XOR Ising Model and Constrained Percolation

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Ising Model

Figure: Ising Model on the Square Grid
XOR Ising Model

- $\sigma_1, \sigma_2 \in \{\pm 1\}^{V(L_1)}$: two IID Ising spins on $L_1 = (V(L_1), E(L_1))$. 
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- $\sigma_1, \sigma_2 \in \{ \pm 1 \}^{V(L_1)}$: two IID Ising spins on $L_1 = (V(L_1), E(L_1))$.
- Both $\sigma_1$ and $\sigma_2$ has probability measure obtained from the weak limit under periodic boundary conditions.
**XOR Ising Model**

- \( \sigma_1, \sigma_2 \in \{\pm 1\}^{\mathcal{V}(\mathbb{L}_1)} \): two IID Ising spins on \( \mathbb{L}_1 = (\mathcal{V}(\mathbb{L}_1), E(\mathbb{L}_1)) \).
- Both \( \sigma_1 \) and \( \sigma_2 \) has probability measure obtained from the weak limit under periodic boundary conditions.
- \( \sigma_{\text{XOR}}(v) = \sigma_1(v)\sigma_2(v) \), for \( v \in \mathcal{V}(\mathbb{L}_1) \).
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- $\sigma_{XOR}(v) = \sigma_1(v)\sigma_2(v)$, for $v \in V(L_1)$.
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- Both \(\sigma_1\) and \(\sigma_2\) has probability measure obtained from the weak limit under periodic boundary conditions.
- \(\sigma_{XOR}(v) = \sigma_1(v)\sigma_2(v)\), for \(v \in V(\mathbb{L}_1)\).
- Contours: present edges of \(\mathbb{L}_2\) separating different states of \(\sigma_{XOR}\).
- Conjecture (D. Wilson): the contours of critical XOR Ising converges to \(CLE(4)\) in the scaling limit.
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- Contours: present edges of $\mathbb{L}_2$ separating different states of $\sigma_{\text{XOR}}$.
- Conjecture (D. Wilson): the contours of critical XOR Ising converges to $\text{CLE}(4)$ in the scaling limit.
- (B. de Tiliere and C. Boutilier) Contours of $\sigma_{\text{XOR}} \overset{d}{=} \text{level lines of dimer height functions on the square-octagon lattice}$
Graph
Constrained Percolation

$G = (\mathbb{Z}^2, E)$.
Constrained Percolation

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- Examples: perfect matching, polygon model, 6V model...
- Auxiliary lattices: $\mathbb{L}_1, \mathbb{L}_2$. 
Figure: Local configurations of the constrained percolation around a black square. Dotted lines mark contours separating 0’s and 1’s. White (resp. black) disks represent 0’s (resp. 1’s).
Constrained Percolation

Figure: Constrained percolation configuration and associated contour configuration. Red lines represent primal contours. Blue lines represent dual contours. Green and yellow vertices represent two different states of vertices.
Perfect Matching on Square-octagon Lattice

Figure: Square-octagon lattice: solid lines are edges of the square-octagon lattice; dashed lines are edges of $G$. 
Probability Measure

$\mu$: probability measure on $\Omega'$ satisfying assumptions

1. $2\mathbb{Z} \times 2\mathbb{Z}$-translation-invariant;

and further assumptions
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Introduction

Constrained Percolation

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3. \( \mathcal{H} \)-translation-invariant, where \( \mathcal{H} \) is the subgroup of \( \mathbb{Z} \times \mathbb{Z} \) generated by \((1, 1)\) and \((1, -1)\);
4. symmetric, i.e. let \( \theta : \Omega \to \Omega \) be a map satisfying \( \theta(\omega)(v) = 1 - \omega(v) \), for each \( v \in \mathbb{Z}^2 \), then \( \mu \) is invariant under \( \theta \), that is, for any event \( A \), \( \mu(A) = \mu(\theta(A)) \).
Clustering and Clusters

- **Cluster**: maximal connected set of vertices of $G$ in which every vertex has the same state.
Contours and Clusters

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- Contour: maximal connected set of present edges of $\mathbb{L}_1$ or $\mathbb{L}_2$.
- Example: 0-cluster, 1-cluster.
- Example: $\mathbb{L}_1$-contour, $\mathbb{L}_2$-contour.
Percolation

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- (Coniglio, Nappi, Peruggi, Russo) Critical 2D ferromagnetic Ising model has no infinite “+”- or “−” clusters.
Constrained Percolation

Theorem

Let $\mu$ be a probability measure on the constrained percolation state space $\Omega'$, satisfying (1) – (4). Let $A$ be the event that there exists a unique infinite cluster consisting of 1’s, then

$$\mu(A) = 0.$$
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Proof

(2) $\Rightarrow \mu(A) = 0$ or $\mu(A) = 1$. 

(3) $\Rightarrow \mu(B_1) = \mu(B_2)$. 

(2) $\Rightarrow \mu(B_1) = \mu(B_2) = 0$ or $\mu(B_1) = \mu(B_2) = 1$. 

$\mu(B_1 \cup B_2) = 1 \Rightarrow \mu(B_1) = \mu(B_2) = 1 \Rightarrow \mu(B_1 \cap B_2 \cap A \cap A') = 1$. 

Let $A'$ be the event that there exists a unique infinite cluster consisting of 0's, (4) $\Rightarrow \mu(A) = \mu(A')$. 

Assume $\mu(A) = 1$, then $\mu(A \cap A') = 1$.
Proof

(2) ⇒ \( \mu(A) = 0 \) or \( \mu(A) = 1 \).

Let \( A' \) be the event that there exists a unique infinite cluster consisting of 0’s, (4) ⇒ \( \mu(A) = \mu(A') \).
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1. (2) $\Rightarrow \mu(A) = 0$ or $\mu(A) = 1$.
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4. $\Rightarrow \mu(B_1) = \mu(B_2) = 0$ or $\mu(B_1) = \mu(B_2) = 1$.
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4. Let $B_1$ (resp. $B_2$) be the event that there exists an infinite $\mathbb{L}_1$ (resp. $\mathbb{L}_2$) contour incident to both the infinite 0-cluster and infinite 1-cluster, then $\mu(A \cap A') = 1 \Rightarrow \mu(B_1 \cup B_2) = 1$. 
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\( \mu(B_1 \cup B_2) = 1 \Rightarrow \mu(B_1) = \mu(B_2) = 1 \Rightarrow \mu(B_1 \cap B_2 \cap A \cap A') = 1 \).
\[ x \in C_1 \quad y \in C_2 \quad z \in C_1 \quad w \in C_2 \quad \ell_{xy} \subseteq \xi_1 \quad \ell_{zx} \subseteq C_1 \quad \ell_{wy} \subseteq C_2 \quad \ell_{zw} \subseteq \xi_0 \]

Figure: Infinite clusters and incident contours

\[ \mu(B_1 \cap B_2 \cap A \cap A') = 0 \Rightarrow \mu(A) = 0. \]
Question: Given (1)-(4), does the Theorem hold for unconstrained percolation as well?
Unconstrained Percolation

- Question: Given (1)-(4), does the Theorem hold for unconstrained percolation as well?
- Answer: No.
Unconstrained Percolation

- Example:
Unconstrained Percolation

- Example:
  - $\left( X_m \right)_{m \in \mathbb{Z}}$ and $\left( Y_n \right)_{m \in \mathbb{Z}}$: two independent sequence of i.i.d Bernoulli random variables, each of which taking value 1 with probability $1/2$. 

- A run of $X$ (or $Y$): the maximal nonempty interval of $\mathbb{Z}$ on which the corresponding variables are all equal.

- A run rectangle: $I \times J \subseteq \mathbb{Z}^2$, where $I$ is a run of $X$ and $J$ is a run of $Y$.

- A site rectangle: a run rectangle $I \times J$ where both $I$ and $J$ are runs of 1s.

- A bond rectangle: a run rectangle $I \times J$, exactly one of which is a run of 1s.

- $G$: random graph whose vertex set is the set of all site rectangles and edge set is the set of all bond rectangles. $G$ is isomorphic to the 2D square grid $\mathbb{Z}^2$.

- Take a uniform spanning tree of $G$. Assign each vertex in $G$ value 1 if and only if it lies on the uniform spanning tree of $G$.

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Unconstrained Percolation

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Introduction

Constrained Percolation

Unconstrained Percolation

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- Question: Given (1)-(4), is it possible that there exist more than one infinite “1” clusters?
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- Question: Given (1)-(4), is it possible that there exist more than one infinite “1” clusters?
- Answer: Yes.
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Answer: Yes.

Example: each row of $\mathbb{Z}^2$ is independently assigned all 0’s with probability $\frac{1}{2}$, and all 1’s with probability $\frac{1}{2}$. 
Let $\nu_1$ (resp. $\nu_2$) be the corresponding marginal distribution of $\mu$ on bond configurations of $\mathbb{L}_1$ (resp. $\mathbb{L}_2$). Assume $\nu_1$ has finite energy in the following sense: let $S$ be a face of $\mathbb{L}_1$, and $\partial S$ be the set of four sides of the square $S$. Let $\phi \in \Phi_1$. Define $\phi_S$ to be the configuration obtained by switching the states of each element of $\partial S$, i.e. $\phi_S(e) = 1 - \phi(e)$, if $e \in \partial S$; and $\phi_S(e) = \phi(e)$ otherwise. Let $E$ be an event, define

$$E_S = \{\phi_S : \phi \in E\}. \quad (1)$$

Then $\nu_1(E_S) > 0$, whenever $\nu_1(E) > 0$. 

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Figure: Change of contour configurations
Unique Infinite Contour under Weaker Conditions.

Theorem

(A. Holroyd and Z. Li) Let $\mu$ be a probability measure on $\Omega'$, satisfying Assumptions (1),(2),(5), then $\mu$-a.s. there is at most one $\mathbb{L}_1$-contour.
Unique Infinite Contour under Weaker Conditions.

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- The number of ends of a connected graph is the supremum over its finite subgraphs of the number of infinite components that remain after removing the subgraph.
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- For invariant percolation on amenable graphs, any infinite cluster has at most two ends (Benjamini, Lyons, Peres, Schramm).

- If there exists more than one $\mathbb{L}_1$-contour, by finite energy we can merge multiple $\mathbb{L}_1$-contours to form an infinite $\mathbb{L}_1$ contour with more than two ends.
Non-existence of Infinite Clusters and Contours

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- there are no infinite contours;
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- at most one infinite $\mathbb{L}_1$-contour and at most one infinite $\mathbb{L}_2$-contour.
- $B_1$ (resp. $B_2$): there exists exactly 1 infinite $\mathbb{L}_1$-contour (resp. $\mathbb{L}_2$-contour.)
Proof.

- at most one infinite $\mathbb{L}_1$-contour and at most one infinite $\mathbb{L}_2$-contour.
- $B_1$ (resp. $B_2$): there exists exactly 1 infinite $\mathbb{L}_1$-contour (resp. $\mathbb{L}_2$-contour.)
- $\mu(B_1) = \mu(B_2) = 0$ or $\mu(B_1) = \mu(B_2) = 1$. 

If $\mu(B_1) = \mu(B_2) = 1$, there exists an infinite cluster incident to both the infinite $\mathbb{L}_1$-contour and $\mathbb{L}_2$-contour, then $\mu(A_0) = \mu(A_1) = 1$.

Same argument as before to obtain a contradiction.
Proof.

- at most one infinite $\mathbb{L}_1$-contour and at most one infinite $\mathbb{L}_2$-contour.
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Critical XOR Ising Model

Both $\sigma_1$, $\sigma_2$ has coupling constants $J_h$ ($J_v$) on horizontal (vertical) edges satisfying

$$\exp(-2J_h) + \exp(-2J_v) + \exp(-2J_h - 2J_v) = 1.$$ 

Then $\sigma_1$ and $\sigma_2$ are critical.

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For the critical XOR-Ising model,

1. almost surely there are no infinite “+”-clusters, and no infinite “−”-clusters;
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Non-critical XOR Ising Model

- High-temperature: both $\sigma_1, \sigma_2$ has coupling constants $J_h$ ($J_v$) on horizontal (vertical) edges satisfying

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- Low-temperature: both $\sigma_1, \sigma_2$ has coupling constants $J_h (J_v)$ on horizontal (vertical) edges satisfying

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Probability Measures

Contours in XOR Ising model $d$ Level lines of dimer height functions on the square-octagon lattice.
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High-temperature XOR Ising Model

Theorem

Let $p_c$ be the critical probability for the i.i.d Bernoulli site percolation on the square grid ($p_c > \frac{1}{2}$). Let $h_0 > 0$ be a positive number obtained from $p_c$ by the following identity

$$\frac{e^{h_0}}{e^{h_0} + e^{-h_0}} = p_c.$$

Consider a high-temperature XOR Ising model on the square grid satisfying $2(J_h + J_v) < h_0$, then almost surely
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1. there are no infinite “+”-clusters or infinite “−”-clusters;
2. there exists exactly one infinite contour.
Low-temperature XOR Ising Model

Theorem

In the low temperature XOR Ising model, almost surely there are no infinite contours.
Let \( J'_h > 0, J'_v > 0 \) be obtained from \( J_h, J_v \) by

\[
\exp(-2J_h) + \exp(-2J'_v) + \exp(-2J_h - 2J'_v) = 1; \quad (2)
\]

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\exp(-2J_h) + \exp(-2J'_v) + \exp(-2J_h - 2J'_v) = 1. \quad (3)
\]

If we assign the coupling constant \( J'_h \) to each horizontal edge of the square grid, and \( J'_v \) to each vertical edge, then we obtain a low-temperature XOR Ising model in which the total number of infinite “+”-clusters and “−”-clusters is exactly one almost surely.
Planar Lattices

- planar $[m, 4, n, 4]$ lattice: each vertex is incident to four faces of degree $m, 4, n, 4$. 

- If $m + n = 2$, the lattice can be embedded to the Euclidean plane; example: $m = n = 4$ (square grid); $m = 3, n = 6$.

- If $m + n > 2$, the lattice can be embedded into the sphere, and the lattice is finite.

- If $m + n < 2$, the lattice can be embedded into the hyperbolic plane, and the lattice is non-amenable.
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[3, 4, 7, 4] Lattice in hyperbolic plane
Constrained Percolation on $[m, 4, n, 4]$ lattice.

- Same constraint required around each square face.
Thank you!