt{N}}{N^{1/6}} \leq s\right) = F_2(s)$$

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- Can also be used to study finer statistics.

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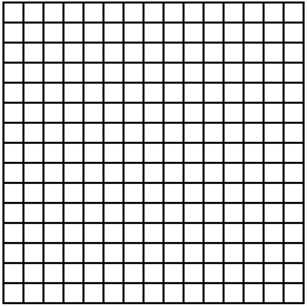
Beyond mean-field models



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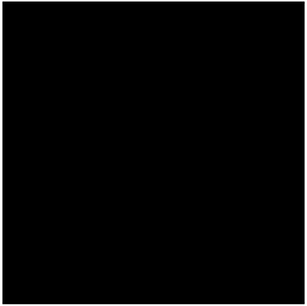

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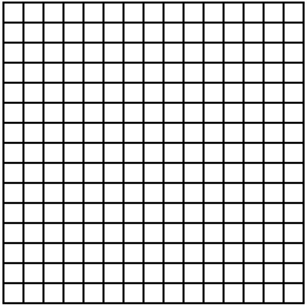

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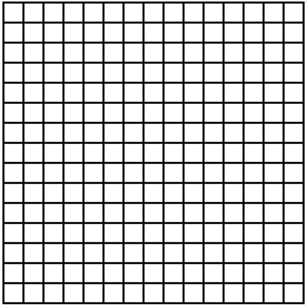

- Random Schrödinger operators:

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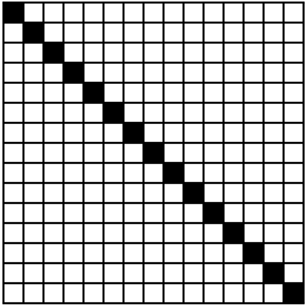

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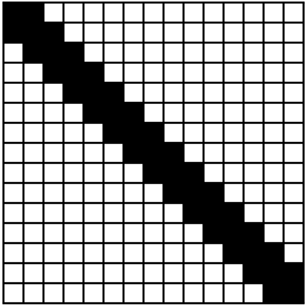

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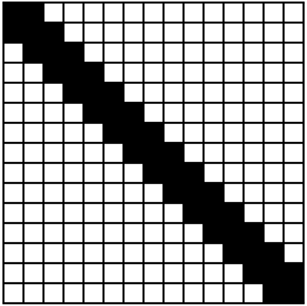

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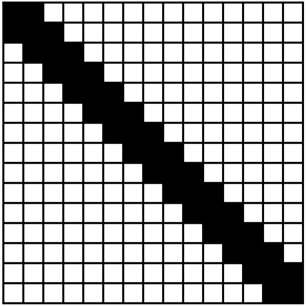

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- We call Ξ_N a *random band matrix*.

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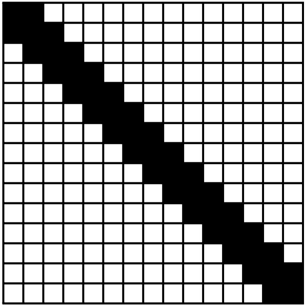

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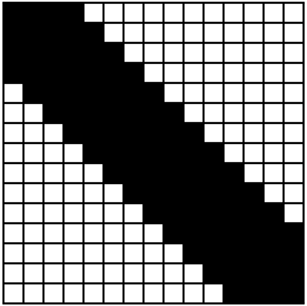

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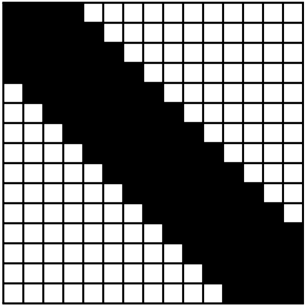

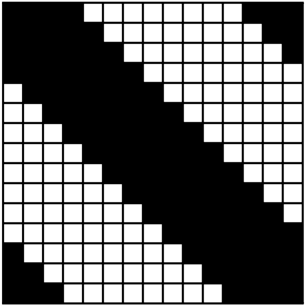

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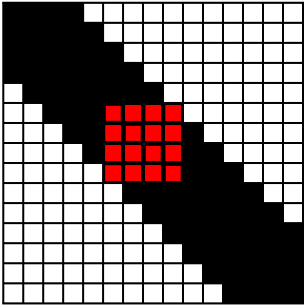

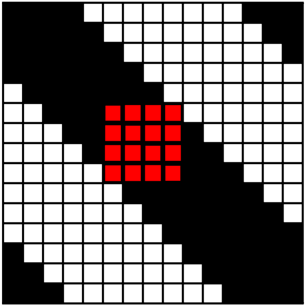

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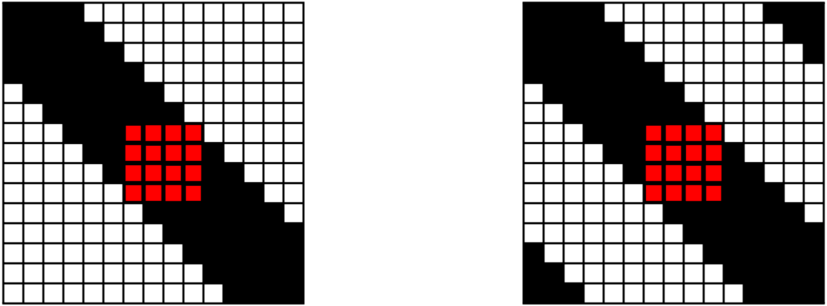

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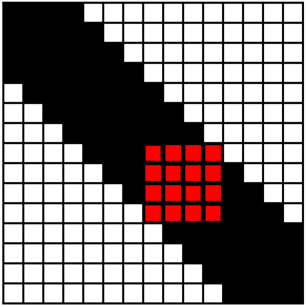

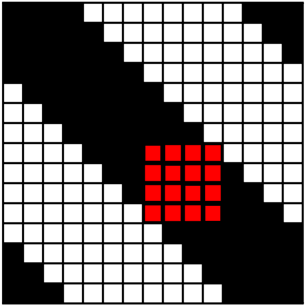

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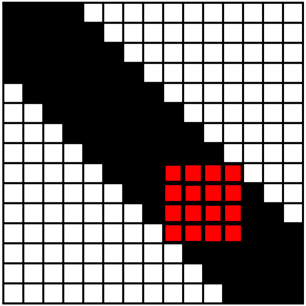

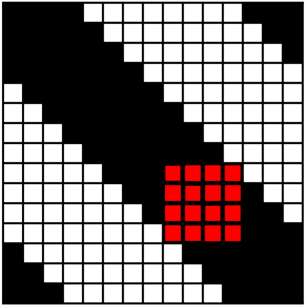

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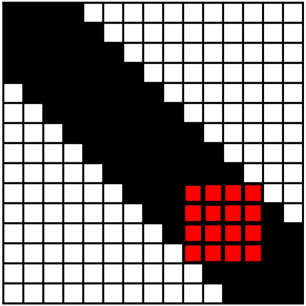

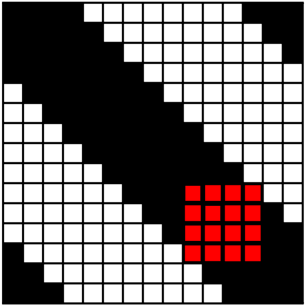

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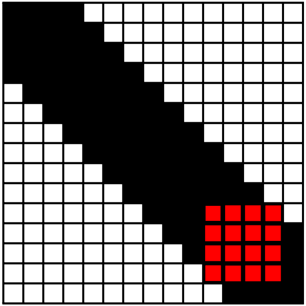

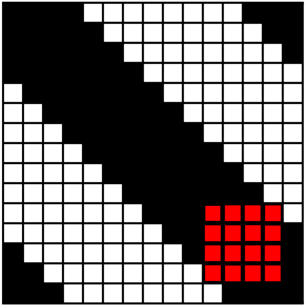

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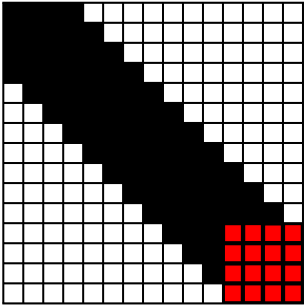

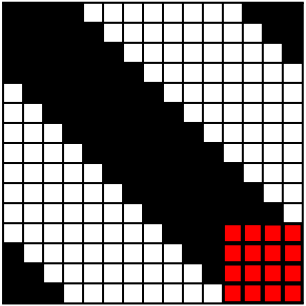

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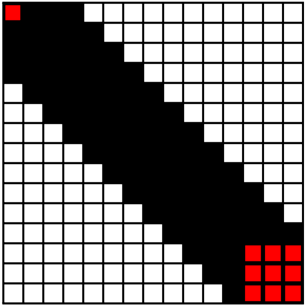

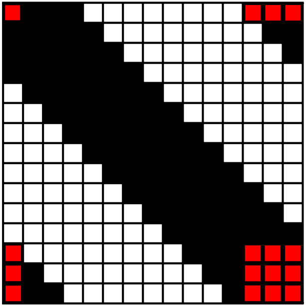

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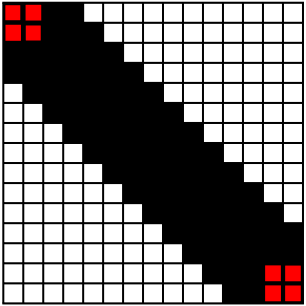

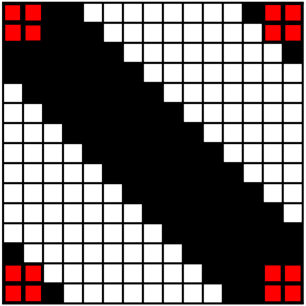

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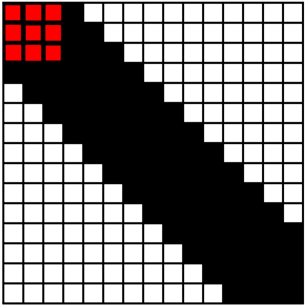

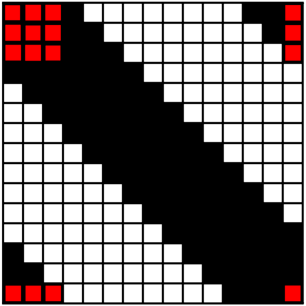

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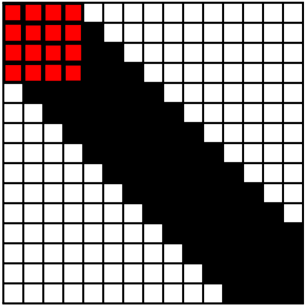

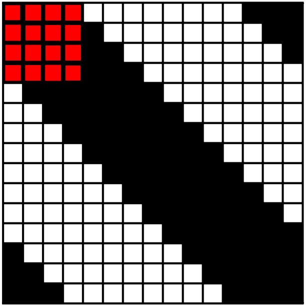

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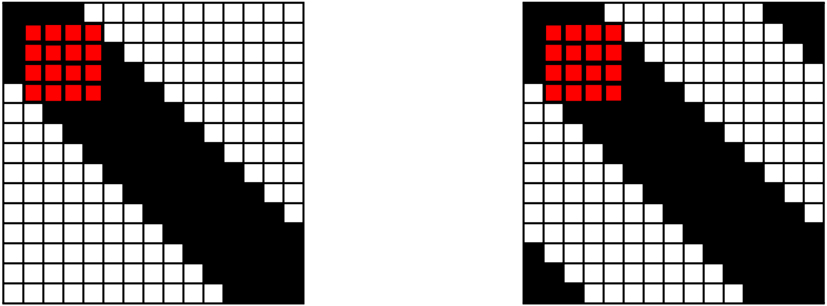

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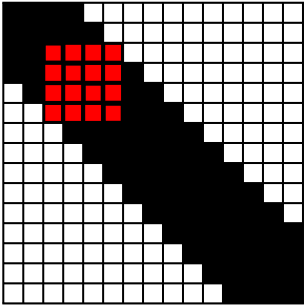

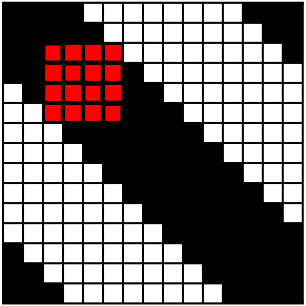

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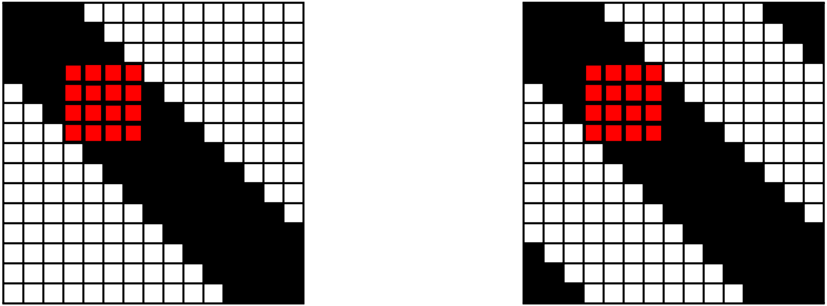

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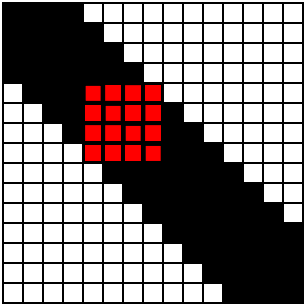

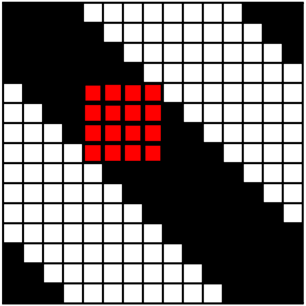

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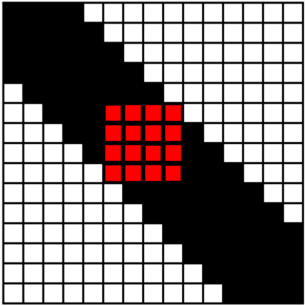

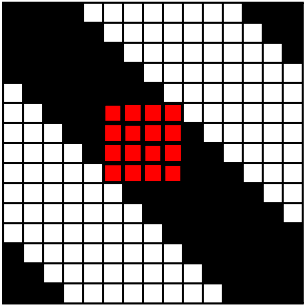

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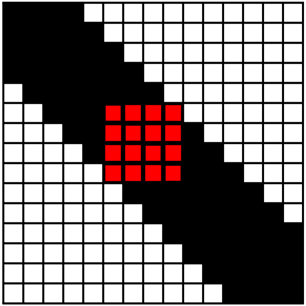
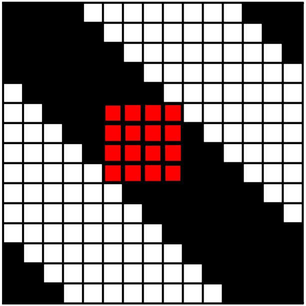


$$|i-j| \leq b_N$$



$$\min\{|i-j|, N-|i-j|\} \leq b_N$$

Random band matrices

Random band matrices

Global behavior:

Theorem (BMP91)

$$\mathbb{P}(\lim_{N\to\infty}\mu(\boldsymbol{\Xi}_N)=\mu_{\mathcal{SC}})=1 \iff b_N \gg 1.$$

Random band matrices

Global behavior:

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$$\mathbb{P}(\lim_{N\to\infty} \mu(\Xi_N) = \mu_{SC}) = 1 \iff b_N \gg 1.$$

Largest eigenvalue:

Theorem (Sod10,BvH16)

If the entries $X_{i,j}$ are sub-Gaussian, then

$$\mathbb{P}(\lim_{N\to\infty} \lambda_1(\Xi_N) = 2) = 1 \iff b_N \gg \log(N).$$

Moreover,

$$\lim_{N\to\infty} \mathbb{P}(N^{2/3}(\lambda_1(\Xi_N) - 2) \leq s) = F_\beta(s) \iff b_N \gg N^{5/6}.$$

Random band matrices

Random band matrices

Universality:

Conjecture (CGI90, CMI90, FLW91, WFL91, FM91)

Sharp transition around the critical rate $\sqrt{N}$:

- *(strong disorder) Poisson local statistics and eigenvector localization for $b_N \ll \sqrt{N}$;*
- *(weak disorder) random matrix theory local statistics and eigenvector delocalization for $b_N \gg \sqrt{N}$.*

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Progress:

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Progress:

- Eigenvector localization for $b_N \ll N^{1/8}$ (Schenker 2009)

Random band matrices

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Progress:

- Eigenvector localization for $b_N \ll N^{1/7}$ (PSSS 2017)

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Progress:

- Eigenvector localization for $b_N \ll N^{1/7}$ (PSSS 2017)
- Eigenvector delocalization for $b_N \gg N^{6/7}$ (EK 2011)

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Progress:

- Eigenvector localization for $b_N \ll N^{1/7}$ (PSSS 2017)
- Eigenvector delocalization for $b_N \gg N^{4/5}$ (EKYY 2013)

Random band matrices

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Progress:

- Eigenvector localization for $b_N \ll N^{1/7}$ (PSSS 2017)
- Eigenvector delocalization and QUE for $b_N \gg N^{3/4+\alpha}$ (BYY 2018)

Random band matrices

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- Localization implies Poisson statistics (Min96)

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- QUE implies random matrix theory local statistics (BEYY17)

Free probability

Free probability

Definition

A $*$-*probability space* $(\mathcal{A}, \varphi)$ is a pair consisting of a unital $*$-algebra $\mathcal{A}$ over $\mathbb{C}$ together with a unital linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ such that $\varphi(a^* a) \geq 0$ for any $a \in \mathcal{A}$.

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Examples

- $(L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$
- $(\mathrm{Mat}_N(\mathbb{C}), \frac{1}{N}\,\mathrm{Tr})$
- $(\mathrm{Mat}_N(L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P})), \mathbb{E}\frac{1}{N}\,\mathrm{Tr})$
- $(\mathcal{B}(\mathcal{H}), \langle \cdot\xi, \xi\rangle)$
- $(\mathbb{C}[G], \langle \cdot\delta_e, \delta_e\rangle)$

Free probability

Definition

We say that subalgebras $(\mathcal{A}_i)_{i\in I}$ of a $*$-probability space $(\mathcal{A}, \varphi)$ are *classically independent* if $[\mathcal{A}_i, \mathcal{A}_j] = 0$ for $i \neq j$ and

$$\varphi\left(\prod_{j=1}^{n} a_j\right) = \prod_{j=1}^{n} \varphi(a_j), \qquad \forall a_j \in \mathcal{A}_{i(j)},$$

whenever the indices $(i(j))_{j=1}^{n}$ are distinct.

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whenever the indices $(i(j))_{j=1}^{n}$ are distinct.

Examples

- In $(L^{\infty-}(\Omega,\mathcal{F},\mathbb{P}),\mathbb{E})$, the subalgebras $(L^{\infty-}(\Omega,\mathcal{F}_i,\mathbb{P}))_{i\in I}$ for independent σ-algebras $(\mathcal{F}_i)_{i\in I}$
- In $(C[G],\langle\cdot\delta_e,\delta_e\rangle)$, the subalgebras $(C[G_i])_{i\in I}$ for the direct product $G=\times_{i\in I}G_i$.

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Free probability

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We say that subalgebras $(\mathcal{A}_i)_{i\in I}$ of a $*$-probability space $(\mathcal{A}, \varphi)$ are *freely independent* if

$$\varphi\Big(\prod_{j=1}^{n} \mathring{a}_j\Big) = 0, \qquad \forall a_j \in \mathcal{A}_{i(j)},$$

where the indices are consecutively distinct $i(1) \neq i(2) \neq \cdots \neq i(n)$ and $\mathring{a}_j = a_j - \varphi(a_j)$.

Examples

- (Voi91) Unitarily invariant random matrix ensembles in the large N limit; (Dyk93) mean-field matrix ensembles in the large N limit.
- In $(C[G], \langle \cdot \delta_e, \delta_e \rangle)$, the subalgebras $(C[G_i])_{i\in I}$ for the free product $G = *_{i\in I}\, G_i$.

Free convolution

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Observation: the joint distribution of freely independent random variables is completely determined by the marginal distributions in a universal way.

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Definition

Given two probability measures μ and ν on the real line, one defines the *free convolution* $\mu \boxplus \nu$ as the distribution of $X + Y$ for X and Y self-adjoint and freely independent with $X \overset{d}{=} \mu$ and $Y \overset{d}{=} \nu$.

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Question: How do we actually compute the free convolution in practice?

Theorem (BV93)

Let $G_\mu(z) = \int \frac{1}{z-t}\,\mu(dt)$ be the Cauchy transform of μ and $F_\mu(z) = \frac{1}{G_\mu(z)}$ its reciprocal. Then

$$F_\mu^{-1}(z) + F_\nu^{-1}(z) = z + F_{\mu\boxplus\nu}^{-1}(z).$$

Free convolution

Example

Let $\mathbf{W}_N$ be a Wigner matrix and $\mathbf{D}_N$ a diagonal matrix with i.i.d. Rademacher random variables. Then

$$\mu(\mathbf{W}_N) \xrightarrow{w} \frac{1}{2\pi}(4-x^2)_+^{1/2}\,dx \quad \text{and} \quad \mu(\mathbf{D}_N) \xrightarrow{w} \frac{1}{2}\delta_{\pm 1}.$$

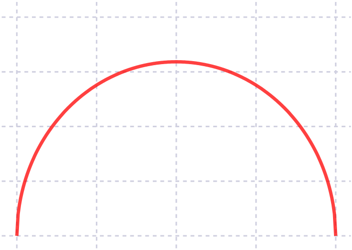
The asymptotic freeness of $\mathbf{W}_N$ and $\mathbf{D}_N$ further tells us that

$$\mu(\mathbf{W}_N+\mathbf{D}_N) \xrightarrow{w} \left(\frac{1}{2\pi}(4-x^2)_+^{1/2}\,dx\right) \boxplus \left(\frac{1}{2}\delta_{\pm 1}\right),$$

which is absolutely continuous with density

$$\frac{1}{2\pi\sqrt{3}}\left(\frac{\sqrt[3]{27x-2x^3+3\sqrt{3}\sqrt{27x^2-4x^4}}}{\sqrt[3]{2}} - \frac{\sqrt[3]{2}x^2}{\sqrt[3]{27x-2x^3+3\sqrt{3}\sqrt{27x^2-4x^4}}}\right).$$

Free convolution



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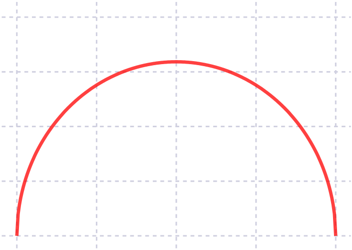
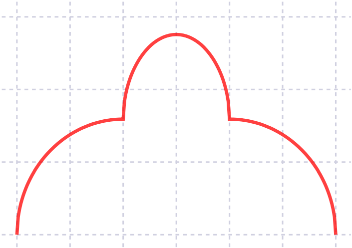
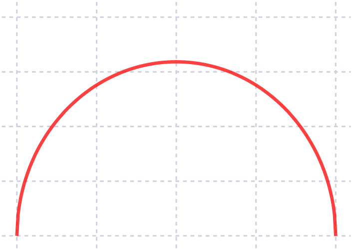
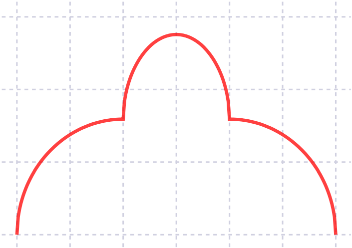
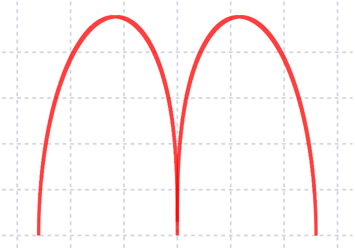




Free convolution







Finite-rank perturbations

Theorem (Péc06, FP07, CDMF09, PRS13)

Let $\mathbf{W}_N$ *be a Wigner matrix. For a localized perturbation* $\theta \mathbf{E}_N^{(1,1)}$ *with* $\theta > 1$, *the largest eigenvalue of the perturbed model*

$$\mathbf{M}_N = \mathbf{W}_N + \theta \mathbf{E}_N^{(1,1)}$$

satisfies

$$\lambda_1(\mathbf{M}_N) \overset{\text{a.s.}}{\to} \theta + \theta^{-1}.$$

Furthermore,

$$\sqrt{N}(1-\theta^{-2})^{-1}\Big(\lambda_1(\mathbf{M}_N) - (\theta + \theta^{-1})\Big) \overset{d}{\to} \mu * \mathcal{N}(0, v_\theta),$$

where

$$v_\theta = \frac{1}{\beta}\left(\frac{\kappa_4(\mu)}{\theta^2} + \frac{2}{\theta^2 - 1}\right).$$

Finite-rank perturbations

Theorem (Péc06, FP07, CDMF09, PRS13)

Let $\mathbf{W}_N$ *be a Wigner matrix. For a delocalized perturbation* $\frac{\theta}{N}\mathbf{J}_N$ *with* $\theta > 1$, *the largest eigenvalue of the perturbed model*

$$\mathbf{M}_N = \mathbf{W}_N + \frac{\theta}{N}\mathbf{J}_N$$

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Furthermore,

$$\sqrt{N}(1-\theta^{-2})^{-1}\Big(\lambda_1(\mathbf{M}_N) - (\theta + \theta^{-1})\Big) \overset{d}{\to} \mathcal{N}(0, 2\beta^{-1}).$$

Free probability for finite-rank perturbations

Free probability for finite-rank perturbations

Observation: $\mathbf{W}_N$ and $\theta\mathbf{E}_N^{(1,1)}$ are asymptotically free with

$$\mu(\mathbf{W}_N) \xrightarrow{w} \frac{1}{2\pi}(4-x^2)_+^{1/2}\,dx \quad \text{and} \quad \mu(\theta\mathbf{E}_N^{(1,1)}) \xrightarrow{w} \delta_0.$$

So,

$$\mu(\mathbf{M}_N) \xrightarrow{w} \left(\frac{1}{2\pi}(4-x^2)_+^{1/2}\,dx\right) \boxplus \delta_0 = \frac{1}{2\pi}(4-x^2)_+^{1/2}\,dx.$$

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Second observation (BBCDMFF):

$$G_{\mu_{SC}}(z) = \frac{z-\sqrt{z^2-4}}{2};$$
$$F_{\mu_{SC}}(z) = \frac{2}{z-\sqrt{z^2-4}};$$
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Infinitesimal free probability

Definition (BS12)

An *infinitesimal probability space* $(\mathcal{A}, \varphi, \varphi')$ is a $*$-probability space $(\mathcal{A}, \varphi)$ with an additional linear functional $\varphi' : \mathcal{A} \to \mathbb{C}$ such that $\varphi'(1) = 0$. We say that subalgebras $(\mathcal{A}_i)_{i \in I}$ are *infinitesimally free* if they are free in $(\mathcal{A}, \varphi)$ and

$$\varphi'(a_1 \cdots a_k) = \sum_{j=1}^{k} \varphi(a_1 \cdots a_{j-1} \varphi'(a_j) a_{j+1} \cdots a_k)$$

for $a_j \in \mathcal{A}_{i(j)}$ and consecutively distinct indices $i(1) \neq i(2) \neq \cdots \neq i(k)$. Equivalently, for $\varphi_t = \varphi + t\varphi'$,

$$\varphi_t((a_1 - \varphi_t(a_1)) \cdots (a_k - \varphi_t(a_k))) = O(t^2).$$

Infinitesimal free probability

Example

Assume that

$$\varphi(P) = \lim_{N\to\infty} \varphi_N(P) = \lim_{N\to\infty} \mathbb{E}\frac{1}{N}\operatorname{Tr}[P(\mathcal{A}_N)]$$

exists for any non-commutative polynomial P in the family of random matrices $\mathcal{A}_N$. If the limit

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also exists for any non-commutative polynomial P, then we can define an infinitesimal probability space $(\mathcal{A}, \varphi, \varphi')$.

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$$G_{\nu_3}(z) = G_{\nu_1}(\omega_1(z))\omega_1'(z) + G_{\nu_2}(\omega_2(z))\omega_2'(z).$$

Example (Shl18)

Assume that $\theta > 1$. Then

$$(\mu_{\mathcal{SC}}, \nu_{\text{GUE}}) \boxplus (\delta_0, \delta_\theta - \delta_0) = \left(\mu_{\mathcal{SC}}, \delta_{\theta+\theta^{-1}} - \frac{\theta(x-2\theta)}{2\pi\sqrt{4-x^2}(\theta(x-\theta)-1)}\, dx\right).$$

For the GOE, one adds the signed measure of mass 0

$$\frac{1}{2}\left(\frac{1}{2}\delta_{x=\pm 2} - \frac{1}{\pi(4-x^2)_+^{1/2}}\, dx\right).$$

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Theorem (EM16)

Assume that $\mathbf{W}_N$ *is a Wigner matrix such that the upper-triangular entries (resp., the diagonal entries) have a common distribution. Then* $\mathbf{W}_N$ *has a limiting infinitesimal distribution given by* $\nu = \frac{1}{2}\left[\frac{\mathbb{1}\{\beta=1\}}{2}\delta_{\pm 2} + \nu_{\text{ac}}\right]$*, where*

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Lemma (Au19+)

Let $\mathbf{W}_N$ be a Wigner matrix as above. Then $\mathbf{W}_N$, the matrix units, and the normalized all-ones matrix $\frac{1}{N}\mathbf{J}_N$ are asymptotically infinitesimally free.

Free probability for random band matrices

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Theorem (BMP91)

Let

$$\Xi_N(i,j) = \mathbb{1}\{\min(|i-j|, N-|i-j|) \le b_N\}\frac{X_{i,j}}{\sqrt{2b_N+1}}$$

be a periodic random band matrix with band width $b_N \gg 1$. *Then* $\mu(\Xi_N)$ *converges weakly almost surely to the semicircle distribution* $\mu_{\mathcal{SC}}$.

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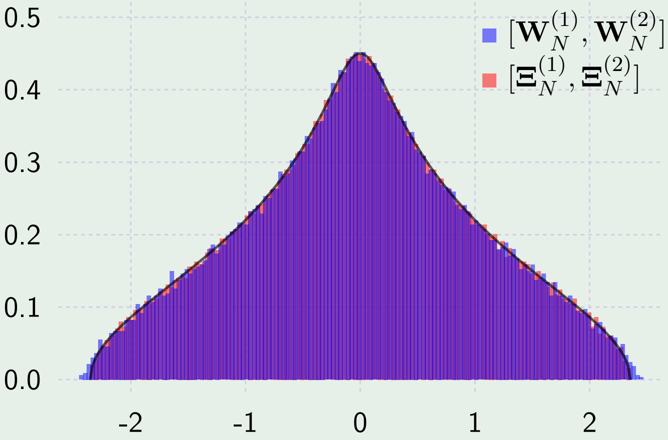
Theorem (Au18)

Informally, free probability still works for random band matrices (e.g., asymptotic freeness for independent random band matrices).

Periodically banded GUE matrices

Example



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Remark

Rate of the band width does not play a role.

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Independent copies of Ξ_N of the same band width are asymptotically infinitesimally free iff the band widths satisfy $b_N \gg \sqrt{N}$.

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Independent copies of Ξ_N *of the same band width are asymptotically infinitesimally free iff the band widths satisfy* $b_N \gg \sqrt{N}$*. Moreover,* Ξ_N *is asymptotically infinitesimally free from the matrix units and the normalized all-ones matrix.*

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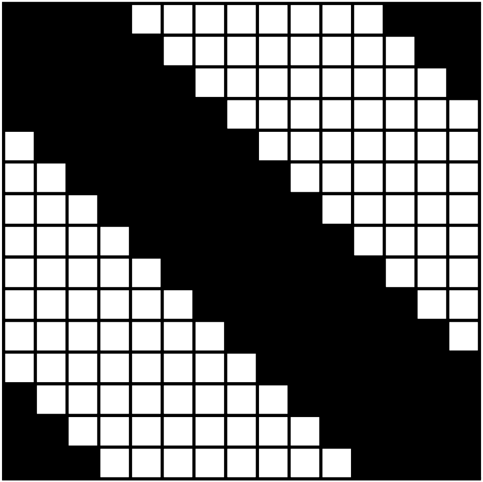
Independent copies of Ξ_N of the same band width are asymptotically infinitesimally free iff the band widths satisfy $b_N \gg \sqrt{N}$. Moreover, Ξ_N is asymptotically infinitesimally free from the matrix units and the normalized all-ones matrix.

Corollary

The infinitesimal distribution of the perturbed model $\Xi_N + \theta \mathbf{E}_N^{(1,1)}$ (resp., $\Xi_N + \frac{\theta}{N}\mathbf{J}_N$) is identical to the perturbed GUE for band widths $b_n \gg \sqrt{N}$. In particular, we find outliers at the classical locations.

Random band matrices



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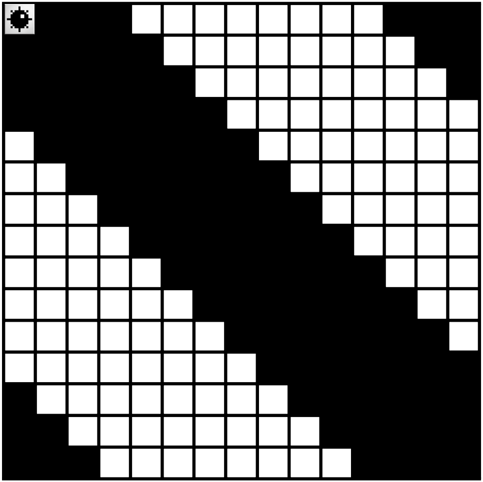

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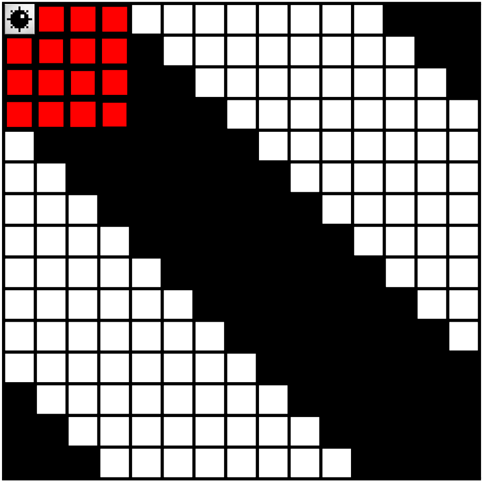
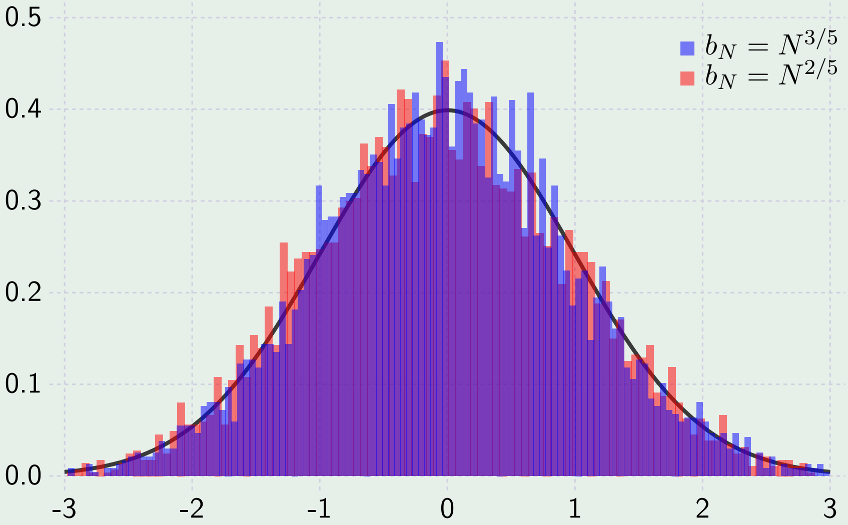


Example $\left(F_{N,1}(b_n) = \frac{\sqrt{2b_N+1}}{\sqrt{\frac{\theta^2-1^2}{\theta^2}}}\left[\lambda_1(\mathbf{\Xi}_N + \theta\mathbf{E}_N^{(1,1)}) - (\theta + \frac{1}{\theta})\right]\right)$



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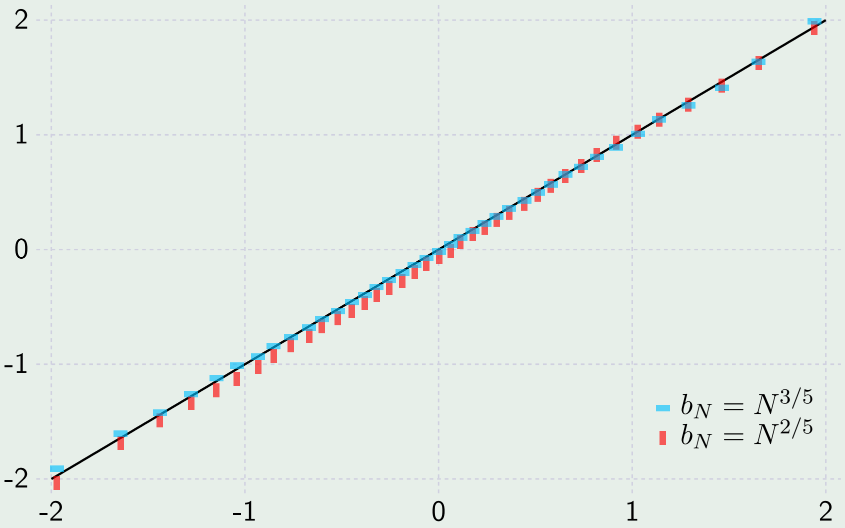

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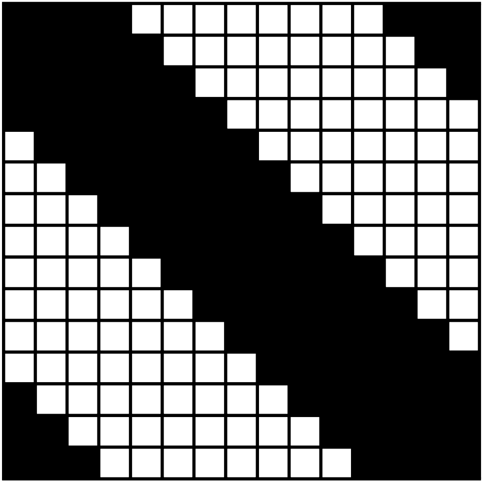

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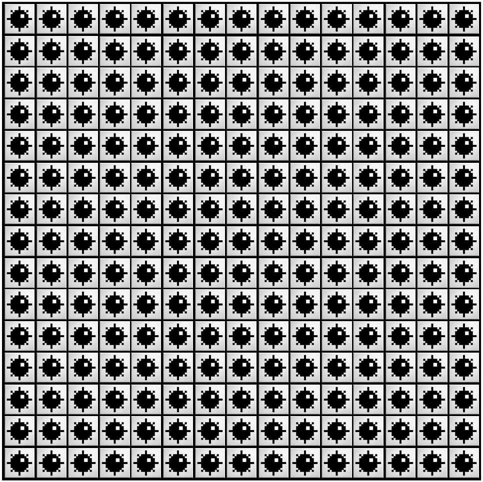

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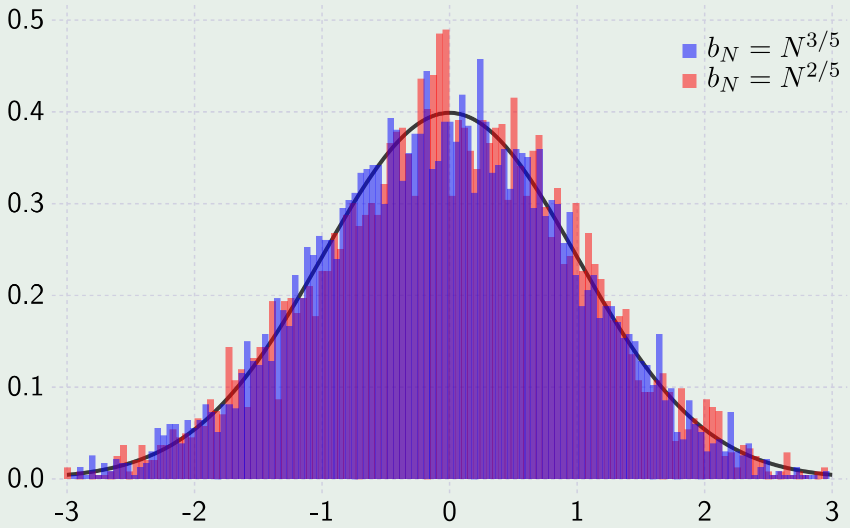

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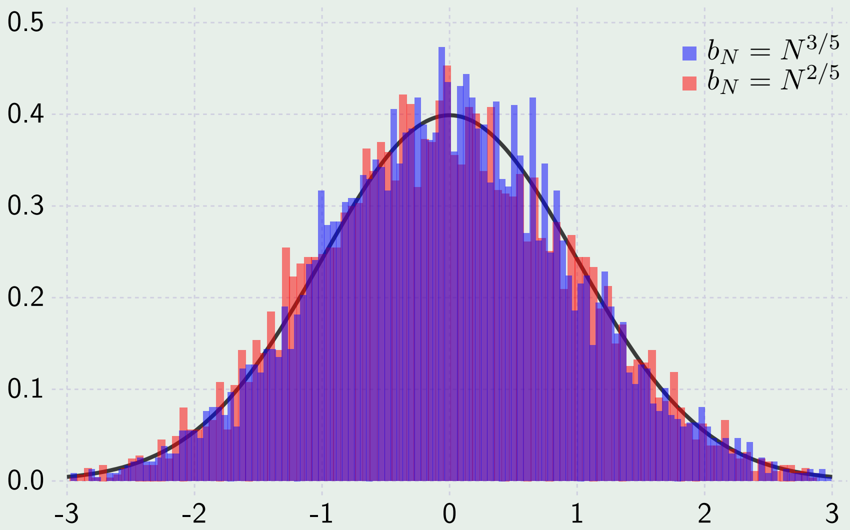

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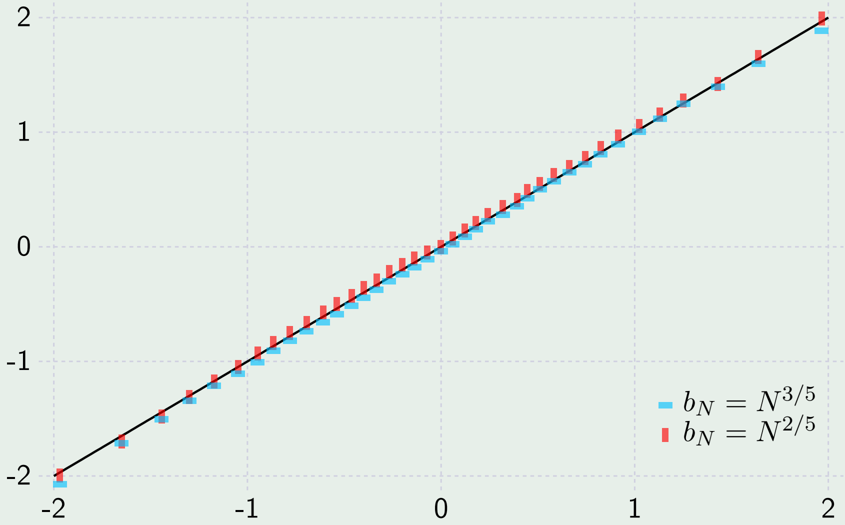

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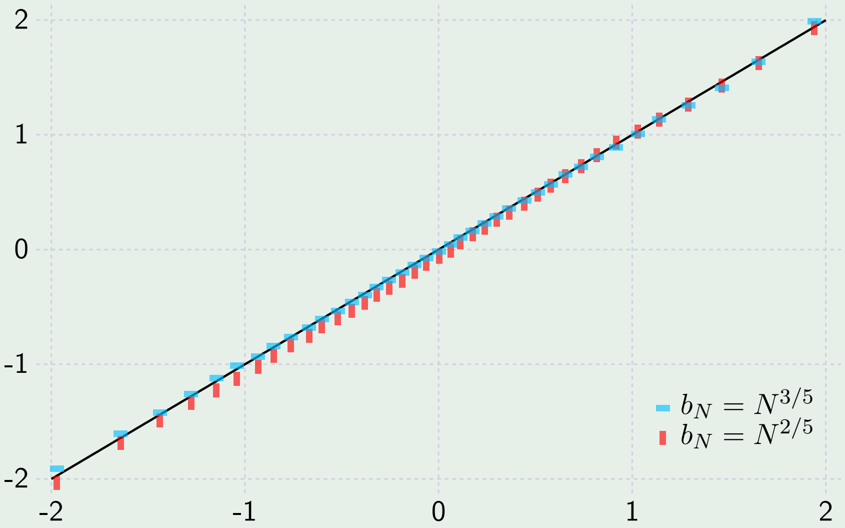

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