XOR Ising Model and Constrained Percolation

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Ising Model

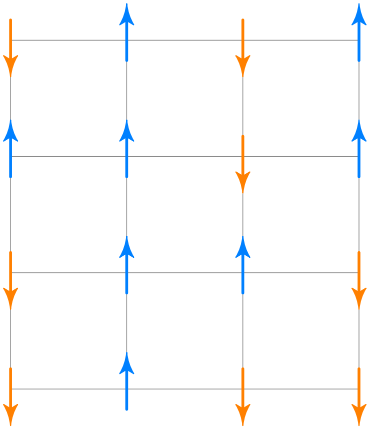

Figure: Ising Model on the Square Grid

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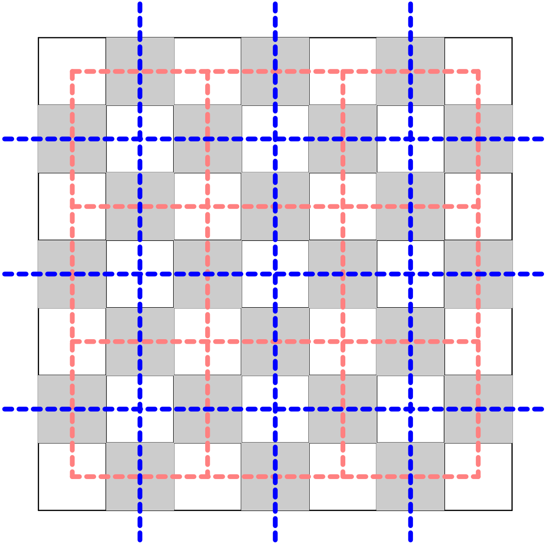
- (B. de Tiliere and C. Boutilier) Contours of $\sigma_{XOR} \overset{d}{=}$ level lines of dimer height functions on the square-octagon lattice

Graph



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- Auxiliary lattices: $\mathbb{L}_1$, $\mathbb{L}_2$.

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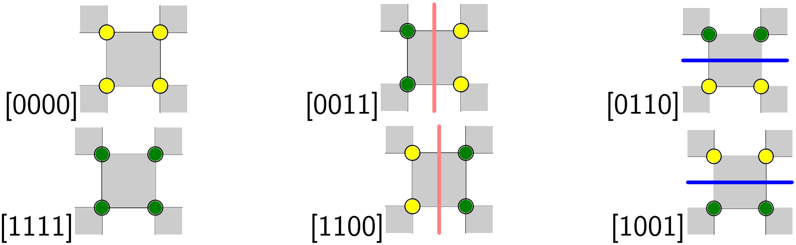

Figure: Local configurations of the constrained percolation around a black square. Dotted lines mark contours separating 0's and 1's. White (resp. black) disks represent 0's (resp. 1's).

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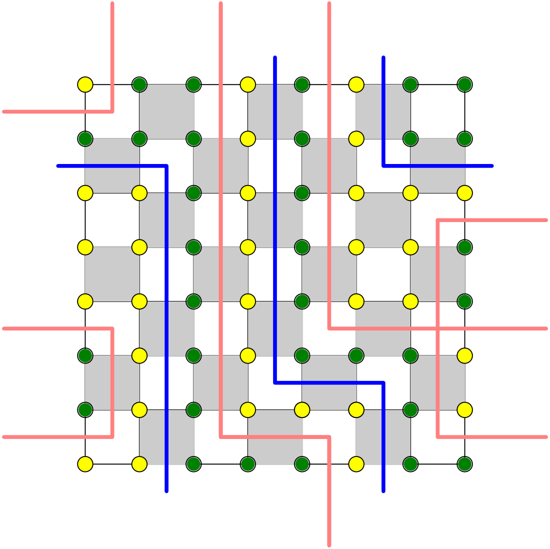

Perfect Matching on Square-octagon Lattice

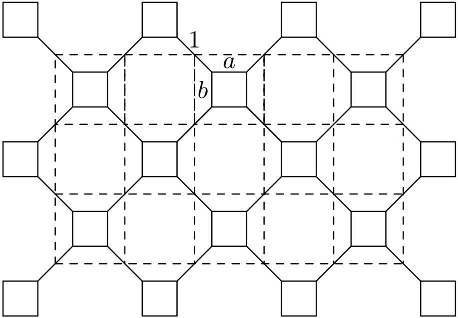

Figure: Square-octagon lattice: solid lines are edges of the square-octagon lattice; dashed lines are edges of G.

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3. $\mathcal{H}$-translation-invariant, where $\mathcal{H}$ is the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by $(1, 1)$ and $(1, -1)$;
4. symmetric, i.e. let $\theta : \Omega \to \Omega$ be a map satisfying $\theta(\omega)(v) = 1 - \omega(v)$, for each $v \in \mathbb{Z}^2$, then μ is invariant under θ, that is, for any event A, $\mu(A) = \mu(\theta(A))$.

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- (Coniglio, Nappi, Peruggi, Russo)Critical 2D ferromagnetic Ising model has no infinite "$+$"- or "$-$" clusters.

Constrained Percolation

Theorem

Let μ be a probability measure on the constrained percolation state space Ω', satisfying $(1) - (4)$. Let $\mathcal{A}$ be the event that there exists a unique infinite cluster consisting of 1's, then

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- No assumption of stochastic monotonicity or FKG inequality.

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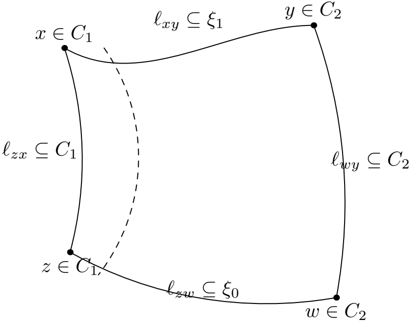

Figure: Infinite clusters and incident contours

- $\mu(B_1 \cap B_2 \cap \mathcal{A} \cap \mathcal{A}') = 0 \Rightarrow \mu(\mathcal{A}) = 0.$

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  - There is a unique infinite "1" cluster in the random construction.

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- Example: each row of $\mathbb{Z}^2$ is independently assigned all 0’s with probability $\frac{1}{2}$, and all 1’s with probability $\frac{1}{2}$.

Finite Energy

5. Let ν_1 (resp. ν_2) be the corresponding marginal distribution of μ on bond configurations of $\mathbb{L}_1$ (resp. $\mathbb{L}_2$). Assume ν_1 has finite energy in the following sense: let S be a face of $\mathbb{L}_1$, and ∂S be the set of four sides of the square S. Let $\phi \in \Phi_1$. Define ϕ_S to be the configuration obtained by switching the sates of each element of ∂S, i.e. $\phi_S(e) = 1 - \phi(e)$, if $e \in \partial S$; and $\phi_S(e) = \phi(e)$ otherwise. Let E be an event, define

$$E_S = \{\phi_S : \phi \in E\}. \tag{1}$$

Then $\nu_1(E_S) > 0$, whenever $\nu_1(E) > 0$.

Finite Energy

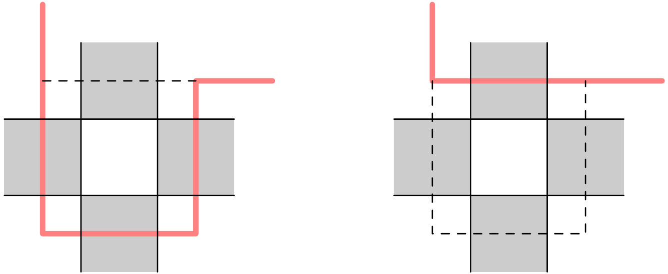

Figure: Change of contour configurations

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Theorem

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- If there exists more than one $\mathbb{L}_1$-contour, by finite energy we can merge multiple $\mathbb{L}_1$-contours to form an infinite $_1$ contour with more than two ends.

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- A_0 (resp. A_1): there exists an infinite 0-cluster (infinite 1-cluster) incident to both the infinite $\mathbb{L}_1$-contour and the infinite $\mathbb{L}_2$-contour.

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- at most one infinite $\mathbb{L}_1$-contour and at most one infinite $\mathbb{L}_2$-contour.
- B_1 (resp. B_2): there exists exactly 1 infinite $\mathbb{L}_1$-contour (resp. $\mathbb{L}_2$-contour.)
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- Same argument as before to obtain a contradiction.

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Critical XOR Ising Model

Both σ_1, σ_2 has coupling constants J_h (J_v) on horizontal (vertical) edges satisfying

$$\exp(-2J_h) + \exp(-2J_v) + \exp(-2J_h - 2J_v) = 1.$$

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For the critical XOR-Ising model,

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- High-temperature: both σ_1, σ_2 has coupling constants J_h (J_v) on horizontal (vertical) edges satisfying

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3. Critical XOR Ising model $\Rightarrow$ Constrained percolation measure satisfies (1)-(5).
4. Non-critical XOR Ising model $\Rightarrow$ Constrained percolation measure satisfies (1),(2),(4),(5).

High-temperature XOR Ising Model

Theorem

Let p_c be the critical probability for the i.i.d Bernoulli site percolation on the square grid ($p_c > \frac{1}{2}$). Let $h_0 > 0$ be a positive number obtained from p_c by the following identity

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1. *there are no infinite "$+$"-clusters or infinite "$-$"-clusters;*
2. *there exists exactly one infinite contour.*

Low-temperature XOR Ising Model

Theorem

In the low temperature XOR Ising model, almost surely there are no infinite contours.
Let $J_h' > 0$, $J_v' > 0$ *be obtained from* J_h, J_v *by*

$$\exp(-2J_h) + \exp(-2J_v') + \exp(-2J_h - 2J_v') = 1; \tag{2}$$

$$\exp(-2J_h) + \exp(-2J_v') + \exp(-2J_h - 2J_v') = 1. \tag{3}$$

If we assign the coupling constant J_h' *to each horizontal edge of the square grid, and* J_v' *to each vertical edge, then we obtain a low-temperature XOR Ising model in which the total number of infinite "+"-clusters and "−"-clusters is exactly one almost surely.*

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- planar $[m, 4, n, 4]$ lattice: each vertex is incident to four faces of degree $m, 4, n, 4$.

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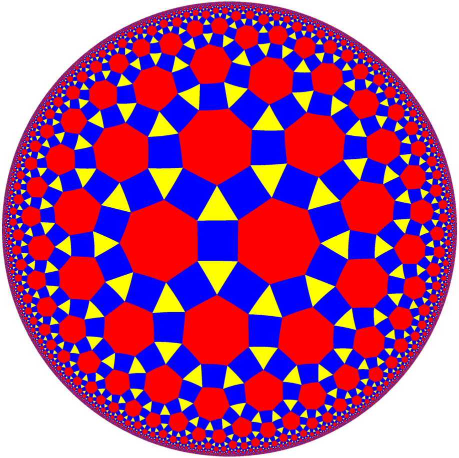
- If $\frac{1}{m} + \frac{1}{n} < \frac{1}{2}$, the lattice can be embedded into the hyperbolic plane, and the lattice is non-amenable.

$[3, 4, 7, 4]$ Lattice in hyperbolic plane



Constrained Percolation on $[m, 4, n, 4]$ lattice.

- Same constraint required around each square face.

Thank you!