Inequalities for posets

Asymptotic Algebraic Combinatorics, UCLA

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- Hook Length Formula:

$$e(\mathcal{P}) = \frac{n!}{\prod_{x \in \mathcal{P}} h(x)}$$

true when $\mathcal{P}$ is a Young diagram, or an arborescence

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Sketch: choose i.i.d. $\eta(x) \in [0, 1]$ for all $x \in \mathcal{P}$.
$E(x) : \eta(x) = \min_{y \in I(x)} \eta(y)$. Then

$$\frac{e(\mathcal{P})}{n!} = \mathrm{prob} \cap_x E(x) = \prod_{i=1}^{n} \mathrm{prob}(E(x_i)|E(x_1), \ldots, E(x_{i-1})) \geqslant$$

$$\prod_{i=1}^{n} \mathrm{prob}(E(x_i)) = \prod_{i=1}^{n} \frac{1}{|I(x_i)|}$$

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$\mathcal{P}$ — Young diagram, possibly multidimensional, an arborescence, product of arborescences etc.

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- $A = (a_1, \dots, a_n)$, $B = (b_1, \dots, b_n)$,

$$(b_1, \dots, b_n) \in \operatorname{conv}\left\{ \left(a_{\pi(1)}, \dots, a_{\pi(n)}\right) : \pi \in S_n \right\}$$

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- ($d = 1$) for all $k = 1, 2, \ldots$ the sum of k max elements in A is not less than in B
- ($d = 1$) B is obtained from A by the smoothing together changes $(x, y) \to (x + \varepsilon, y - \varepsilon)$, $0 < \varepsilon \leqslant (y - x)/2$

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$$A = \sum \lambda_i P_i \leqslant \lambda_1 P_1 + \ldots + \lambda_k P_k$$

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- (Malamud, Pereira, 2003, conjectured by de Bruijn, Springer in 1947) $z_1, \ldots, z_n$ — roots of $F(z)$, $w_1, \ldots, w_{n-1}$ of $F'(z)$.
  $$(n-1) \cdot \{z_1, \ldots, z_n\} \succ n \cdot \{w_1, \ldots, w_{n-1}\}$$

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(Sra 2015, conjectured by Cuttler, Greene and Skandera in 2011)

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Take a set X of squares. Need: Y, $|Y| = |X|$, $\sum_Y h \geqslant \sum_X h^*$

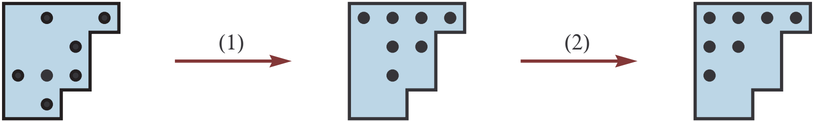

Figure: Shuffling of 7 squares in $\lambda = (4, 3, 3, 2)$.

Chains and antichains covering

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$$\frac{n!}{\prod |C_i|!} \geqslant e(\mathcal{P}) \geqslant \prod |A_i|! \qquad (*)$$

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Theorem (Bochkov, P. 2019):

$$\frac{n!}{\prod c_i!} \geqslant e(P) \geqslant \prod_i a_i!$$

Corollary. $(*)$ holds for arbitrary antichains.

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$$e(\mathcal{P}) \geqslant \sum_{x \in A} e(\mathcal{P} - x)$$

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Therefore by induction proposition we have $e(\mathcal{P} - x) \geqslant \frac{1}{a_1} \prod_i a_i!$, for all $x \in A$. Sum up over all $x \in A$ and get $e(\mathcal{P}) \geqslant \prod_i a_i!$,

Upper bound

Lemma. A is the antichain of maximal elements. $c_1 \geqslant c_2 \geqslant \dots$ are the chain GKF parameters of $\mathcal{P}$. Then the elements of A may be enumerated as $x_1, x_2, \dots, x_{|A|}$ so that for all $i = 1, 2, \dots, |A|$ there exist i chains whose maximal elements belong to the set $\{x_1, \dots, x_i\} \cup (\mathcal{P} \setminus A)$ with total size $c_1 + \dots + c_i$.

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The multiset $\{r_1, r_2, \dots\}$ of GKF chain parameters of $\mathcal{P} \setminus \{x_i\}$ majorizes the multiset $\{c_1, c_2, \dots, c_{j-1}, c_j - 1, c_{j+1}, \dots\}$. By Karamata $\prod r_j! \geqslant \frac{1}{c_j} \prod_j c_j!$, and by induction

$$e(\mathcal{P} - x_i) \leqslant \frac{(n-1)!}{\prod r_j!} \leqslant \frac{c_j(n-1)!}{\prod c_j!}.$$

Sum up this by all i and apply $e(\mathcal{P}) = \sum_i e(\mathcal{P} - x_i)$.

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For any two conjugate partitions $a_1 \geqslant a_2 \geqslant \ldots$ and $c_1 \geqslant c_2 \geqslant \ldots$ of the positive integer n onto positive integer parts the inequality

$$\prod_i a_i! \prod_i c_i! \geqslant \left(\frac{n}{eH_n}\right)^n = n! e^{-n\log\log n + o(n)}$$

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$$\prod_i c_i! = \prod_k k^{|i:c_i \geqslant k|} = \prod_k k^{a_k},$$

thus the inequality to prove is

$$\prod_k k^{a_k} a_k! \geqslant \left(\frac{n}{eH_n}\right)^n$$

What does it give for Young diagrams?

Antichain GKF parameters: sizes of antidiagonals
$\{x : h^*(x) = \mathrm{const}\}$

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$\tilde{h}(x)$: put 1's on the maximal antidiagonal, 2 on the second largest antidiagonal etc. So we get

$$\prod \tilde{h}(x) \leqslant \prod h(x) \leqslant \prod h^*(x)$$

for any Young diagram.

Thank you!