Inequalities for posets
Asymptotic Algebraic Combinatorics, UCLA

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Posets

A poset, $(P, \leq)$ — a partially ordered set, $|P| = n < \infty$

- **Chain**: a linearly ordered subset $a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_k$

- **Antichain**: a subset of mutually incomparable elements

- **Linear extension**: a way to introduce new inequalities so that the new order is linear: $P = \{x_1, x_2, \ldots, x_n\}$, $x_i \leq x_j \Rightarrow i \leq j$.

- **Hook**: $H(x) = \{y \geq x: [x, y] \text{ is a chain}\}$, $h(x) = |H(x)|$.

- **Antihook**: $H^*(x) = \{y \leq x: [y, x] \text{ is a chain}\}$, $h^*(x) = |H^*(x)|$.

- **Hook Length Formula**: $e(P) = \frac{n!}{\prod_{x \in P} h(x)}$ true when $P$ is a Young diagram, or an arborescence.
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$$e(\mathcal{P}) = \frac{n!}{\prod_{x \in \mathcal{P}} h(x)}$$

true when $\mathcal{P}$ is a Young diagram, or an arborescence.
Hammett–Pittel inequality
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if \( I(x) = \{y : y \geq x\} \) (note that \( |I(x)| \geq |H(x)| \)), then

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|e(\mathcal{P})| \geq \frac{n!}{\prod_{x \in \mathcal{P}} |I(x)|}
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Sketch: choose i.i.d. \( \eta(x) \in [0, 1] \) for all \( x \in \mathcal{P} \).
Hammett–Pittel inequality

if \( I(x) = \{y : y \geq x\} \) (note that \(|I(x)| \geq |H(x)|\)), then

\[ |e(P)| \geq \frac{n!}{\prod_{x \in P} |I(x)|} \]

Sketch: choose i.i.d. \( \eta(x) \in [0, 1] \) for all \( x \in \mathcal{P} \).

\( E(x) : \eta(x) = \min_{y \in I(x)} \eta(y) \). Then

\[ \frac{e(P)}{n!} = \text{prob} \cap_{x} E(x) = \prod_{i=1}^{n} \text{prob}(E(x_i)|E(x_1), \ldots, E(x_{i-1})) \geq \prod_{i=1}^{n} \text{prob}(E(x_i)) = \prod_{i=1}^{n} \frac{1}{|I(x_i)|} \]
Hooks vs. antihooks

Morales – Pak – Panova, Swanson (2018): for Young diagram
\[ \prod_{x \in \mathcal{P}} h(x) \leq \prod_{x \in \mathcal{P}} h^*(x) \].

Pak – P. – Sokolov (2019): for any convex \( f \),
\[ \sum_{x \in \mathcal{P}} f(h(x)) \geq \sum_{x \in \mathcal{P}} f(h^*(x)) \].

\( \mathcal{P} \) — Young diagram, possibly multidimensional, an arborescence, product of arborescences etc.
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Majorization

A, B—two finite multisets in \( \mathbb{R} \) (or \( \mathbb{R}^d \)).

\[ A \succ B \]: ▶ (Karamata inequality) for any convex \( f \),

\[ \sum A f \geq \sum B f \](⇒|\( A \)|=|\( B \)|, \( \sum A = \sum B \))

▶ \( A = (a_1, \ldots, a_n), B = (b_1, \ldots, b_n) \),

\( (b_1, \ldots, b_n) \in \text{conv}\{ (a_\pi(1), \ldots, a_\pi(n)) : \pi \in S_n \} \)

▶ there exists a bistochastic matrix \( M \) such that

\[ b_i = \sum_j m_{ij} a_j \]

▶ (\( d = 1 \)) for all \( k = 1, 2, \ldots \), the sum of \( k \) max elements in \( A \) is not less than in \( B \)

▶ (\( d = 1 \)) \( B \) is obtained from \( A \) by the smoothing together changes (\( x, y \)→(\( x + \varepsilon, y - \varepsilon \)), \( 0 < \varepsilon \leq \frac{(y - x)}{2} \))
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- ($d = 1$) $B$ is obtained from $A$ by the smoothing together changes $(x, y) \rightarrow (x + \varepsilon, y - \varepsilon), 0 < \varepsilon \leq (y - x)/2$
Examples

Schur, 1923: A is an $n \times n$ Hermitian matrix, then eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ majorize diagonal elements.

$\lambda_k \geq 0 \geq \lambda_k + 1,\ldots,
\sum \lambda_i P_i \leq \lambda_1 P_1 + \ldots + \lambda_k P_k d_1 + \ldots + d_k \leq \text{tr}(\lambda_1 P_1 + \ldots + \lambda_k P_k) = \lambda_1 + \ldots + \lambda_k$.

Malamud, Pereira, 2003, conjectured by de Bruijn, Springer in 1947: $z_1, \ldots, z_n$ — roots of $F(z)$, $w_1, \ldots, w_{n-1}$ of $F'(z)$.

$(n-1) \cdot \{z_1, \ldots, z_n\} \succ n \cdot \{w_1, \ldots, w_{n-1}\}$.
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Important applications
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$x_i \geq 0, \lambda \succ \nu$, then

(Muirhead 1903) $\operatorname{Sym} x_1^{\lambda_1} \ldots x_n^{\lambda_n} \geq \operatorname{Sym} x_1^{\nu_1} \ldots x_n^{\nu_n}$. 

Kostka numbers: $K_{\mu, \lambda} \leq K_{\mu, \nu}$
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\[ (\text{Sra 2015, conjectured by Cuttler, Greene and Skandera in 2011}) \]

\[ \frac{s_\lambda(x_1, \ldots, x_n)}{s_\lambda(1, \ldots, 1)} \geq \frac{s_\nu(x_1, \ldots, x_n)}{s_\nu(1, \ldots, 1)} \]
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Kostka numbers: \( K_{\mu,\lambda} \leq K_{\mu,\nu} \)
Why do hooks majorize antihooks?

Take a set $X$ of squares. Need: $Y$, $|Y| = |X|$, $\sum Y h \geq \sum X h$. (1) (2)

Figure: Shuffling of 7 squares in $\lambda = (4, 3, 3, 2)$. 
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**Figure:** Shuffling of 7 squares in $\lambda = (4, 3, 3, 2)$. 
Chains and antichains covering

\[ \mathcal{P} = A_1 \sqcup A_2 \sqcup A_3 \ldots, \] where \( A_i \) is the antichain of elements with rank \( i \).
Chains and antichains covering

\( P = A_1 \sqcup A_2 \sqcup A_3 \ldots \), where \( A_i \) is the antichain of elements with rank \( i \). Also \( P = C_1 \sqcup C_2 \sqcup C_3 \ldots \), \( C_i \) are chains. Then

\[
\frac{n!}{\prod |C_i|!} \geq e(P) \geq \prod |A_i|!
\]  

(\*)

Theorem (GK, F). \((a_i)\) and \((c_i)\) are conjugate partitions.

Theorem (Bochkov, P. 2019):

\[
\frac{n!}{\prod |C_i|!} \geq e(P) \geq \prod |A_i|!
\]

Corollary. (\*)(\*) holds for arbitrary antichains.
Chains and antichains covering

\[ \mathcal{P} = A_1 \sqcup A_2 \sqcup A_3 \ldots, \text{ where } A_i \text{ is the antichain of elements with rank } i. \text{ Also } \mathcal{P} = C_1 \sqcup C_2 \sqcup C_3 \ldots, \text{ } C_i \text{ are chains. Then} \]

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\frac{n!}{\prod |C_i|!} \geq e(\mathcal{P}) \geq \prod |A_i|!
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Greene – Kleitman – Fomin parameters of \( \mathcal{P} \).
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Greene – Kleitman – Fomin parameters of \( \mathcal{P} \). \( c_1 \) — maximal size of a chain,
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Greene – Kleitman – Fomin parameters of \( \mathcal{P} \). \( c_1 \) — maximal size of a chain, \( c_1 + c_2 \) — maximal number of elements covered by 2 chains, etc.
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\[ \mathcal{P} = A_1 \sqcup A_2 \sqcup A_3 \ldots, \text{ where } A_i \text{ is the antichain of elements with rank } i. \]  Also \( \mathcal{P} = C_1 \sqcup C_2 \sqcup C_3 \ldots, \) \( C_i \) are chains. Then

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Greene – Kleitman – Fomin parameters of \( \mathcal{P}. \) \( c_1 \) — maximal size of a chain, \( c_1 + c_2 \) — maximal number of elements covered by 2 chains, etc. \( c_1 \geq c_2 \geq \ldots; a_1 \geq a_2 \geq \ldots: \) the same for antichains.

**Theorem** (GK,F). \( (a_i) \) and \( (c_i) \) are conjugate partitions.
Chains and antichains covering

\[ \mathcal{P} = A_1 \sqcup A_2 \sqcup A_3 \ldots, \text{ where } A_i \text{ is the antichain of elements with rank } i. \text{ Also } \mathcal{P} = C_1 \sqcup C_2 \sqcup C_3 \ldots, \text{ } C_i \text{ are chains. Then}
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Greene – Kleitman – Fomin parameters of \( \mathcal{P} \). \( c_1 \) — maximal size of a chain, \( c_1 + c_2 \) — maximal number of elements covered by 2 chains, etc. \( c_1 \geq c_2 \geq \ldots; a_1 \geq a_2 \geq \ldots \): the same for antichains.

**Theorem (GK,F).** \( (a_i) \) and \( (c_i) \) are conjugate partitions.

**Theorem (Bochkov, P. 2019):**

\[
\frac{n!}{\prod c_i!} \geq e(\mathcal{P}) \geq \prod_i a_i!
\]

**Corollary.** \((*)\) holds for arbitrary antichains.
Lower bound


\[ A \] — antichain,
\[ e(P) \geq \sum_{x \in A} e(P-x) \]

\[ A \] — maximal antichain,
\[ |A| = a_1 \]. Fix \( x \in A \).

Let \( r_1 \geq r_2 \ldots \) denote antichain GKF parameters for \( P-x \).

Then \( \{r_1, r_2, \ldots\} \succ \{a_1-1, a_2, a_3, \ldots\} \).

By Karamata for \( f(x) = x! \prod r_i! \geq (a_1-1)! \prod i > 1 a_i! = 1 \)

Therefore by induction proposition we have \( e(P-x) \geq 1 a_1 \prod i a_i! \),
for all \( x \in A \).

Sum up over all \( x \in A \) and get \( e(P) \geq \prod i a_i! \),
Lower bound

Edelman, Hibi, Stanley 1989, Sidorenko 1992. \( A \) — antichain,

\[
e(\mathcal{P}) \geq \sum_{x \in A} e(\mathcal{P} - x)
\]

\( A \) — maximal antichain, \(|A| = a_1\). Fix \( x \in A \). Let \( r_1 \geq r_2 \ldots \) denote antichain GKF parameters for \( \mathcal{P} - x \).

\[ e(P) \geq \sum_{x \in A} e(P - x) \]

$A$ — maximal antichain, $|A| = a_1$. Fix $x \in A$. Let $r_1 \geq r_2 \ldots$ denote antichain GKF parameters for $P - x$. Then \( \{r_1, r_2, \ldots\} \succ \{a_1 - 1, a_2, a_3, \ldots\} \).

$$e(P) \geq \sum_{x \in A} e(P - x)$$

$A$ — maximal antichain, $|A| = a_1$. Fix $x \in A$. Let $r_1 \geq r_2 \ldots$ denote antichain GKF parameters for $P - x$. Then

$\{r_1, r_2, \ldots\} \succ \{a_1 - 1, a_2, a_3, \ldots\}$. By Karamata for $f(x) = x!$

$$\prod r_i! \geq (a_1 - 1)! \prod a_i! = \frac{1}{a_1} \prod_{i > 1} a_i!$$
Lower bound


\[ e(\mathcal{P}) \geq \sum_{x \in A} e(\mathcal{P} - x) \]

A — maximal antichain, \(|A| = a_1\). Fix \(x \in A\). Let \(r_1 \geq r_2 \ldots\) denote antichain GKF parameters for \(\mathcal{P} - x\). Then \(\{r_1, r_2, \ldots\} \succ \{a_1 - 1, a_2, a_3, \ldots\}\). By Karamata for \(f(x) = x!\)

\[ \prod r_i! \geq (a_1 - 1)! \prod_{i>1} a_i! = \frac{1}{a_1} \prod_{i} a_i! \]

Therefore by induction proposition we have \(e(\mathcal{P} - x) \geq \frac{1}{a_1} \prod_{i} a_i!\), for all \(x \in A\). Sum up over all \(x \in A\) and get \(e(\mathcal{P}) \geq \prod_{i} a_i!\),
**Upper bound**

**Lemma.** $A$ is the antichain of maximal elements. $c_1 \geq c_2 \geq \ldots$ are the chain GKF parameters of $\mathcal{P}$. Then the elements of $A$ may be enumerated as $x_1, x_2, \ldots, x_{|A|}$ so that for all $i = 1, 2, \ldots, |A|$ there exist $i$ chains whose maximal elements belong to the set $\{x_1, \ldots, x_i\} \cup (\mathcal{P} \setminus A)$ with total size $c_1 + \ldots + c_i$. 
Lemma. $A$ is the antichain of maximal elements. $c_1 \geq c_2 \geq \ldots$ are the chain GKF parameters of $\mathcal{P}$. Then the elements of $A$ may be enumerated as $x_1, x_2, \ldots, x_{|A|}$ so that for all $i = 1, 2, \ldots, |A|$ there exist $i$ chains whose maximal elements belong to the set \{x_1, \ldots, x_i\} \cup (\mathcal{P} \setminus A)$ with total size $c_1 + \ldots + c_i$.

The multiset \{r_1, r_2, \ldots\} of GKF chain parameters of $\mathcal{P} \setminus \{x_i\}$ majorizes the multiset \{c_1, c_2, \ldots, c_{j-1}, c_j - 1, c_{j+1}, \ldots\}. By Karamata $\prod r_j! \geq \frac{1}{c_j} \prod j c_j!$, and by induction

$$e(\mathcal{P} - x_i) \leq \frac{(n - 1)!}{\prod r_j!} \leq \frac{c_j(n - 1)!}{\prod c_j!}.$$ 

Sum up this by all $i$ and apply $e(\mathcal{P}) = \sum_i e(\mathcal{P} - x_i)$. 
Accuracy of the bounds

The upper and lower bounds \( n! / \prod c_i! \) and \( \prod a_i! \) are quite close: the ratio is always \( e^{O(n \log \log n)} = n! o(1) \) (but alas worse than exponential in \( n \)).

For any two conjugate partitions \( a_1 \geq a_2 \geq \ldots \) and \( c_1 \geq c_2 \geq \ldots \) of the positive integer \( n \) onto positive integer parts the inequality

\[
\prod a_i! \prod c_i! \geq \left( n e^{H_n} \right)^n = n! e^{-n \log \log n + o(n)}
\]

holds (here \( H_n = 1 + 1/2 + \ldots + 1/n = \log n + O(1) \) is a Harmonic sum).
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For any two conjugate partitions $a_1 \geq a_2 \geq \ldots$ and $c_1 \geq c_2 \geq \ldots$ of the positive integer $n$ onto positive integer parts the inequality

$$\prod_i a_i! \prod_i c_i! \geq \left( \frac{n}{eH_n} \right)^n = n!e^{-n \log \log n + o(n)}$$

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$$\prod_i c_i! = \prod_k k^{i: c_i \geq k} = \prod_k k^{a_k},$$

thus the inequality to prove is

$$\prod_k k^{a_k} a_k! \geq \left(\frac{n}{eH_n}\right)^n$$
What does it give for Young diagrams?

Antichain GKF parameters: sizes of antidiagonals
\{x : h^*(x) = \text{const}\}

\[
\frac{n!}{\prod h(x)} = e(\mathcal{P}) \leq \frac{n!}{\prod c_i!} = \frac{n!}{\prod k^a} = \frac{n!}{\prod \tilde{h}(x)}
\]
What does it give for Young diagrams?

Antichain GKF parameters: sizes of antidiagonals
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\[ \frac{n!}{\prod h(x)} = e(P) \leq \frac{n!}{\prod c_i!} = \frac{n!}{\prod k^{a_k}} = \frac{n!}{\prod \tilde{h}(x)} \]

\( \tilde{h}(x) \): put 1’s on the maximal antidiagonal, 2 on the second largest antidiagonal etc.
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\]

\(\tilde{h}(x)\): put 1’s on the maximal antidiagonal, 2 on the second largest antidiagonal etc. So we get

\[
\prod \tilde{h}(x) \leq \prod h(x) \leq \prod h^*(x)
\]

for any Young diagram.
Thank you!