Renormalizable Rectangle Exchange Maps

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Outline

1 Introduction
   - Main results
   - Background

2 Cut-and-Project Domain Exchange Maps (DEM)

3 PV DEMs

4 Renormalization scheme for PV REMs

5 Multistage REMs

6 Further Work on REMs
Domain exchange map (DEM)

- A dynamical system defined on a **smooth Jordan domain** which is a piecewise translation
- Joint work with Richard Kenyon and Ren Yi

\[ T : D_1 \rightarrow D_1 \]
\[ T : x \mapsto x + v \]
Definition

Let $X$ be a **Jordan domain** partitioned into smaller Jordan domains, with disjoint interiors, in two different ways

$$X = \bigcup_{k=0}^{N} A_k = \bigcup_{k=0}^{N} B_k$$

such that for each $k = 0, \ldots, N$, $\exists v_k \in \mathbb{R}^2$ with

$$A_k + v_k = B_k.$$  

A **domain exchange map** is the dynamical system

$$T(x) = x + v_k$$  

for $x \in \partial A_k$.

The map is not defined for points $x \in \bigcup_{k=0}^{N} \partial A_k$. 

Setup: $T : X \rightarrow X$ a dynamical system and $Y \subsetneq X$

**Definition**
A dynamical system is **minimal** if every point has a dense orbit

**Definition**
The **first-return map** $\hat{T}|_Y : Y \rightarrow Y$ is defined by

$$\hat{T}|_Y(p) = T^m(p) \quad \text{where} \quad m = \min\{k \in \mathbb{N} : T^k(p) \in Y\}$$

for $p \in Y$.

If $X = [0, 1]^2$ the DEM is a **Rectangle Exchange Map (REM)**
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for $p \in Y$. If $X = [0, 1]^2$ the DEM is a **Rectangle Exchange Map (REM)**.
**Definition**

A **renormalization scheme** is a proper subset $Y \subseteq X$, a dynamical system $T' : X' \to X'$, and a homeomorphism $\phi : X \to X'$ such that

$$\hat{T}|_Y = \phi^{-1} \circ T' \circ \phi.$$ 

If $T' = T$ the dynamical system is called **self-induced** or **renormalizable**.

**Main Results (A., Kenyon, Yi 2018)**

- Construct minimal DEMs on any domain with equidistributed orbits
- Find an infinite family of renormalizable REMs
- Compose REMs to produce **multistage** REMs with periodic renormalization schemes
Definition

A renormalization scheme is a proper subset $Y \subsetneq X$, a dynamical system $T' : X' \to X'$, and a homeomorphism $\phi : X \to X'$ such that

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- Construct minimal DEMs on any domain with equidistributed orbits
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An interval exchange transformation (IET) $T : X \rightarrow X$ is a 1-dimensional DEM defined on an interval

1. Each probability vector and permutation (of matching length) defines an IET on $[0, 1]$
2. IETs are important examples in ergodic theory
3. Keane’s minimality criterion (’75)
4. $T$ is uniquely ergodic if the only invariant probability measure is a multiple of Lebesgue measure
   - A measure $\mu$ is invariant with respect to $T$ if $\mu(T^{-1}(A)) = \mu(A)$ $\forall A$
   - Unique ergodicity $\implies$ orbits of points uniformly distributed
5. There exist minimal IETs ($n = 4$) which are not uniquely ergodic (Keane ’77)
6. Almost every minimal IET with an irreducible permutation is uniquely ergodic (Masur/Veech ‘82)
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6. Almost every minimal IET with an irreducible permutation is uniquely ergodic (Masur/Veech ‘82)
The Hausdorff dimension of the set of non-uniquely ergodic IETs with \( n \) pieces is \( n - 1 - 1/2 \) (Chaika-Masur ’18).

- Proof uses Rauzy Induction to construct large sets of non-uniquely ergodic IETs.
- **Rauzy Induction** is a renormalization scheme for general IETs (‘79).

Almost every minimal IET with an irreducible permutation that isn’t a rotation is weak mixing (Avila & Forni ’04).

**Open Questions for DEMs**

1. Find examples of minimal DEMs
2. Develop a general renormalization theory
3. Understand ergodic properties
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**Open Questions for DEMs**
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Previous Work on DEMs

- Keane’s minimality condition generalized to REMs (Haller ’81)
  - Difficult to verify in practice
- Piecewise isometries in any dimension have zero topological entropy (Buzzi ‘01)
- Hooper found a 2-dimensional family of renormalizable REMs with periodic points (‘13)
- Schwartz used multigraphs to construct polytope exchange transformations (PETs) in every dimension and developed a renormalization theory for the Octagonal PETs (‘14)
- Yi constructed renormalizable triple lattice PETs (‘17)

Our Goal

Find a large family of minimal DEMs and develop their renormalization theory
Introduction

Cut-and-Project Domain Exchange Maps (DEM)

PV DEMs

Renormalization scheme for PV REMs

Multistage REMs

Further Work on REMs

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Cut-and-Project Sets

- $L$ a full-rank lattice in $\mathbb{R}^3$
- $X$ a domain in the $xy$-plane

Define:

$$\Lambda(X, L) = \{x \in L : \pi_{xy}(x) \in X\}$$

$$P = \{\pi_z(p) : p \in L \text{ and } \pi_{xy}(p) \in X\}.$$

Definition

$P$ is a **cut-and-project set** if

1. $\pi_z|_L$ is injective
2. $\pi_{xy}(L)$ is dense in $\mathbb{R}^2$. 

Figure: Lattice points $\Lambda(X, L)$
\( \Lambda(X, L) \) has a natural ordering: \( \{ \ldots, x_{-1}, x_0, x_1, \ldots \} \) where
\[
\pi_z(x_i) < \pi_z(x_j) \quad \text{for} \quad i < j
\]

Define \( \tilde{T} : \Lambda(X, L) \to \Lambda(X, L) \) by \( \tilde{T}(x_i) = x_{i+1} \)

\( \tilde{T} \) has finitely many translation vectors \( \mathcal{E} = \{ \eta_i \}_{i=0}^{N} \)

\( \pi_{xy} \circ \tilde{T} \) induces a DEM \( T : X \to X \) with translations \( v_i = \pi_{xy}(\eta_i), \; i = 0, \ldots, N \) with inherited ordering
Proposition

T is a DEM on X

Proof.

Partition X greedily into finitely many sets on which $\pi_{xy} \circ \tilde{T}$ is constant

Figure: Constructing the partition induced by $\pi_{xy} \circ \tilde{T}$

Let $f_v(x) = x + v$ for $x \in \mathbb{R}^2$ denote translation by $v \in \mathbb{R}^2$

$$A_0 = f_{v_0}^{-1}(X) \cap X \quad \text{and} \quad A_k = (f_{v_k}^{-1}(X) \cap X) \setminus \bigcup_{j=0}^{k-1} A_j \quad \text{for} \quad k = 1, \ldots, N$$
Theorem

Every well-defined orbit is dense and equidistributed in \( X \)

Proof:

\[
\begin{align*}
i : X & \rightarrow \mathbb{R}^3/L \\
\Phi_t(x, y, z) &= (x, y, z + t) \mod L \\
\tau_p &= \inf\{t > 0 \mid \Phi_t(i(p)) \in i(X)\}
\end{align*}
\]

- \( T \) is conjugate to the first return map to \( X \) of the vertical linear flow on \( \mathbb{R}^3/L \):

\[
(i \circ T)(p) = (\Phi_{\tau_p} \circ i)(p)
\]

- Orbits of vertical linear flow are dense and equidistributed by Weyl’s Equidistribution Theorem
Constructing renormalizable DEMs

**Idea:** use the algebraic structure of the lattice to find renormalizable cut-and-project DEMs

**Definition**

A Pisot-Vijayaraghavan or PV number is a real algebraic integer with modulus larger than 1 whose Galois conjugates have modulus strictly less than one.

For $n \geq 1$ the leading root of

$$x^3 - (n + 1)x^2 + nx - 1 = 0$$

is a PV number.
Constructing **renormalizable** DEMs

- **Idea:** use the algebraic structure of the lattice to find renormalizable cut-and-project DEMs

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For $n \geq 1$ the leading root of

$$x^3 - (n + 1)x^2 + nx - 1 = 0$$

is a PV number.
- $\lambda = \lambda_3$ a PV number with Galois conjugates $\lambda_1, \lambda_2 \in \mathbb{R}$
- Let $L$ be the Galois embedding $\mathbb{Z}[\lambda] \rightarrow \mathbb{R}^3$

\[ L = \langle (1, 1, 1), (\lambda_1, \lambda_2, \lambda_3), (\lambda_1^2, \lambda_2^2, \lambda_3^2) \rangle \]

\[ \Rightarrow \Lambda(X, L) \text{ is a cut-and-project set} \]

- $\mathbb{Z}[\lambda]$ can be identified with $\mathbb{Z}^3$

\[ (a, b, c) \mapsto a + b\lambda + c\lambda^2. \]

\[ \pi_{xy}(a + b\lambda + c\lambda^2) = (a + b\lambda_1 + c\lambda_1^2, a + b\lambda_2 + c\lambda_2^2) \]
\[ \pi_z(a + b\lambda + c\lambda^2) = a + b\lambda_3 + c\lambda_3^2 \]

**Definition**

A **PV DEM** is a cut-and-project DEM associated to the Galois embedding of a PV number with real Galois conjugates.
\( \lambda = \lambda_3 \) a PV number with Galois conjugates \( \lambda_1, \lambda_2 \in \mathbb{R} \)

- Let \( L \) be the Galois embedding \( \mathbb{Z}[\lambda] \hookrightarrow \mathbb{R}^3 \)

\[
L = \langle (1, 1, 1), (\lambda_1, \lambda_2, \lambda_3), (\lambda_1^2, \lambda_2^2, \lambda_3^2) \rangle
\]

\( \implies \) \( \Lambda(X, L) \) is a cut-and-project set

- \( \mathbb{Z}[\lambda] \) can be identified with \( \mathbb{Z}^3 \)

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(a, b, c) \mapsto a + b\lambda + c\lambda^2.
\]

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\pi_{xy}(a + b\lambda + c\lambda^2) = (a + b\lambda_1 + c\lambda_1^2, a + b\lambda_2 + c\lambda_2^2)
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\[
\pi_z(a + b\lambda + c\lambda^2) = a + b\lambda_3 + c\lambda_3^2
\]

**Definition**

A **PV DEM** is a cut-and-project DEM associated to the Galois embedding of a PV number with real Galois conjugates.
Choose the domain $X = [0, 1] \times [0, 1] \implies$ DEMs are REMs with rectilinear tiles

$$S = \left\{ M_n = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -n & n + 1 \end{bmatrix}, \ n \geq 6 \right\}$$

$$\text{det}(\lambda I - M_n) = \lambda^3 - (n + 1)\lambda^2 + n\lambda - 1$$

Fact: $0 < \lambda_1(M_n) < \lambda_2(M_n) < 1 < \lambda_3(M_n)$

$\lambda_3(M_n)$ is a PV number and determines a PV REM $T_{M_n}$

**Theorem**

*The PV REMs $\{ T_{M_n} \}_{n \geq 6}$ all have the same combinatorics*
Coordinates of the tiles for the PV REM $T_{Mn}$, $\lambda_i = \lambda_i(M_n)$

$A_0 = [1 - \lambda_1, 1] \times [1 - \lambda_2, 1]$
$A_1 = [0, 1 - \lambda_1] \times [0, 1 - \lambda_2]$
$A_2 = ([1 - 2\lambda_1, 1 - \lambda_1] \times [1 - \lambda_2, 2 - 2\lambda_2]) \cup ([1 - \lambda_1, 1] \times [0, 1 - \lambda_2])$
$A_3 = [0, 3\lambda_1 - \lambda_1^2] \times [-1 + 3\lambda_2 - \lambda_2^2, 1]$
$A_4 = [3\lambda_1 - \lambda_1^2, 1 - \lambda_1] \times [2\lambda_2 - \lambda_2^2, 1]$
$A_5 = [0, 2\lambda_1 - \lambda_1^2] \times [1 - \lambda_2, -1 + 3\lambda_2 - \lambda_2^2]$
$A_6 = ([1 - 2\lambda_1, 3\lambda_1 - \lambda_1^2] \times [2 - 2\lambda_2, -1 + 3\lambda_2 - \lambda_2^2])$
\hspace{1cm} $\cup ([2\lambda_1 - \lambda_1^2, 1 - 2\lambda_1] \times [1 - \lambda_2, -1 + 3\lambda_2 - \lambda_2^2])$
\hspace{1cm} $\cup ([3\lambda_1 - \lambda_1^2, 1 - \lambda_1] \times [2 - 2\lambda_2, 2\lambda_2 - \lambda_2^2]).$
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Figure: PV REM $T_{M_6}$ and the partition induced by the first return map $\hat{T}_{M_6}|_Y$ to $Y = A_0$
Theorem

Let

\( M \in S \) a matrix with eigenvalues \( 0 < \lambda_1 < \lambda_2 < 1 < \lambda_3 \)

\( T_M \) the associated PV REM

\( Y \subset X \) the tile in the partition corresponding to the rectangle \([1 - \lambda_1, 1] \times [1 - \lambda_2, 1] \).

\( T_M \) is renormalizable with

\[ \hat{T}_M|_Y = \phi^{-1} \circ T_M \circ \phi \]

where \( \phi : X \rightarrow Y \) is the affine map

\[ \phi : (x, y) \mapsto \left( \frac{x + \lambda_1 - 1}{\lambda_1}, \frac{y + \lambda_2 - 1}{\lambda_2} \right) \].
Proof of the renormalization scheme

\[ \Lambda(X, L) = \{(x, y, z) \in \mathbb{Z}^3 \mid \pi_{xy}(x, y, z) \in X\} \]
\[ \Lambda_Y = \{(x, y, z) \in \mathbb{Z}^3 \mid \pi_{xy}(x, y, z) \in Y\} \]

Define a bijection \( \psi : \Lambda(X, L) \rightarrow \Lambda_Y \)

\[ \psi : \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto (M_n)^T \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \]

which preserves the ordering of \( \Lambda(X, L) \)

\[ \pi_z(\omega_i) < \pi_z(\omega_j) \quad \text{if and only if} \quad \pi_z \circ \psi(\omega_i) < \pi_z \circ \psi(\omega_j). \]
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Multi-stage REMs

Combine PV REMs to form a **multistage** REM whose renormalization scheme has multiple stages.

**Conjecture**

The closure of the parameter space of all renormalizable multistage REMs is a Cantor set in $\mathbb{R}^4$.

**Figure:** The 4-dimensional parameter space of multi-stage REMs
Key lemma

As before let \( S = \left\{ \begin{array}{c}
M_n = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & -n & n+1
\end{bmatrix}, \ n \geq 6
\end{array} \right\} \)

Define the monoid \( \mathcal{M} = \langle S, \cdot \rangle \)

Lemma

If \( W \in \mathcal{M} \) then its eigenvalues \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are real and satisfy the inequalities

\[
0 < \lambda_1 < \lambda_2 < 1 < \lambda_3.
\]

\( \mathcal{M} \) is a Pisot monoid of Pisot matrices
Proof

- Find a change of basis $S$ so that $S^{-1}M_nS$ is primitive (has a positive power with positive entries)
- Perron-Frobenius theorem $\implies \lambda_3 > 1$
- Find a change of basis $Q$ so that $Q^{-1}M_{n^{-1}}Q$ is primitive
- Perron-Frobenius theorem $\implies \lambda_1 > 0$
- Let $P \in \mathcal{M}$ with characteristic polynomial

$$q_P(x) = x^3 - \text{Tr}(P)x^2 + b(P)x - 1$$

- Use induction on the length of the product and Cramer’s rule to show $b(P) < \text{Tr}(P)$
- Conclude $\lambda_2 < 1$
Admissible REMs

- A REM with the same combinatorics as the family of PV REMs associated to $S$
- Let $L$ be the Galois embedding of the eigenvalues of $M_n \in S$
- Let $E = \{\eta_i\}_{i=0}^6$ be the translation vectors associated to the dynamical system $\tilde{T}_{M_n} : \Lambda(X, L) \to \Lambda(X, L)$
- Let $W \in M$ with normalized eigenvectors $\xi_1 = (1, x, x')$ and $\xi_2 = (1, y, y')$, associated to eigenvalues $\lambda_1$ and $\lambda_2$ respectively

The REM $T_W$ defined by the translation vectors

$$V = \{v_i = \pi_{xy}(\eta_i), \text{ for } i = 0, 1, \ldots 6\}$$

where $\pi_{xy} : x \mapsto (x \cdot \xi_1, x \cdot \xi_2)$ is admissible if it has the same combinatorics as $T_{M_n}$
The tiles in the partition are given by

0. $A_0 = [1 - x, 1] \times [1 - y, 1]$

1. $A_1 = [0, 1 - x] \times [0, 1 - y]$

2. $A_2 = ([1 - 2x, 1 - x] \times [1 - y, 2 - 2y]) \cup ([1 - x, 1] \times [0, 1 - y])$

3. $A_3 = [0, 3x - x'] \times [-1 + 3y - y', 1]$

4. $A_4 = [3x - x', 1 - x] \times [2y - y', 1]$

5. $A_5 = [0, 2x - x'] \times [1 - y, -1 + 3y - y']$

6. $A_6 = ([1 - 2x, 3x - x'] \times [2 - 2y, -1 + 3y - y'])$
   $\cup ([2x - x', 1 - 2x] \times [1 - y, -1 + 3y - y'])$
   $\cup ([3x - x', 1 - x] \times [2 - 2y, 2y - y']).$

where

$\xi_1 = (1, x, x')$ and $\xi_2 = (1, y, y').$
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**Theorem**

*Admissible REMs are minimal.*

**Proof.**

Let $T : X \rightarrow X$ be an admissible REM

- Let $\xi_3$ be the eigenvector with eigenvalue $\lambda_3$. Define
  
  $\pi_z : x \mapsto x \cdot \xi_3$ and $P = \{ \pi_z(p) : p \in L \text{ and } \pi_{xy}(p) \in X \}$

- $P$ is a cut-and-project set
- $T$ defines the same REM as the cut-and-project REM constructed using $\Lambda(X, L)$
- Apply proof of minimality for cut-and-project REMs
Let $W = M_{n_L} \cdots M_{n_1} \in \mathcal{M}$ and $T_W$ be an admissible REM.

- $\xi_1$, $\xi_2$ be appropriately normalized eigenvectors of $W$.
- Define $W_1 = M_{n_1}$, $W_2 = M_{n_2} M_{n_1}$, \ldots $W = W_L = M_{n_L} \cdots M_{n_1}$.
- $\xi_i^k = W_k \xi_i$, for $i = 1, 2$.
- Consider projections $\pi_{xy}^k : x \mapsto (x \cdot \xi_1^k, x \cdot \xi_2^k)$.

For each $k$, $V_k = \{ v_i^k = \pi_{xy}^k(\eta_i), \text{ for } i = 0, 1, \ldots 6 \}$ defines a REM $T_k$.

### Definition

An admissible REM $T_W$ is a **multistage REM** if the induced REMs $T_1$, $T_2$, \ldots, $T_{L-1}$, $T_L$ all have the same combinatorics.
Let $T_W$ be a multi-stage REM with $W = M_{n_L} \cdots M_{n_2} M_{n_1}$.

**Theorem**

The multistage REM $T_W$ is renormalizable, i.e., for each $k$ there exists $Y_k \subset X$ and an affine map $\phi_k : Y_k \rightarrow X$ such that

$$\hat{T}_k|_{Y_k} = \phi_k^{-1} \circ T_{k+1} \circ \phi_k.$$  

**Figure:** The multistage REM $T_W$ and associated REMs $T_{W_1}$, $T_{W_2}$, $T_{W_3}$ and $T_{W_4} = T_W$ with $W = M_7 M_7 M_8 M_6$.  

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Each affine map has the form

$$\phi_k : (x, y) \mapsto \left( \frac{x + x_k - 1}{x_k}, \frac{y + y_k - 1}{y_k} \right)$$

where $x_k$ and $y_k$ are the dimensions of the tile in the partition corresponding to the rectangle $[1 - x_k, 1] \times [1 - y_k, 1]$.

$$W_1 = M_{n_1}, W_2 = M_{n_2} M_{n_1}, \ldots W = W_L = M_{n_L} \cdots M_{n_1}$$

$$\xi^k_i = W_k \xi_i, \text{ for } i = 1, 2$$

$$\xi^k_1 = (1, x_k, x'_k) \text{ and } \xi^k_2 = (1, y_k, y'_k)$$
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Extend cut-and-project DEMs to rank 4 lattices and degree 4 Perron numbers

(a) $\pi_\perp(\Lambda(X, L))$
(b) Spanning forest on $\pi_\perp(\Lambda(X, L))$
(c) A second spanning forest on $\pi_\perp(\Lambda(X, L))$

Figure: Lattice determined by roots of $x^4 - 4x^2 + x + 1$
(a) REM induced by previous figure (b)

(b) REM induced by previous figure (c)

Figure: Two REMs associated to $x^4 - 4x^2 + x + 1$
Open problems

- Study the full parameter space of REMs associated to matrices in $\mathcal{M}$
- Find a renormalization scheme for REMs associated to rank 4 lattices
- Use the renormalization scheme to construct minimal but non-uniquely ergodic REMs

**Figure:** The 4-dimensional parameter space of multi-stage REMs

Thank you!