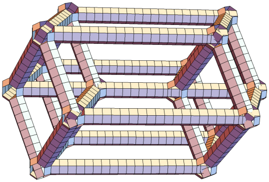
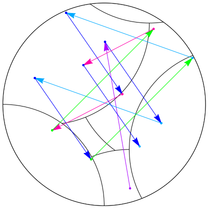
Renormalizable Rectangle Exchange Maps

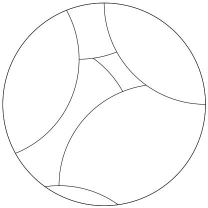
Ian M. Alevy

Department of Mathematics
University of Rochester

February 4, 2020







Outline

1. Introduction
   - Main results
   - Background
2. Cut-and-Project Domain Exchange Maps (DEMs)
3. PV DEMs
4. Renormalization scheme for PV REMs
5. Multistage REMs
6. Further Work on REMs

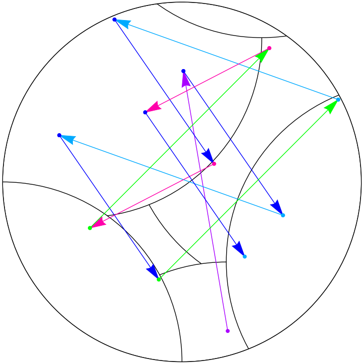
I. Alevy, R. Kenyon and R. Yi, *A Family of Minimal and Renormalizable Rectangle Exchange Maps*,
ArXiv e-prints (March 2018), 1803.06369,
to appear in *Ergodic Theory and Dynamical Systems*.

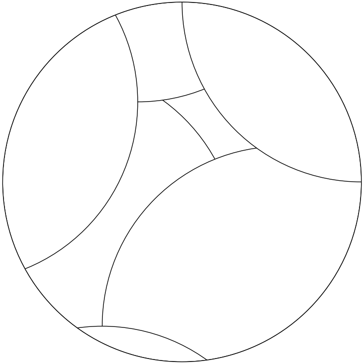
Outline

1. Introduction
   - Main results
   - Background
2. Cut-and-Project Domain Exchange Maps (DEMs)
3. PV DEMs
4. Renormalization scheme for PV REMs
5. Multistage REMs
6. Further Work on REMs

Domain exchange map (DEM)

- A dynamical system defined on a **smooth Jordan domain** which is a piecewise translation
- Joint work with **Richard Kenyon** and **Ren Yi**





$$T : D_1 \to D_1$$

$$T : x \mapsto x + v$$

Definition

Let X be a **Jordan domain** partitioned into smaller Jordan domains, with disjoint interiors, in two different ways

$$X = \bigcup_{k=0}^{N} A_k = \bigcup_{k=0}^{N} B_k$$

such that for each $k = 0, \ldots, N,\ \exists v_k \in \mathbb{R}^2$ with

$$A_k + v_k = B_k.$$

A **domain exchange map** is the dynamical system

$$T(x) = x + v_k \qquad \text{for} \qquad x \in \mathring{A}_k.$$

The map is not defined for points $x \in \bigcup_{k=0}^{N} \partial A_k$.

Setup: $T : X \to X$ a dynamical system and $Y \subsetneq X$

Definition

A dynamical system is **minimal** if every point has a dense orbit

Definition

The **first-return map** $\widehat{T}|_Y : Y \to Y$ is defined by

$$\widehat{T}|_Y(p) = T^m(p) \quad \text{where} \quad m = \min\{k \in \mathbb{N} : T^k(p) \in Y\}$$

for $p \in Y$.

If $X = [0,1]^2$ the DEM is a Rectangle Exchange Map (REM)

Setup: $T : X \to X$ a dynamical system and $Y \subsetneq X$

Definition

A dynamical system is **minimal** if every point has a dense orbit

Definition

The **first-return map** $\widehat{T}|_Y : Y \to Y$ is defined by

$$\widehat{T}|_Y(p) = T^m(p) \quad \text{where} \quad m = \min\{k \in \mathbb{N} : T^k(p) \in Y\}$$

for $p \in Y$.

If $X = [0, 1]^2$ the DEM is a Rectangle Exchange Map (REM)

Setup: $T : X \to X$ a dynamical system and $Y \subsetneq X$

Definition

A dynamical system is **minimal** if every point has a dense orbit

Definition

The **first-return map** $\widehat{T}|_Y : Y \to Y$ is defined by

$$\widehat{T}|_Y(p) = T^m(p) \quad \text{where} \quad m = \min\{k \in \mathbb{N} : T^k(p) \in Y\}$$

for $p \in Y$.

If $X = [0,1]^2$ the DEM is a Rectangle Exchange Map (REM)

Definition

A **renormalization scheme** is a proper subset $Y \subsetneq X$, a dynamical system $T' : X' \to X'$, and a homeomorphism $\phi : X \to X'$ such that

$$\widehat{T}|_Y = \phi^{-1} \circ T' \circ \phi.$$

If $T' = T$ the dynamical system is called **self-induced** or **renormalizable**.

Main Results (A., Kenyon, Yi 2018)

- Construct minimal DEMs on any domain with equidistributed orbits
- Find an infinite family of renormalizable REMs
- Compose REMs to produce **multistage** REMs with periodic renormalization schemes

Definition

A **renormalization scheme** is a proper subset $Y \subsetneq X$, a dynamical system $T' : X' \to X'$, and a homeomorphism $\phi : X \to X'$ such that

$$\widehat{T}|_Y = \phi^{-1} \circ T' \circ \phi.$$

If $T' = T$ the dynamical system is called **self-induced** or **renormalizable**.

Main Results (A., Kenyon, Yi 2018)

- Construct minimal DEMs on any domain with equidistributed orbits
- Find an infinite family of renormalizable REMs
- Compose REMs to produce **multistage** REMs with periodic renormalization schemes

Background

An **interval exchange transformation (IET)** $T : X \to X$ is a 1-dimensional DEM defined on an interval

1. Each probability vector and permutation (of matching length) defines an IET on $[0, 1]$
2. IETs are important examples in ergodic theory
3. Keane's minimality criterion ('75)
4. T is uniquely ergodic if the only invariant probability measure is a multiple of Lebesgue measure
    1. A measure μ is invariant with respect to T if $\mu(T^{-1}(A)) = \mu(A) \quad \forall A$
    2. Unique ergodicity $\implies$ orbits of points uniformly distributed
5. There exist minimal IETs ($n = 4$) which are not uniquely ergodic (Keane '77)
6. Almost every minimal IET with an irreducible permutation is uniquely ergodic (Masur/Veech '82)

Background

An **interval exchange transformation (IET)** $T : X \to X$ is a 1-dimensional DEM defined on an interval

1. Each probability vector and permutation (of matching length) defines an IET on $[0, 1]$
2. IETs are important examples in ergodic theory
3. Keane's minimality criterion ('75)
4. T is uniquely ergodic if the only invariant probability measure is a multiple of Lebesgue measure
    1. A measure μ is invariant with respect to T if $\mu(T^{-1}(A)) = \mu(A) \quad \forall A$
    2. Unique ergodicity $\implies$ orbits of points uniformly distributed
5. There exist minimal IETs ($n = 4$) which are not uniquely ergodic (Keane '77)
6. Almost every minimal IET with an irreducible permutation is uniquely ergodic (Masur/Veech '82)

Background

An **interval exchange transformation (IET)** $T : X \to X$ is a 1-dimensional DEM defined on an interval

1. Each probability vector and permutation (of matching length) defines an IET on $[0,1]$
2. IETs are important examples in ergodic theory
3. Keane's minimality criterion ('75)
4. T is uniquely ergodic if the only invariant probability measure is a multiple of Lebesgue measure
    1. A measure μ is invariant with respect to T if $\mu(T^{-1}(A)) = \mu(A) \quad \forall A$
    2. Unique ergodicity $\implies$ orbits of points uniformly distributed
5. There exist minimal IETs ($n = 4$) which are not uniquely ergodic (Keane '77)
6. Almost every minimal IET with an irreducible permutation is uniquely ergodic (Masur/Veech '82)

- The Hausdorff dimension of the set of non-uniquely ergodic IETs with n pieces is $n - 1 - 1/2$ (Chaika-Masur '18)
  - Proof uses Rauzy Induction to construct large sets of non-uniquely ergodic IETs
  - Rauzy Induction is a renormalization scheme for general IETs ('79)
- Almost every minimal IET with an irreducible permutation that isn't a rotation is weak mixing (Avila & Forni '04)

Open Questions for DEMs

1. Find examples of minimal DEMs
2. Develop a general renormalization theory
3. Understand ergodic properties

- The Hausdorff dimension of the set of non-uniquely ergodic IETs with n pieces is $n - 1 - 1/2$ (Chaika-Masur '18)
  - Proof uses Rauzy Induction to construct large sets of non-uniquely ergodic IETs
  - Rauzy Induction is a renormalization scheme for general IETs ('79)
- Almost every minimal IET with an irreducible permutation that isn't a rotation is weak mixing (Avila & Forni '04)

Open Questions for DEMs

1. Find examples of minimal DEMs
2. Develop a general renormalization theory
3. Understand ergodic properties

Previous Work on DEMs

- Keane’s minimality condition generalized to REMs (Haller ’81)
  - Difficult to verify in practice
- Piecewise isometries in any dimension have zero topological entropy (Buzzi ‘01)
- Hooper found a 2-dimensional family of renormalizable REMs with periodic points (‘13)
- Schwartz used multigraphs to construct polytope exchange transformations (PETs) in every dimension and developed a renormalization theory for the Octagonal PETs (‘14)
- Yi constructed renormalizable triple lattice PETs (‘17)

Our Goal

Find a large family of minimal DEMs and develop their renormalization theory

Outline

1. Introduction
   - Main results
   - Background
2. Cut-and-Project Domain Exchange Maps (DEMs)
3. PV DEMs
4. Renormalization scheme for PV REMs
5. Multistage REMs
6. Further Work on REMs

Cut-and-Project Sets

- L a full-rank lattice in $\mathbb{R}^3$
- X a domain in the xy-plane

Define:

$$\Lambda(X, L) = \{x \in L : \pi_{xy}(x) \in X\}$$
$$P = \{\pi_z(p) \ : \ p \in L \text{ and } \pi_{xy}(p) \in X\}.$$

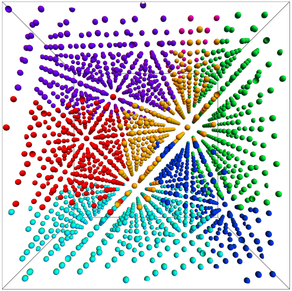


Figure: Lattice points $\Lambda(X, L)$

Definition

P is a **cut-and-project set** if

1. $\pi_z|_L$ is injective
2. $\pi_{xy}(L)$ is dense in $\mathbb{R}^2$.

- $\Lambda(X, L)$ has a natural ordering: $\{\ldots, x_{-1}, x_0, x_1, \ldots\}$ where

$$\pi_z(x_i) < \pi_z(x_j) \quad \text{for} \quad i < j$$

- Define $\widetilde{T} : \Lambda(X, L) \to \Lambda(X, L)$ by $\widetilde{T}(x_i) = x_{i+1}$

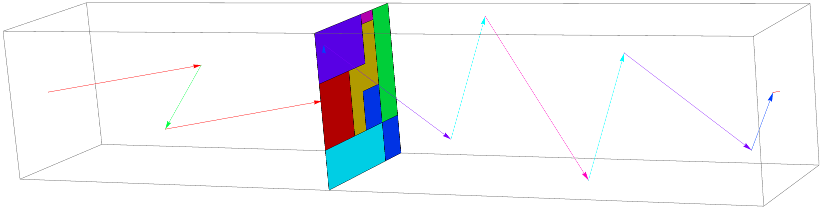


Figure: $\widetilde{T}$ and the partition associated to the induced DEM T

- $\widetilde{T}$ has finitely many translation vectors $\mathcal{E} = \{\eta_i\}_{i=0}^{N}$
- $\pi_{xy} \circ \widetilde{T}$ induces a DEM $T : X \to X$ with translations $v_i = \pi_{xy}(\eta_i)$, $i = 0, \ldots, N$ with inherited ordering

Proposition

T is a DEM on X

Proof.

Partition X greedily into finitely many sets on which $\pi_{xy} \circ \widetilde{T}$ is constant □

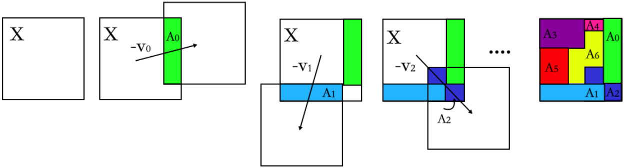


Figure: Constructing the partition induced by $\pi_{xy} \circ \widetilde{T}$

Let $f_v(x) = x + v$ for $x \in \mathbb{R}^2$ denote translation by $v \in \mathbb{R}^2$

$$A_0 = f_{v_0}^{-1}(X) \cap X \quad \text{and} \quad A_k = (f_{v_k}^{-1}(X) \cap X) \setminus \bigcup_{j=0}^{k-1} A_j \quad \text{for} \quad k = 1, \ldots, N$$

Theorem

Every well-defined orbit is dense and equidistributed in X

Proof:

$$
\begin{aligned}
& i : X \to \mathbb{R}^3/L \\
& \Phi_t(x, y, z) = (x, y, z + t) \mod L \\
& \tau_p = \inf\{t > 0 \mid \Phi_t(i(p)) \in i(X)\}
\end{aligned}
$$

- T is conjugate to the first return map to X of the vertical linear flow on $\mathbb{R}^3/L$:

$$(i \circ T)(p) = (\Phi_{\tau_p} \circ i)(p)$$

- Orbits of vertical linear flow are dense and equidistributed by Weyl's Equidistribution Theorem

Outline

1. Introduction
   - Main results
   - Background
2. Cut-and-Project Domain Exchange Maps (DEMs)
3. PV DEMs
4. Renormalization scheme for PV REMs
5. Multistage REMs
6. Further Work on REMs

Constructing renormalizable DEMs

- Idea: use the algebraic structure of the lattice to find renormalizable cut-and-project DEMs

Definition

A **Pisot-Vijayaraghavan or PV number** is a real algebraic integer with modulus larger than 1 whose Galois conjugates have modulus strictly less than one.

For $n \geq 1$ the leading root of

$$x^3 - (n+1)x^2 + nx - 1 = 0$$

is a PV number

Constructing renormalizable DEMs

- Idea: use the algebraic structure of the lattice to find renormalizable cut-and-project DEMs

Definition

A **Pisot-Vijayaraghavan or PV number** is a real algebraic integer with modulus larger than 1 whose Galois conjugates have modulus strictly less than one.

For $n \geq 1$ the leading root of

$$x^3 - (n+1)x^2 + nx - 1 = 0$$

is a PV number

- $\lambda = \lambda_3$ a PV number with Galois conjugates $\lambda_1, \lambda_2 \in \mathbb{R}$
- Let L be the Galois embedding $\mathbb{Z}[\lambda] \hookrightarrow \mathbb{R}^3$

$$L = \langle (1,1,1), (\lambda_1, \lambda_2, \lambda_3), (\lambda_1^2, \lambda_2^2, \lambda_3^2) \rangle$$

$\implies \Lambda(X, L)$ is a cut-and-project set

- $\mathbb{Z}[\lambda]$ can be identified with $\mathbb{Z}^3$

$$(a, b, c) \mapsto a + b\lambda + c\lambda^2.$$

$$\pi_{xy}(a + b\lambda + c\lambda^2) = (a + b\lambda_1 + c\lambda_1^2, a + b\lambda_2 + c\lambda_2^2)$$
$$\pi_z(a + b\lambda + c\lambda^2) = a + b\lambda_3 + c\lambda_3^2$$

Definition

A **PV DEM** is a cut-and-project DEM associated to the Galois embedding of a PV number with real Galois conjugates

- $\lambda = \lambda_3$ a PV number with Galois conjugates $\lambda_1, \lambda_2 \in \mathbb{R}$
- Let L be the Galois embedding $\mathbb{Z}[\lambda] \hookrightarrow \mathbb{R}^3$

$$L = \langle (1,1,1), (\lambda_1, \lambda_2, \lambda_3), (\lambda_1^2, \lambda_2^2, \lambda_3^2) \rangle$$

$\implies \Lambda(X, L)$ is a cut-and-project set

- $\mathbb{Z}[\lambda]$ can be identified with $\mathbb{Z}^3$

$$(a, b, c) \mapsto a + b\lambda + c\lambda^2.$$

$$\pi_{xy}(a + b\lambda + c\lambda^2) = (a + b\lambda_1 + c\lambda_1^2, a + b\lambda_2 + c\lambda_2^2)$$
$$\pi_z(a + b\lambda + c\lambda^2) = a + b\lambda_3 + c\lambda_3^2$$

Definition

A **PV DEM** is a cut-and-project DEM associated to the Galois embedding of a PV number with real Galois conjugates

- Choose the domain $X = [0,1] \times [0,1] \implies$ DEMs are **REMs** with rectilinear tiles

$$\mathcal{S} = \left\{ M_n = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -n & n+1 \end{bmatrix}, \; n \geq 6 \right\}$$

$$\det(\lambda I - M_n) = \lambda^3 - (n+1)\lambda^2 + n\lambda - 1$$

Fact: $0 < \lambda_1(M_n) < \lambda_2(M_n) < 1 < \lambda_3(M_n)$

$\lambda_3(M_n)$ is a PV number and determines a PV REM T_{M_n}

Theorem

The PV REMs $\{T_{M_n}\}_{n \geq 6}$ all have the same combinatorics

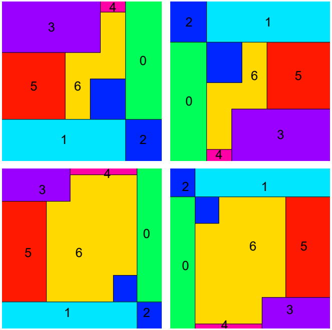


Figure: Two REMs with the same combinatorics

Coordinates of the tiles for the PV REM $T_{M_n}, \ \lambda_i = \lambda_i(M_n)$

$$A_0 = [1 - \lambda_1, 1] \times [1 - \lambda_2, 1]$$
$$A_1 = [0, 1 - \lambda_1] \times [0, 1 - \lambda_2]$$
$$A_2 = ([1 - 2\lambda_1, 1 - \lambda_1] \times [1 - \lambda_2, 2 - 2\lambda_2]) \cup ([1 - \lambda_1, 1] \times [0, 1 - \lambda_2])$$
$$A_3 = [0, 3\lambda_1 - \lambda_1^2] \times [-1 + 3\lambda_2 - \lambda_2^2, 1]$$
$$A_4 = [3\lambda_1 - \lambda_1^2, 1 - \lambda_1] \times [2\lambda_2 - \lambda_2^2, 1]$$
$$A_5 = [0, 2\lambda_1 - \lambda_1^2] \times [1 - \lambda_2, -1 + 3\lambda_2 - \lambda_2^2]$$
$$A_6 = \left([1 - 2\lambda_1, 3\lambda_1 - \lambda_1^2] \times [2 - 2\lambda_2, -1 + 3\lambda_2 - \lambda_2^2]\right) \cup \left([2\lambda_1 - \lambda_1^2, 1 - 2\lambda_1] \times [1 - \lambda_2, -1 + 3\lambda_2 - \lambda_2^2]\right) \cup \left([3\lambda_1 - \lambda_1^2, 1 - \lambda_1] \times [2 - 2\lambda_2, 2\lambda_2 - \lambda_2^2]\right).$$

Outline

1. Introduction
    - Main results
    - Background
2. Cut-and-Project Domain Exchange Maps (DEMs)
3. PV DEMs
4. Renormalization scheme for PV REMs
5. Multistage REMs
6. Further Work on REMs

Renormalization scheme

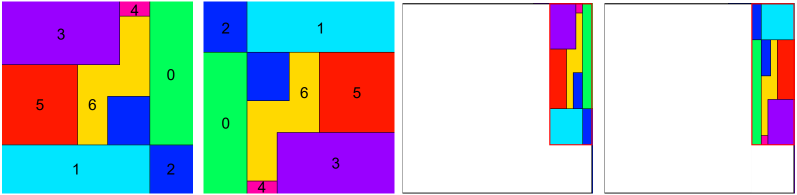


Figure: PV REM T_{M_6} and the partition induced by the first return map $\hat{T}_{M_6}|_Y$ to $Y = A_0$

Theorem

Let

- $M \in \mathcal{S}$ *a matrix with eigenvalues* $0 < \lambda_1 < \lambda_2 < 1 < \lambda_3$
- T_M *the associated PV REM*
- $Y \subset X$ *the tile in the partition corresponding to the rectangle* $[1-\lambda_1, 1] \times [1-\lambda_2, 1]$.

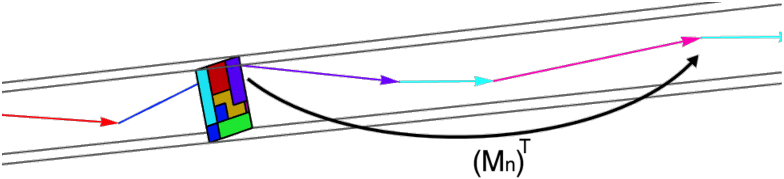
T_M *is* ***renormalizable*** *with*

$$\widehat{T}_M|_Y = \phi^{-1} \circ T_M \circ \phi$$

where $\phi : X \to Y$ *is the affine map*

$$\phi : (x, y) \mapsto \left(\frac{x + \lambda_1 - 1}{\lambda_1}, \frac{y + \lambda_2 - 1}{\lambda_2} \right).$$

Proof of the renormalization scheme



$$\Lambda(X, L) = \{(x, y, z) \in \mathbb{Z}^3 \mid \pi_{xy}(x, y, z) \in X\}$$
$$\Lambda_Y = \{(x, y, z) \in \mathbb{Z}^3 \mid \pi_{xy}(x, y, z) \in Y\}$$

Define a bijection $\Psi : \Lambda(X, L) \to \Lambda_Y$

$$\Psi : \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto (M_n)^T \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

which preserves the ordering of $\Lambda(X, L)$

$$\pi_z(\omega_i) < \pi_z(\omega_j) \quad \text{if and only if} \quad \pi_z \circ \Psi(\omega_i) < \pi_z \circ \Psi(\omega_j).$$

Outline

1. Introduction
   - Main results
   - Background
2. Cut-and-Project Domain Exchange Maps (DEMs)
3. PV DEMs
4. Renormalization scheme for PV REMs
5. Multistage REMs
6. Further Work on REMs

Multi-stage REMs

Combine PV REMs to form a **multistage** REM whose renormalization scheme has multiple stages

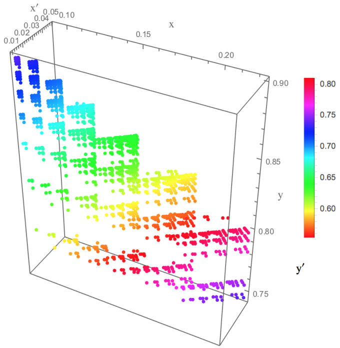


Figure: The 4-dimensional parameter space of multi-stage REMs

Conjecture

The closure of the parameter space of all renormalizable multistage REMs is a Cantor set in $\mathbb{R}^4$.

Key lemma

As before let $\mathcal{S} = \left\{ M_n = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -n & n+1 \end{bmatrix},\ n \geq 6 \right\}$

Define the monoid $\mathcal{M} = \langle \mathcal{S}, \cdot \rangle$

Lemma

If $W \in \mathcal{M}$ then its eigenvalues λ_1, λ_2 and λ_3 are real and satisfy the inequalities

$$0 < \lambda_1 < \lambda_2 < 1 < \lambda_3.$$

$\mathcal{M}$ is a **Pisot monoid** of **Pisot matrices**

Proof

- Find a change of basis S so that $S^{-1}M_nS$ is primitive (has a positive power with positive entries)
- Perron-Frobenius theorem $\implies \lambda_3 > 1$
- Find a change of basis Q so that $Q^{-1}M_n^{-1}Q$ is primitive
- Perron-Frobenius theorem $\implies \lambda_1 > 0$
- Let $P \in \mathcal{M}$ with characteristic polynomial

$$q_P(x) = x^3 - \mathrm{Tr}(P)x^2 + b(P)x - 1$$

- Use induction on the length of the product and Cramer's rule to show $b(P) < \mathrm{Tr}(P)$
- Conclude $\lambda_2 < 1$

Admissible REMs

- A REM with the same combinatorics as the family of PV REMs associated to $\mathcal{S}$
- Let L be the Galois embedding of the eigenvalues of $M_n \in \mathcal{S}$
- Let $\mathcal{E} = \{\eta_i\}_{i=0}^{6}$ be the translation vectors associated to the dynamical system $\widetilde{T}_{M_n} : \Lambda(X, L) \to \Lambda(X, L)$
- Let $W \in \mathcal{M}$ with normalized eigenvectors

$$\xi_1 = (1, x, x') \quad \text{and} \quad \xi_2 = (1, y, y'),$$

  associated to eigenvalues λ_1 and λ_2 respectively

The REM T_W defined by the translation vectors

$$V = \{v_i = \pi_{xy}(\eta_i), \text{ for } i = 0, 1, \ldots 6\}$$

where $\pi_{xy} : x \mapsto (x \cdot \xi_1,\ x \cdot \xi_2)$ is admissible if it has the same combinatorics as T_{M_n}

The tiles in the partition are given by

0. $A_0 = [1-x, 1] \times [1-y, 1]$
1. $A_1 = [0, 1-x] \times [0, 1-y]$
2. $A_2 = ([1-2x, 1-x] \times [1-y, 2-2y]) \cup ([1-x, 1] \times [0, 1-y])$
3. $A_3 = [0, 3x-x'] \times [-1+3y-y', 1]$
4. $A_4 = [3x-x', 1-x] \times [2y-y', 1]$
5. $A_5 = [0, 2x-x'] \times [1-y, -1+3y-y']$
6. $A_6 = \big([1-2x, 3x-x'] \times [2-2y, -1+3y-y']\big) \cup \big([2x-x', 1-2x] \times [1-y, -1+3y-y']\big) \cup \big([3x-x', 1-x] \times [2-2y, 2y-y']\big).$

where

$$\xi_1 = (1, x, x') \quad \text{and} \quad \xi_2 = (1, y, y').$$

Theorem

Admissible REMs are minimal.

Proof.

Let $T : X \to X$ be an admissible REM

- Let ξ_3 be the eigenvector with eigenvalue λ_3. Define

$$\pi_z : x \mapsto x \cdot \xi_3 \text{ and } P = \{\pi_z(p) \ : \ p \in L \text{ and } \pi_{xy}(p) \in X\}$$

- P is a cut-and-project set
- T defines the same REM as the cut-and-project REM constructed using $\Lambda(X, L)$
- Apply proof of minimality for cut-and-project REMs

□

- Let $W = M_{n_L} \cdots M_{n_1} \in \mathcal{M}$ and T_W be an admissible REM
- ξ_1, ξ_2 be appropriately normalized eigenvectors of W
- Define $W_1 = M_{n_1}, W_2 = M_{n_2} M_{n_1}, \ldots W = W_L = M_{n_L} \cdots M_{n_1}$
- $\xi_i^k = W_k \xi_i$, for $i = 1, 2$
- Consider projections $\pi_{xy}^k : x \mapsto (x \cdot \xi_1^k, x \cdot \xi_2^k)$

For each k ,$V_k = \{v_i^k = \pi_{xy}^k(\eta_i),$ for $i = 0, 1, \ldots 6\}$ defines a REM T_k

Definition

An admissible REM T_W is a **multistage REM** if the induced REMs $T_1, T_2, \ldots, T_{L-1}, T_L$ all have the same combinatorics

Let T_W be a multi-stage REM with $W = M_{n_L} \cdots M_{n_2} M_{n_1}$.

Theorem

The multistage REM T_W is renormalizable, i.e., for each k there exists $Y_k \subset X$ and an affine map $\phi_k : Y_k \to X$ such that

$$\widehat{T}_k|_{Y_k} = \phi_k^{-1} \circ T_{k+1} \circ \phi_k .$$

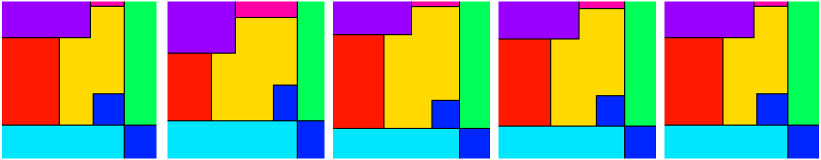


Figure: The multistage REM T_W and associated REMs $T_{W_1}, T_{W_2}, T_{W_3}$ and $T_{W_4} = T_W$ with $W = M_7 M_7 M_8 M_6$.

Each affine map has the form

$$\phi_k : (x, y) \mapsto \left(\frac{x + x_k - 1}{x_k}, \frac{y + y_k - 1}{y_k} \right)$$

where x_k and y_k are the dimensions of the tile in the partition corresponding to the rectangle $[1 - x_k, 1] \times [1 - y_k, 1]$.

$$W_1 = M_{n_1}, W_2 = M_{n_2} M_{n_1}, \ldots W = W_L = M_{n_L} \cdots M_{n_1}$$

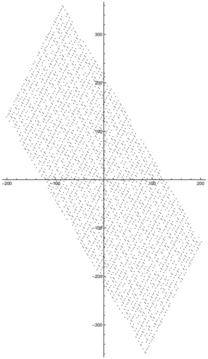
$$\xi_i^k = W_k \xi_i, \text{ for } i = 1, 2$$

$$\xi_i^k = (1, x_k, x_k') \text{ and } \xi_2^k = (1, y_k, y_k')$$

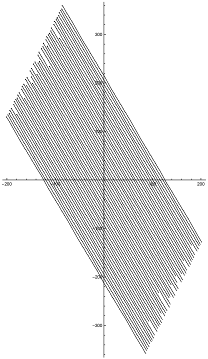
Outline

1. Introduction
   - Main results
   - Background
2. Cut-and-Project Domain Exchange Maps (DEMs)
3. PV DEMs
4. Renormalization scheme for PV REMs
5. Multistage REMs
6. Further Work on REMs

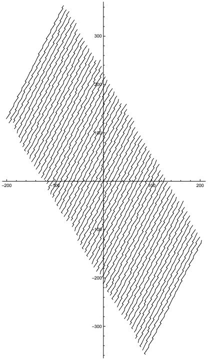
Extend cut-and-project DEMs to rank 4 lattices and degree 4 **Perron numbers**



(a) $\pi_\perp(\Lambda(X, L))$

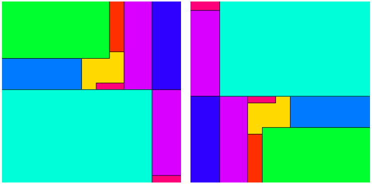


(b) Spanning forest on $\pi_\perp(\Lambda(X, L))$

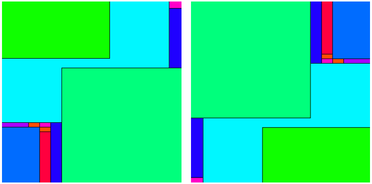


(c) A second spanning forest on $\pi_\perp(\Lambda(X, L))$

Figure: Lattice determined by roots of $x^4 - 4x^2 + x + 1$



(a) REM induced by previous figure (b)



(b) REM induced by previous figure (c)

Figure: Two REMs associated to $x^4 - 4x^2 + x + 1$

Open problems

- Study the full parameter space of REMs associated to matrices in $\mathcal{M}$
- Find a renormalization scheme for REMs associated to rank 4 lattices
- Use the renormalization scheme to construct minimal but non-uniquely ergodic REMs

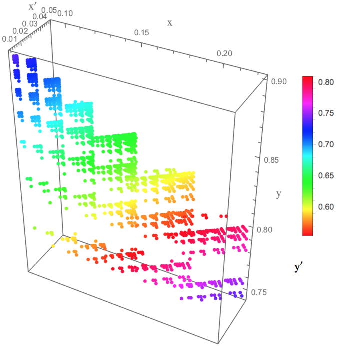


Figure: The 4-dimensional parameter space of multi-stage REMs

Thank you!