# Random Young diagrams and the approximate factorization property

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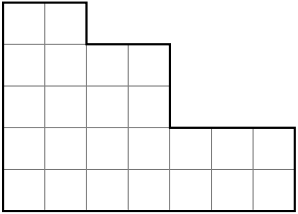
# Setup

**Definition**

- A partition $\pi$ of the integer $n$ ($\pi \vdash n$, or $\pi \in \mathcal{P}_n$): a finite non-increasing sequence of positive integers $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_k$, such that $|\pi| := \sum_i \pi_i = n$;
- Graphical representation by a Young diagram $\lambda \in \mathbb{Y}_n$ of size $n$.



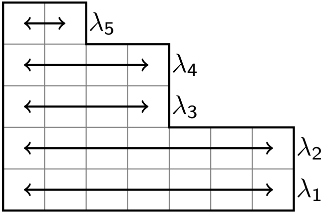
**Problem**

*We want to study asymptotic behaviour of random Young diagrams $\mathbb{Y}_n$, when their size $n$ is tending to infinity.*

# Setup

## Example

- $\pi = (7, 7, 4, 4, 2) \vdash 24$,
- Represented by a Young diagram $\lambda$ with $\ell(\lambda) = 5$ rows.

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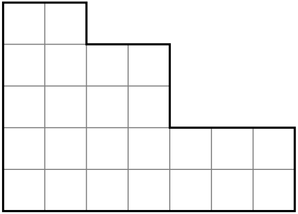

## Problem

*We want to study asymptotic behaviour of random Young diagrams $\mathbb{Y}_n$, when their size $n$ is tending to infinity.*

# Random Young diagrams and the symmetric group

$\mathbb{P}_n$ a probability measure on the set of Young diagrams $\mathbb{Y}_n$.

$$\updownarrow$$

$\chi_n : \mathfrak{S}_n \to \mathbb{R}$ a normalized central positive definite function on the symmetric group $\mathfrak{S}_n$ (called a reducible character):

$$\chi_n(\pi) = \sum_{\lambda \in \mathbb{Y}_n} \mathbb{P}_n(\lambda) \chi_\lambda(\pi)$$

- $\rho_\lambda$ - an irreducible representation of $\mathfrak{S}_n$,
- $\chi_\lambda$ - an associated irreducible character, i.e.

$$\chi_\lambda(\pi) := \frac{\operatorname{Tr} \ \rho_\lambda(\pi)}{\operatorname{Tr} \ \rho_\lambda(\mathrm{id})}.$$

**Conclusion**

*In order to understand random Young diagrams, we can studied associated reducible characters.*

# Examples (from the representation theory)

## Example

- The Plancherel measure

$$\chi(\pi) := \begin{cases} 1 & \text{if } \pi = 1^n, \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad \mathbb{P}_\chi(\lambda) := \frac{(\dim \rho_\lambda)^2}{n!}$$

- the Schur-Weyl measure

$$\chi(\pi) := N^{\ell(\pi)-|\pi|} \quad \leftrightarrow \quad \mathbb{P}_\chi(\lambda) := \frac{\dim E_\lambda}{N^n},$$

where $(\mathbb{C}^N)^{\otimes n} = \bigoplus_{\lambda \vdash n} E_\lambda$.

# The Plancherel measure - law of large numbers

We can describe the Plancherel measure more combinatorially:

- as a push-forward of the uniform measure on $\mathfrak{S}_n$ through RSK
- using hook-length formula:

$$\mathbb{P}(\lambda) = \frac{n!}{\prod_{(x,y)\in\lambda} h_\lambda^2(x,y)},$$

where $h_\lambda(x,y)$ - hook length of a cell $(x,y) \in \lambda$.

**Theorem (Logan–Shepp, Vershik–Kerov 1977, informal statement)**

*Let $\lambda_n \in \mathbb{Y}_n$ be a random Young diagram sampled with the Plancherel distribution $\mathbb{P}_n$. Then the sequence $(\lambda_n)$ of Young diagrams converges to some limit shape in the limit $n \to \infty$ when the number of the boxes tends to infinity.*

# Vershik-Kerov, Logan-Shepp limit shape

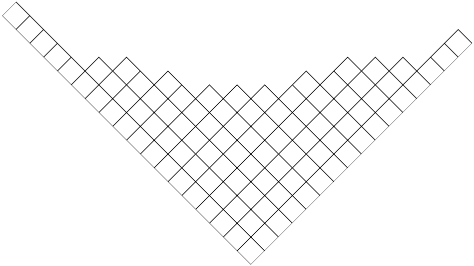

Figure: Scaled random Young diagram of size 100 distributed according with Plancherel measure

# Vershik-Kerov, Logan-Shepp limit shape

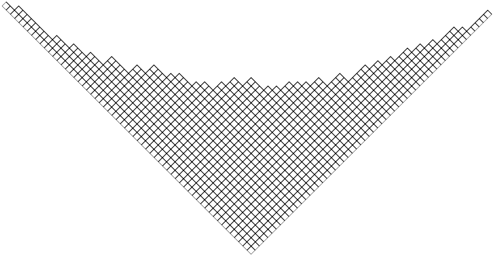

Figure: Scaled random Young diagram of size 1000 distributed according with Plancherel measure

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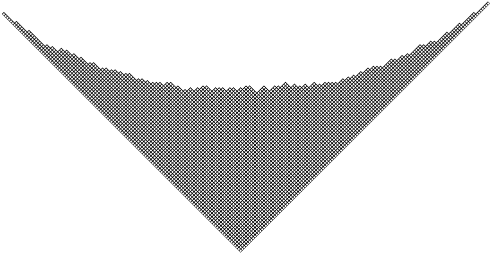

Figure: Scaled random Young diagram of size 5000 distributed according with Plancherel measure

# Vershik-Kerov, Logan-Shepp limit shape

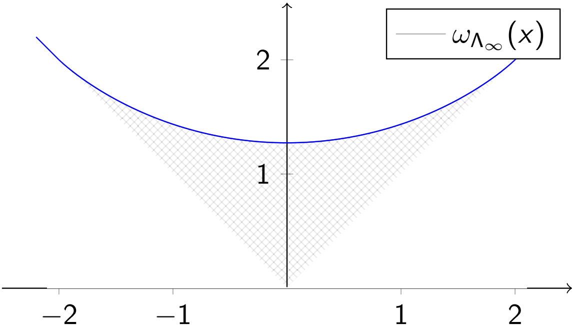


Figure: $\omega_{\Lambda_\infty}(x) = \begin{cases} |x| & \text{if } |x| \geq 2; \\ \frac{2}{\pi}\left(x \cdot \arcsin\frac{x}{2} + \sqrt{4-x^2}\right) & \text{otherwise.} \end{cases}$

# Anisotropic Young diagrams

**Definition**

Anisotropic Young diagram $T_{w,h}(\lambda)$ - polygon obtained from the Young diagram $\lambda$ by a horizontal stretching of ratio $w$ and a vertical stretching of ratio $h$ (each box $1 \times 1$ is replaced by a box of dimension $w \times h$).




In order to study the shape of random Young diagrams $\lambda_n \in \mathbb{Y}_n$ sampled by the Plancherel measure, the right scaling is the following:

$$\Lambda_n := T_{\sqrt{\frac{1}{n}},\sqrt{\frac{1}{n}}}\lambda_n.$$

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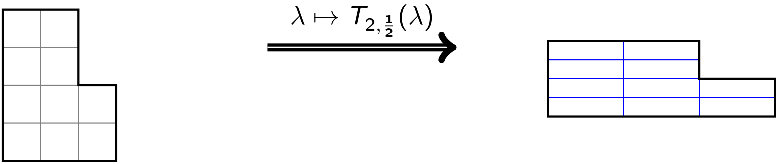


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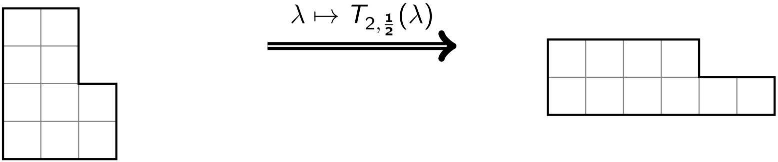


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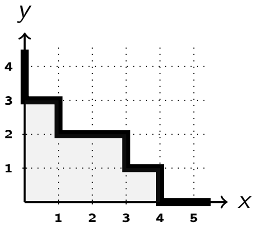
# Young diagrams as continuous objects

French convention:



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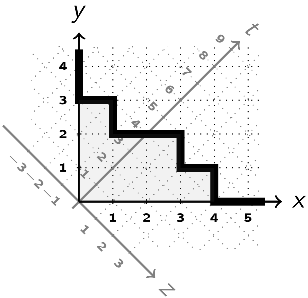

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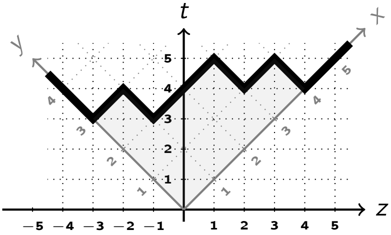

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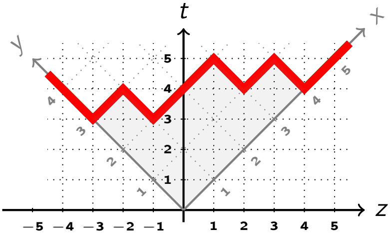

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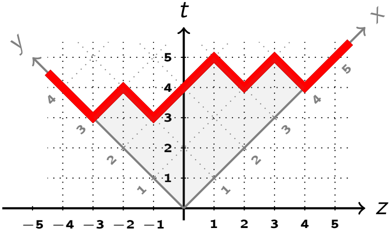
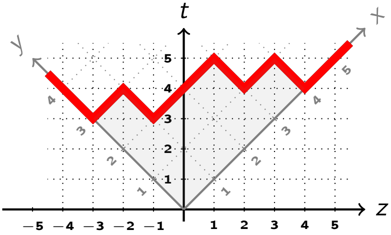



## Definition

A profile of a Young diagram $\lambda$ is a function $\omega_\lambda : \mathbb{R} \to \mathbb{R}_+$ such that its graph is a profile of $\lambda$ drawn in Russian convention.

# Young diagrams as continuous objects

Russian convention:




**Theorem (Logan–Shepp, Vershik–Kerov 1977 (revisited))**

*Let $\lambda_n$ be a random Young diagram sampled with the Plancherel distribution $\mathbb{P}_n$. Then there exists a deterministic function $\omega_{\Lambda_\infty} : \mathbb{R} \to \mathbb{R}$ with the property that*

$$\lim_{n\to\infty} \omega_{\Lambda_n} = \omega_{\Lambda_\infty},$$

*where the convergence holds true with respect to the supremum norm, in probability.*

# The approximate factorization property

We extend the domain of $\chi_n \colon \mathcal{P}_n \to \mathbb{R}$ to the set $\bigsqcup_{0 \leq k \leq n} \mathcal{P}_k$ of partitions of sufficiently small numbers by setting

$$\chi_n(\pi) := \chi_n(\pi, 1^{n-|\pi|}) \qquad \text{for } |\pi| \leq n.$$

Biane defined characters with the approximate factorization property:

- the characters do not decay too slow:

$$\chi_n(\pi) = O(n^{-\frac{|\pi|-\ell(\pi)}{2}}),$$

- the characters should approximately factorize, i.e.

$$\chi_n(\pi_1 \cdot \pi_2) - \chi_n(\pi_1) \cdot \chi_n(\pi_2) = O\left(n^{-\frac{|\pi_1|+\pi_2-\ell(\pi_1)-\ell(\pi_2)-2}{2}}\right).$$

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**Theorem (Biane 2001)**

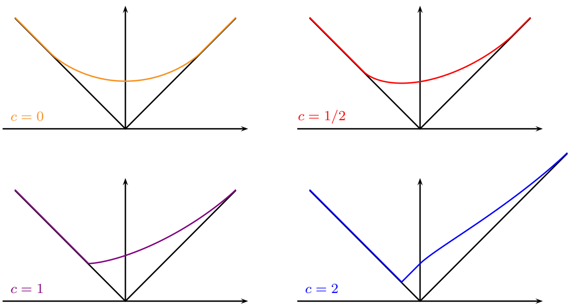
*Let $\chi_n \colon \mathcal{P}_n \to \mathbb{R}$, $n \geq 1$ be a family of reducible characters with the approximate factorization property. Let $\lambda_n$ be a random Young diagram with the probability distribution $\mathbb{P}_{\chi_n}$. Then there exists some deterministic function $\omega_{\Lambda_\infty} : \mathbb{R} \to \mathbb{R}$ with the property that*

$$\lim_{n\to\infty} \omega_{\Lambda_n} = \omega_{\Lambda_\infty},$$

*where the convergence holds true with respect to the supremum norm, in probability.*

# Back to our examples

- The Plancherel measure has the character with the approximate factorization property → Logan-Shepp, Vershik–Kerov result.
- the Schur-Weyl measure has the character given by $\chi(\pi) := N^{\ell(\pi)-|\pi|}$. Let $\frac{\sqrt{n}}{N} \to c \in [0,\infty)$. Then $\chi$ has the approximate factorization property and the limit shape $\omega_{\Lambda_\infty}$ is given by an explicit curve $\omega_c$ (Biane 2001):



# Fluctuations

**Problem**

*How to "measure" fluctuations around the limit shape $\omega_{\Lambda_\infty}$?*

We know that $\omega_{\Lambda_n} \to \omega_{\Lambda_\infty}$, so we define

$$\Delta_n := \sqrt{n}\left(\omega_{\Lambda_n} - \omega_{\Lambda_\infty}\right).$$

We would like to show that $\Delta_n$ converges to some function $\Delta_\infty$, so informally speaking,

$$\omega_{\Lambda_n} \approx \omega_{\Lambda_\infty} + \frac{1}{\sqrt{n}}\Delta_\infty.$$

We need to study suitable test functions:

$$Y_k := \frac{k-1}{2}\int u^{k-2}\,\Delta_n(u)du, \quad k \geq 2.$$

# the Plancherel measure - central limit theorem

**Theorem (Kerov 1993)**

*Let $\lambda_n$ be a random Young diagram sampled with the Plancherel distribution $\mathbb{P}_n$.*

*Then the random vector $\Delta_n$ converges in distribution to some (non-centered) Gaussian random vector $\Delta_\infty$ as $n \to \infty$.*

*Equivalently, the family of random variables $(Y_k)_{k \geq 2}$ converges as $n \to \infty$ to a (non-centered) Gaussian distribution.*

# Characters with the approximate factorization property revisited - cumulants

Note that

- $\chi_n(\pi) = \mathbb{E}\big(\chi_{(\circ)}(\pi)\big)$,
- $\chi_n(\pi_1 \cdot \pi_2) - \chi_n(\pi_1) \cdot \chi_n(\pi_2) = \mathrm{Var}\left(\chi_{(\circ)}(\pi)\right)$

of the irreducible characters $\chi_\lambda(\pi)$ taken with the probability $\mathbb{P}_{\chi_n}(\lambda)$.

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Cumulants $\kappa_\ell^{\mathbb{E}}(x_1, \dots, x_\ell)$ of random variables $x_1, \dots, x_\ell$ - natural generalization of a variance:

$$\begin{cases} \mathbb{E}(x_1) = \kappa_1^{\mathbb{E}}(x_1), \\ \mathbb{E}(x_1 x_2) = \kappa_2^{\mathbb{E}}(x_1, x_2) + \kappa_1^{\mathbb{E}}(x_1)\kappa_1^{\mathbb{E}}(x_2), \\ \mathbb{E}(x_1 x_2 x_3) = \kappa_3^{\mathbb{E}}(x_1, x_2, x_3) + \kappa_1^{\mathbb{E}}(x_1)\kappa_2^{\mathbb{E}}(x_2, x_3) \\ \qquad + \kappa_1^{\mathbb{E}}(x_2)\kappa_2^{\mathbb{E}}(x_1, x_3) + \kappa_1^{\mathbb{E}}(x_3)\kappa_2^{\mathbb{E}}(x_1, x_2) \\ \qquad + \kappa_1^{\mathbb{E}}(x_1)\kappa_1^{\mathbb{E}}(x_2)\kappa_1^{\mathbb{E}}(x_3), \\ \vdots \end{cases}$$

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Cumulants $\kappa_\ell^\chi(\pi_1 \dots \pi_\ell)$ of random variables $\chi_{(\circ)}(\pi_1), \dots, \chi_{(\circ)}(\pi_\ell)$ - natural generalization of a variance:

$$\begin{cases} \chi(\pi_1) = \kappa_1^\chi(\pi_1), \\ \chi(\pi_1\pi_2) = \kappa_2^\chi(\pi_1,\pi_2) + \kappa_1^\chi(\pi_1)\ \kappa_1^\chi(\pi_2), \\ \chi(\pi_1\pi_2\pi_3) = \kappa_3^\chi(\pi_1,\pi_2,\pi_3) + \kappa_1^\chi(\pi_1)\ \kappa_2^\chi(\pi_2,\pi_3) \\ \qquad + \kappa_1^\chi(\pi_2)\ \kappa_2^\chi(\pi_1,\pi_3) + \kappa_1^\chi(\pi_3)\ \kappa_2^\chi(\pi_1,\pi_2) \\ \qquad + \kappa_1^\chi(\pi_1)\ \kappa_1^\chi(\pi_2)\ \kappa_1^\chi(\pi_3), \\ \vdots \end{cases}$$

# Characters with the approximate factorization property revisited

Śniady redefined characters with the approximate factorization property:

$$\begin{cases} \chi_n(\pi) = \kappa_1^{\chi}(\pi_1) = O(n^{-\frac{\|\pi\|}{2}}), \\ \chi_n(\pi_1 \cdot \pi_2) - \chi_n(\pi_1) \cdot \chi_n(\pi_2) = \kappa_2^{\chi}(\pi_1, \pi_2) = O\left(n^{-\frac{\|\pi_1\| + \|\pi_2\| - 2}{2}}\right) \end{cases}$$

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Śniady redefined characters with the approximate factorization property:

$$\kappa_\ell^\chi(\pi_1, \dots, \pi_\ell) = O\left(n^{-\frac{\|\pi_1\| + \cdots + \|\pi_\ell\| - 2(\ell - 1)}{2}}\right).$$

## Theorem (Śniady 2006)

*Let $\chi_n \colon \mathcal{P}_n \to \mathbb{R}, n \geq 1$ be a family of reducible characters with the approximate factorization property. Let $\lambda_n$ be a random Young diagram with the probability distribution $\mathbb{P}_{\chi_n}$.*

*Then the random vector $\Delta_n$ converges in distribution to some (non-centered) Gaussian random vector $\Delta_\infty$ as $n \to \infty$.*

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# Symmetric functions vs. representation theory

- Power-sum symmetric functions $p_\lambda$:

$$p_k = \sum_i x_i^k, \quad p_\lambda = \prod_i p_{\lambda_i}.$$

- Schur symmetric functions $s_\lambda$:

$$s_\lambda = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \chi_\lambda(\pi)\; p_\pi = \sum_{\pi \in \mathcal{P}_n} \frac{\chi_\lambda(\pi)}{z_\pi}\; p_\pi,$$

where $z_\pi = \prod_i m_i(\pi)!\, i^{m_i(\pi)}$.

**Definition**

Hall scalar product:

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda,$$

Schur symmetric functions $s_\lambda$:

- obtained from monomial symmetric functions ordered by lexicographic order by Gram-Schmidt orthonormalization process.

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# Jack symmetric functions

## Definition

Deformation of Hall scalar product:

$$\langle p_\lambda, p_\mu \rangle_\alpha = \alpha^{\ell(\lambda)} \delta_{\lambda,\mu} z_\lambda .$$

Jack symmetric functions $J_\lambda^{(\alpha)}$:

- obtained from monomial symmetric functions ordered by lexicographic order by Gram-Schmidt orthonormalization process and multiplied by explicit constant $c^{(\alpha)}(\lambda) = \prod_{\square \in \lambda} h_\alpha(\square)$;

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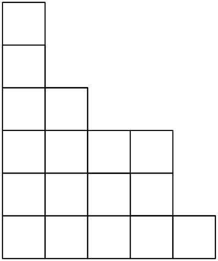
Jack symmetric functions for $\alpha = 1$ :

- obtained from monomial symmetric functions ordered by lexicographic order by Gram-Schmidt orthonormalization process and multiplied by explicit constant $c^{(1)}(\lambda) = \frac{n!}{\dim(\lambda)}$;

# Jack symmetric functions - combinatorial formula



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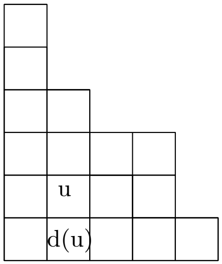

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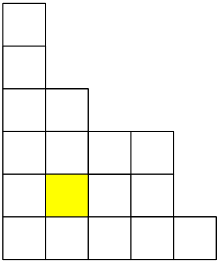

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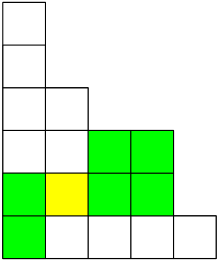
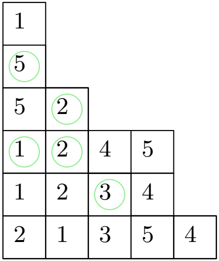


# Jack symmetric functions - combinatorial formula

**Theorem (Knop, Sahi 1997)**

$$J_\lambda^{(\alpha)} = \sum_{\sigma:\lambda^t \to \mathbb{N}_+, \textit{non-attacking}} x^\sigma \prod_{\sigma(\square)=\sigma(d(\square))} (\alpha(\ell(\square)+1)+a(\square)+1).$$

# Jack symmetric functions - combinatorial formula




$$x_1^4 x_2^4 x_3^2 x_4^3 x_5^4 \times (2\alpha+2)(2\alpha+3)(4\alpha+4)(\alpha+1)(2\alpha+1)$$

# Jack symmetric functions - combinatorial formula

**Theorem (Knop, Sahi 1997)**

$$J_\lambda^{(\alpha)} = \sum_{\sigma:\lambda^t \to \mathbb{N}_+,\textit{non-attacking}} x^\sigma \prod_{\sigma(\square)=\sigma(d(\square))} (\alpha(\ell(\square)+1)+a(\square)+1).$$

$$J_\lambda^{(\alpha)} = \lim_{t\to 1} \frac{J_\lambda(x;t^\alpha,t)}{(1-t)^{|\lambda|}}.$$

# Jack deformation

Fix $\alpha \in \mathbb{R}_{>0}$ and expand Jack polynomials $J_\lambda^{(\alpha)}$ in power-sum basis:

$$J_\lambda^{(\alpha)} = \sum_\pi \theta_\pi^{(\alpha)}(\lambda)\ p_\pi.$$

We define irreducible Jack character $\chi_\lambda^{(\alpha)}$:

$$\chi_\lambda^{(\alpha)}(\pi) := \alpha^{-\frac{\|\pi\|}{2}}\ \frac{z_\pi}{n!}\ \theta_\pi^{(\alpha)}(\lambda),$$

where $\|\pi\| := |\pi| - \ell(\pi)$.

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where $\|\pi\| := |\pi| - \ell(\pi)$.

We call $\chi : \mathcal{P}_n \to \mathbb{R}$ a reducible Jack character, if it is a convex combination of irreducible Jack characters.

# Jack deformation - examples

## Example

- Jack-Plancherel measure

$$\chi(\pi) := \begin{cases} 1 & \text{if } \pi = 1^n, \\ 0 & \text{otherwise} \end{cases} \leftrightarrow \mathbb{P}_\chi(\lambda) := \frac{n!}{\prod_{(x,y)\in\lambda} h_\alpha(x,y) h'_\alpha(x,y)}$$

- Jack-Schur-Weyl measure

$$\chi(\pi) := N^{\ell(\pi)-|\pi|} = N^{-\|\pi\|} \quad \leftrightarrow$$

$$\mathbb{P}_\chi(\lambda) := n! \prod_{(x,y)\in\lambda} \frac{N + \sqrt{\alpha}(x-1) - \sqrt{\alpha}^{-1}(y-1)}{N \cdot h_\alpha(x,y) h'_\alpha(x,y)}$$

$$= n! \prod_{(x,y)\in\lambda} \frac{N + (\sqrt{\alpha}\, x - \sqrt{\alpha}^{-1}\, y) + (\sqrt{\alpha}^{-1} - \sqrt{\alpha})}{N \cdot h_\alpha(x,y) h'_\alpha(x,y)}.$$

# Jack deformation - examples

## Example

- Jack-Plancherel measure

$$\chi(\pi) := \begin{cases} 1 & \text{if } \pi = 1^n, \\ 0 & \text{otherwise} \end{cases} \leftrightarrow \mathbb{P}_\chi(\lambda) := \frac{n!}{\prod_{(x,y)\in\lambda} h_\alpha(x,y) h'_\alpha(x,y)}$$

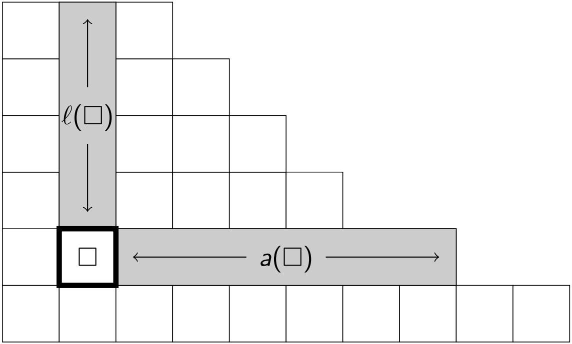
- Jack-Schur-Weyl measure

$$\chi(\pi) := N^{\ell(\pi)-|\pi|} = N^{-\|\pi\|} \quad \leftrightarrow$$

$$\mathbb{P}_\chi(\lambda) := n! \prod_{(x,y)\in\lambda} \frac{N + \sqrt{\alpha}(x-1) - \sqrt{\alpha}^{-1}(y-1)}{N \cdot h_\alpha(x,y) h'_\alpha(x,y)}$$

$$= n! \prod_{(x,y)\in\lambda} \frac{N + \overbrace{\left(\sqrt{\alpha}\, x - \sqrt{\alpha}^{-1}\, y\right)}^{c_\alpha(x,y)} + \overbrace{\left(\sqrt{\alpha}^{-1} - \sqrt{\alpha}\right)}^{\gamma}}{N \cdot h_\alpha(x,y) h'_\alpha(x,y)}.$$

# Jack deformation of hook-length formula




$$h_\alpha(\square) := \sqrt{\alpha}\ a(\square) + \sqrt{\alpha}^{-1}\ \ell(\square) + \sqrt{\alpha},$$
$$h'_\alpha(\square) := \sqrt{\alpha}\ a(\square) + \sqrt{\alpha}^{-1}\ \ell(\square) + \sqrt{\alpha}^{-1}.$$

# Asymptotic shape of large Jack-deformed Young diagrams

**Theorem (D., Śniady 2019; $\alpha = 1$ Biane 2001)**

*For each n let $\chi_n \colon \mathcal{P}_n \to \mathbb{R}$ be a reducible Jack character, and let $\alpha = \alpha(n)$ be such that*

$$\gamma := \sqrt{\alpha}^{-1} - \sqrt{\alpha} = g\sqrt{n} + g' + o(1)$$

*for some $g, g' \in \mathbb{R}$.*
*Let $\lambda_n$ be a random Young diagram with the probability distribution $\mathbb{P}_{\chi_n}$ associated with reducible Jack-characters $\chi := \chi_n$ with the approximate factorization property.*
*Then there exists some deterministic function $\omega_{\Lambda_\infty} : \mathbb{R} \to \mathbb{R}$ with the property that*

$$\lim_{n\to\infty} \omega_{\Lambda_n} = \omega_{\Lambda_\infty}, \quad \Lambda_n = T_{\sqrt{\frac{\alpha}{n}}, \sqrt{\frac{1}{\alpha n}}}(\lambda_n)$$

*where the convergence holds true with respect to the supremum norm, in probability.*

# Central limit theorem

**Theorem (D., Śniady 2019; $\alpha = 1$ Śniady 2006)**

*For each $n$ let $\chi_n \colon \mathcal{P}_n \to \mathbb{R}$ be a reducible Jack character, and let $\alpha = \alpha(n)$ be such that*

$$\gamma := \sqrt{\alpha}^{-1} - \sqrt{\alpha} = g\sqrt{n} + g' + o(1)$$

*for some $g, g' \in \mathbb{R}$.*
*Let $\lambda_n$ be a random Young diagram with the probability distribution $\mathbb{P}_{\chi_n}$ associated with reducible Jack-characters $\chi := \chi_n$ with the approximate factorization property.*

*Then the random vector $\Delta_n$ converges in distribution to some (non-centered) Gaussian random vector $\Delta_\infty$ as $n \to \infty$.*

*Equivalently, the family of random variables $(Y_k)_{k \geq 2}$ converges as $n \to \infty$ to a (non-centered) Gaussian distribution.*

# Examples

We recall that $\gamma = g\sqrt{n} + g' + o(1)$.

## Example

When $\alpha > 0$ is fixed, that is $g = 0$ then the limit shape $\omega_{\Lambda_\infty}$ does not depend on $\alpha$!.

- Jack-Plancherel measure (D., Féray 2016)

$$\omega_{\Lambda_\infty}(x) = \begin{cases} |x| & \text{if } |x| \geq 2; \\ \frac{2}{\pi}\left(x \cdot \arcsin\frac{x}{2} + \sqrt{4 - x^2}\right) & \text{otherwise.} \end{cases}$$

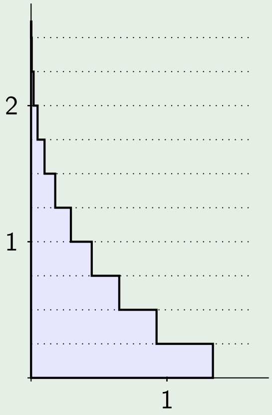
- Jack-Schur-Weyl measure with $\frac{\sqrt{n}}{N} \to c \in [0, \infty)$

$$\omega_{\Lambda_\infty}(x) - \text{explicit function depending on } c.$$

# Examples

We recall that $\gamma = g\sqrt{n} + g' + o(1)$.

## Example




An interesting choice is when $\alpha(n) = \frac{1}{c^2 n}$ for some $c > 0$, that is $g = c, g' = 0$. Then the anisotropic Young diagram $\Lambda_n$ is a collection of rectangles of the same height $g$ and of the widths $\frac{\lambda_1}{gn}, \frac{\lambda_2}{gn}, \dots,$ and the limit shape $\omega_{\Lambda_\infty}$ clearly depends on $g$!

The limit shape of random Young diagrams distributed according to the Jack–Plancherel measure in the double scaling limit for $c = \frac{1}{4}$.

# Approximate factorization property revisited

## Examples (Of measures with the AFP, thus CLT)

- Jack-Plancherel measure ($\alpha > 0$ fixed, D., Féray 2016)

$$\chi_n(\pi) := \begin{cases} 1 & \text{if } \pi = 1^n, \\ 0 & \text{otherwise} \end{cases} \quad \kappa_\ell^\chi(\pi_1, \dots, \pi_\ell) = \begin{cases} 1 & \text{if } \ell = 1, \pi_1 = 1^k, \\ 0 & \text{otherwise} \end{cases}$$

- Jack-Schur-Weyl measure ($\frac{\sqrt{n}}{N} \to c \in [0, \infty)$)

$$\chi_n(\pi) := N^{-\|\pi\|} \quad \kappa_\ell^\chi(\pi_1, \dots, \pi_\ell) = \begin{cases} N^{-\|\pi_\ell\|} & \text{if } \ell = 1, \\ 0 & \text{otherwise.} \end{cases}$$

# More examples

## Theorem

*Let $(\chi^1_n), (\chi^2_n)$ be two families of reducible Jack characters with the approximate factorization property. Then all the families consists of reducible Jack characters with the approximate factorization property:*

- *the restriction* $(\chi^i_{q,n}) := \left( (\chi^i_{q_n})^{\downarrow^{q_n}_n} \right)$, *where* $q_n \geq n$ *and* $\lim_{n\to\infty} \frac{q_n}{n} = q$;
- *the induction* $(\chi^i_{q,n}) := \left( (\chi^i_{q_n})^{\uparrow^{q_n}_n} \right)$, *where* $q_n \leq n$ *and* $\lim_{n\to\infty} \frac{q_n}{n} = q$;
- *the outer product*
$$(\chi_n) := \left( \chi^1_{q^{(1)}_n} \circ \chi^2_{q^{(2)}_n} \right),$$
*where* $q^{(1)}_n + q^{(2)}_n = n$ *and the limits* $q^{(i)} := \lim_{n\to\infty} \frac{q^{(i)}_n}{n}$ *exist;*
- *the tensor product*
$$(\chi_n) := \left( \chi^1_n \cdot \chi^2_n \right).$$

# The outer product vs. structure constants

The outer product

$$(\chi_n) := \left( \chi^{1}_{q_n^{(1)}} \circ \chi^{2}_{q_n^{(2)}} \right),$$

is defined through the multiplication of Jack polynomials:

$$J^{(\alpha)}_{\lambda} \cdot J^{(\alpha)}_{\mu} = \sum_{\nu} c^{\nu}_{\lambda,\mu}(\alpha)\ J^{(\alpha)}_{\nu}.$$

**Conjecture (Stanley 1989)**

$c^{\nu}_{\lambda,\mu}(\alpha) \cdot \langle J^{(\alpha)}_{\nu}, J^{(\alpha)}_{\nu} \rangle$ *are polynomials in $\alpha$ with nonnegative integer coefficients!*

# The main tool - algebraic combinatorics

Our main tool for proving above theorems are certain results on the structure of the algebra of polynomial functions $\mathscr{P}$.

We define the normalized Jack character $\mathrm{Ch}_\pi^{(\alpha)} : \mathbb{Y} \to \mathbb{Q}[\sqrt{\alpha}, \sqrt{\alpha}^{-1}]$:

$$\mathrm{Ch}_\pi^{(\alpha)}(\lambda) := \begin{cases} |\lambda|^{\underline{|\pi|}} \, \chi_\lambda^{(\alpha)}(\pi) & \text{if } |\lambda| \geq |\pi|; \\ 0 & \text{if } |\lambda| < |\pi|. \end{cases}$$

- Basis: $\mathscr{P} = \{\gamma^k \,\mathrm{Ch}_\pi : k \in \mathbb{N}, \pi \in \mathcal{P}\}$.
- Gradation: $\deg(\gamma^k \,\mathrm{Ch}_\pi) = k + \|\pi\|$.

# Understanding normalized Jack characters

We define the free cumulants:

$$\mathcal{R}_k^{(\alpha)}(\lambda) := \lim_{n\to\infty} \frac{\mathrm{Ch}_{(k)}(T_{n,n}(\lambda))}{n^{k+1}}, \quad k \geq 2.$$

**Proposition**

*Functionals* $\mathcal{R}_2^{(\alpha)}, \mathcal{R}_3^{(\alpha)}, \dots \in \mathscr{P}$, *they generate* $\mathscr{P}$ *and* $\deg(\mathcal{R}_k^{(\alpha)}) = k$.

**Problem**

*Express the Jack character* $\mathrm{Ch}_\mu$ *in terms of free cumulants* $\mathcal{R}_2^{(\alpha)}, \mathcal{R}_3^{(\alpha)}, \dots$.

# Kerov polynomials and a conjecture of Lassalle

A polynomial $K_\mu(\mathcal{R}_2^{(\alpha)}, \mathcal{R}_3^{(\alpha)}, \dots)$ expressing the Jack character $\mathrm{Ch}_\mu$ in terms of free cumulants is called Kerov polynomial.

$$
\begin{aligned}
K_{(1)}^{(\alpha)} &= \mathcal{R}_2, \\
K_{(2)}^{(\alpha)} &= \mathcal{R}_3 + \gamma \mathcal{R}_2, \\
K_{(3)}^{(\alpha)} &= \mathcal{R}_4 + 3\gamma \mathcal{R}_3 + (1 + 2\gamma^2)\mathcal{R}_2, \\
K_{(4)}^{(\alpha)} &= \mathcal{R}_5 + 6\gamma \mathcal{R}_4 + \gamma \mathcal{R}_2^2 + (5 + 11\gamma^2)\mathcal{R}_3 + (7\gamma + 6\gamma^3)\mathcal{R}_2, \\
K_{(5)}^{(\alpha)} &= \mathcal{R}_6 + 10\gamma \mathcal{R}_5 + 5\gamma \mathcal{R}_3 \mathcal{R}_2 + (15 + 35\gamma^2)\mathcal{R}_4 + (5 + 10\gamma^2)\mathcal{R}_2^2 \\
&\quad + (55\gamma + 50\gamma^3)\mathcal{R}_3 + (8 + 46\gamma^2 + 24\gamma^4)\mathcal{R}_2,
\end{aligned}
$$

where $\gamma = \sqrt{\alpha}^{-1} - \sqrt{\alpha}$.

# Kerov polynomials and a conjecture of Lassalle

A polynomial $K_\mu(\mathcal{R}_2^{(\alpha)}, \mathcal{R}_3^{(\alpha)}, \dots)$ expressing the Jack character $\mathrm{Ch}_\mu$ in terms of free cumulants is called Kerov polynomial.

$$K_{(1)}^{(\alpha)} = \mathcal{R}_2,$$

$$K_{(2)}^{(\alpha)} = \mathcal{R}_3 + \gamma\mathcal{R}_2,$$

$$K_{(3)}^{(\alpha)} = \mathcal{R}_4 + 3\gamma\mathcal{R}_3 + (1+2\gamma^2)\mathcal{R}_2,$$

$$K_{(4)}^{(\alpha)} = \underbrace{\mathcal{R}_5 + 6\gamma\mathcal{R}_4 + \gamma\mathcal{R}_2^2 + 11\gamma^2\mathcal{R}_3 + 6\gamma^3\mathcal{R}_2}_{\mathrm{Ch}_{(4)}^{\mathbf{top}}} + 5\mathcal{R}_3 + 7\gamma\mathcal{R}_2,$$

$$K_{(5)}^{(\alpha)} = \mathcal{R}_6 + 10\gamma\mathcal{R}_5 + 5\gamma\mathcal{R}_3\mathcal{R}_2 + (15+35\gamma^2)\mathcal{R}_4 + (5+10\gamma^2)\mathcal{R}_2^2 + (55\gamma + 50\gamma^3)\mathcal{R}_3 + (8 + 46\gamma^2 + 24\gamma^4)\mathcal{R}_2,$$

where $\gamma = \sqrt{\alpha}^{-1} - \sqrt{\alpha}$.

# Kerov polynomials and a conjecture of Lassalle

A polynomial $K_\mu(\mathcal{R}_2^{(\alpha)}, \mathcal{R}_3^{(\alpha)}, \dots)$ expressing the Jack character $\mathrm{Ch}_\mu$ in terms of free cumulants is called Kerov polynomial.

## Conjecture (Lassalle, 2009)

*Let $k \geq 1$ be a positive integer. Then $K_{(k)}$ is a polynomial in $\gamma, \mathcal{R}_2^{(\alpha)}, \mathcal{R}_3^{(\alpha)}, \dots$ with positive, integer coefficients.*

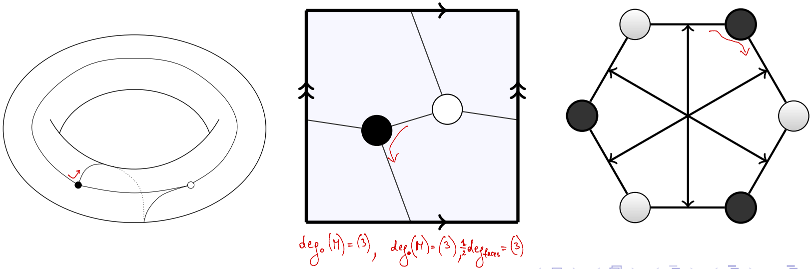
- $K_{(k)}^{(\alpha)}$ is a polynomial in $\gamma, \mathcal{R}_2^{(\alpha)}, \mathcal{R}_3^{(\alpha)}, \dots$ with rational coefficients (D., Féray 2016),
- $\mathrm{Ch}_{(k)}^{\mathrm{top}}$ is a weighted generating function of some bipartite unicellular maps (Czyżewska-Jankowska, Śniady 2017).

## Remark

*Originally, conjecture of Lassalle was stated rather vaguely, since he used a different normalization of Kerov polynomials and he suggested that there exists a way to express it as a polynomial in free cumulants and $\alpha, \beta := 1 - \alpha$ with non-negative, integer coefficients.*

# Maps

- (Bipartite) map $M$ is a connected (bipartite) graph embedded into a surface in a way that the complement of the image is homeomorphic to the collection of open discs called faces.
- Map is rooted if there is a ditingueshed corner of the map.
- The degree of a vertex/face is the number of adjacent corners.
- $(\deg_\circ, \deg_\bullet, \frac{\deg_{\text{faces}}}{2}) \to (\mu, \nu, \lambda)$.



# Maps and Jack symmetric functions revisited

- $p_i = \sum_{n\geq 0} x_n^i, \quad q_i = \sum_{n\geq 0} y_n^i, \quad r_i = \sum_{n\geq 0} z_n^i;$
- $F(t; b, \boldsymbol{p}, \boldsymbol{q}) = \sum_{n\geq 0} t^n \sum_{\lambda\vdash n} \frac{J_\lambda^{(\mathbf{1}+b)}(\boldsymbol{x}) J_\lambda^{(\mathbf{1}+b)}(\boldsymbol{y})}{\langle J_\lambda^{(\mathbf{1}+b)}, J_\lambda^{(\mathbf{1}+b)}\rangle}.$

**Theorem (Cauchy formula - Stanley 1989)**

$$F(t; b, \boldsymbol{p}, \boldsymbol{q}) = \sum_{n\geq 0} t^n \sum_{\lambda\vdash n} \frac{p_\lambda q_\lambda}{(1+b)^{\ell(\lambda)} z_\lambda} = \prod_{i,j} (1 - x_i y_j)^{-1/(1+b)}.$$

# Maps and Jack symmetric functions revisited

- $p_i = \sum_{n \geq 0} x_n^i, \quad q_i = \sum_{n \geq 0} y_n^i, \quad r_i = \sum_{n \geq 0} z_n^i;$
- $F(t; b, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) = \sum_{n \geq 0} t^n \sum_{\lambda \vdash n} \frac{J_\lambda^{(\mathbf{1}+b)}(\boldsymbol{x}) J_\lambda^{(\mathbf{1}+b)}(\boldsymbol{y}) J_\lambda^{(\mathbf{1}+b)}(\boldsymbol{z})}{\langle J_\lambda^{(\mathbf{1}+b)}, J_\lambda^{(\mathbf{1}+b)} \rangle}.$

**Conjecture (the $b$-conjecture - Goulden, Jackson 1996)**

$$\Phi(t; b, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) := (1+b) t \partial_t \log\left(F(t; b, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r})\right) =$$
$$\sum_M t^{e(M)} b^{\mathrm{MON}(M)} \prod_{f \in F(M)} p_{\deg(f)/2} \prod_{v \in V_\circ(M)} q_{\deg(v)} \prod_{v \in V_\bullet(M)} r_{\deg(v)},$$

*where we sum over all rooted, bipartite maps and* $\mathrm{MON} : \mathrm{BipMaps} \to \mathbb{N}$ *is a statistic (Measure Of Nonorientability) such that* $\mathrm{MON}(M) = 0 \iff M$ *is orientable.*

# What is known?

- Brown, Jackson 2007: there exists a statistic MON on the set of maps (not necessarily bipartite) s.t.

$$\Phi\Big|_{\boldsymbol{p}=(1,1,\dots),\boldsymbol{r}=(0,1,0,0,\dots)} = \sum_M t^{e(M)} b^{\mathrm{MON}(M)} \prod_{v\in V(M)} q_{\deg(v)},$$

- La Croix 2009: there exists a statistic MON on the set of maps (not necessarily bipartite) s.t.

$$\Phi\Big|_{\boldsymbol{p}=(P,P,\dots),\boldsymbol{r}=(0,1,0,0,\dots)} = \sum_M t^{e(M)} b^{\mathrm{MON}(M)} P^{f(M)} \prod_{v\in V(M)} q_{\deg(v)},$$

- Kannunikov, Vassilieva 2015: the coefficient $[t^n p_n q_\mu r_n]\Phi$ is a polynomial in $b$ with nonnegative integer coefficients,
- D., Féray 2017: the coefficient $[t^n p_\mu q_\nu r_\lambda]\Phi$ is a polynomial in $b$ with rational coefficients,
- Kanunnikov, Promyslov, Vassilieva 2018 the normalized coefficient $a(\mu)\cdot[t^n p_\mu q_\nu r_n]\Phi$ is a polynomial in $b$ with integer coefficients, where $a(\mu) = \prod_i \mu_i!\,\mathrm{Aut}(\mu)$.

# A bit more, but still something to do...

**Theorem (Chapuy, D. 2020+)**

*There exists a statistic* MON : BipMaps $\to \mathbb{N}$ *s.t.*

$$\Phi\Big|_{\mathbf{r}=(R,R\dots)} = \sum_{M} t^{e(M)} b^{\mathrm{MON}(M)} R^{v_\bullet(M)} \prod_{f \in F(M)} p_{\deg(f)/2} \prod_{v \in V_\circ(M)} q_{\deg(v)}.$$

- geometric interpretation in terms of weighted branched coverings of the sphere
- $b$-deformation of classical (single or double) Hurwitz numbers obtained as a specialization $b = 0$.

# Perspectives

- Limit shape of the Jack-Plancherel measure (or other measures given by convex characters) in the double scaling limit?
- Covariance of normal distribution in the double scaling limit = the top-degree of normalized Jack characters indexed by two rows = the combinatorics of unhandled maps with two faces.
- Joint distribution of properly normalized $\left(\lambda_{(n)}\right)_1 \geq \left(\lambda_{(n)}\right)_2 \geq \dots$ with respect to Jack-Plancherel measure = Tracy-Widom $\beta$ (Guionnet, Huang 2019). What about convex characters with AFP?
- Understand better a relation between our model and other models of discrete $\beta$-ensembles (Moll, Borodin-Gorin-Guionnet)

# Thank you

THANK YOU FOR YOUR ATTENTION!