Partition identities and $A_n^{(1)}$ crystals

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Outline

1. Partition identities coming from representation theory
2. Infinite family of identities generalising Primc's identity
3. Representation theoretic consequences
4. Connection with Capparelli's identity

Integer partitions

Definition

A *partition* π of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \dots, \lambda_m$ such that $\lambda_1 + \cdots + \lambda_m = n$. The integers $\lambda_1, \dots, \lambda_m$ are called the *parts* of the partition.

Example

There are 5 partitions of 4:

$$4, 3+1, 2+2, 2+1+1 \text{ and } 1+1+1+1.$$

Generating functions

Notation : $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), n \in \mathbb{N} \cup \{\infty\}$.

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Let $Q(n,k)$ be the number of partitions of n into k distinct parts. Then

$$\begin{aligned} 1 + \sum_{n\geq 1}\sum_{k\geq 1} Q(n,k) z^k q^n &= (1+zq)(1+zq^2)(1+zq^3)(1+zq^4)\cdots \\ &= (-zq;q)_\infty. \end{aligned}$$

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Let $p(n,k)$ be the number of partitions of n into k parts. Then

$$\begin{aligned} 1 + \sum_{n\geq 1}\sum_{k\geq 1} p(n,k) z^k q^n &= \prod_{n\geq 1}\left(1 + zq^n + z^2q^{2n} + \cdots\right) \\ &= \frac{1}{(zq;q)_\infty}. \end{aligned}$$

Generating functions

More generally:

- The generating function for partitions into distinct parts congruent to $k \mod N$ is
$$(-zq^k; q^N)_\infty.$$
- The generating function for partitions into parts congruent to $k \mod N$ is
$$\frac{1}{(zq^k; q^N)_\infty}.$$

So the general shape of a generating function for partitions with congruence conditions is

$$\frac{(-z_1 q^{k_1}; q^{N_1})_\infty \cdots (-z_s q^{k_s}; q^{N_s})_\infty}{(z_1' q^{k_1'}; q^{N_1'})_\infty \cdots (z_r' q^{k_r'}; q^{N_r'})_\infty}.$$

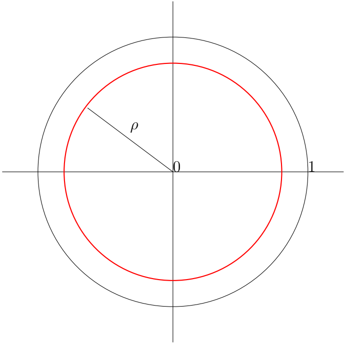
Asymptotics

Partitions with congruence conditions are easy to study asymptotically via the circle method.

For all $n \in \mathbb{N}$,

$$p(n) = \frac{1}{2i\pi} \oint_\gamma \frac{q^{-n-1}}{(q;q)_\infty}\, dq,$$

where γ is any circle centred at the origin with radius $\rho < 1$.



Theorem (Hardy–Ramanujan 1918)

As $n \to \infty$,

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

The Rogers–Ramanujan identities

Theorem (Rogers 1894, Rogers–Ramanujan 1919)

$$\sum_{n=0}^{\infty} RR_1(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty},$$

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Theorem (Partition version)

For every positive integer n, the number of partitions of n such that the difference between two consecutive parts is at least 2 is equal to the number of partitions of n into parts congruent to 1 or 4 modulo 5.

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Theorem (Asymptotics)

As $n \to \infty$,

$$RR_1(n) \sim \frac{1}{4 \cdot 15^{1/4} \cdot n^{3/4} \cdot \sin(\pi/5)} \exp\left(\frac{2\pi\sqrt{n}}{\sqrt{15}}\right).$$

The Rogers–Ramanujan identities

Theorem (Second Rogers–Ramanujan identity)

$$\sum_{n=0}^{\infty} RR_2(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty},$$

Theorem (Partition version)

For every positive integer n, the number of partitions of n such that the difference between two consecutive parts is at least 2 *and the smallest part is $\geq$* 2 *is equal to the number of partitions of n into parts congruent to* 2 *or* 3 *modulo* 5*.*

Theorem (Asymptotics)

As $n \to \infty$,

$$RR_2(n) \sim \frac{1}{4 \cdot 15^{1/4} \cdot n^{3/4} \cdot \sin(2\pi/5)} \exp\left(\frac{2\pi\sqrt{n}}{\sqrt{15}}\right).$$

Some definitions on Lie algebras

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra with Cartan subalgebra $\mathfrak{h}$. The corresponding (derived) affine Lie algebra $\hat{\mathfrak{g}}$ is constructed as

$$\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where $\mathbb{C}[t, t^{-1}]$ is the complex vector space of Laurent polynomials in the indeterminate t, and $\mathbb{C}c$ is $\hat{\mathfrak{g}}$'s center (one-dimensional).

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If V is an irreducible highest weight module of $\hat{\mathfrak{g}}$, the central element c acts on V by multiplication by a scalar k, which is called the **level** of V. The **character** $\mathrm{ch}(V)$ of V is defined as

$$\mathrm{ch}(V) = \sum_{\mu} \dim(V_\mu) e^{\mu},$$

where the sum is over the weights μ of V,
$V_\mu := \{v \in V : \forall H \in \mathfrak{h}, \quad H \cdot v = \mu(H)v\}$ is a weight space,
and e^μ is a formal exponential satisfying $e^\mu e^{\mu'} = e^{\mu+\mu'}$.

Representation theoretic interpretation

Lepowsky and Wilson 1984: representation theoretic interpretation

$$\frac{1}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q;q^2)_\infty}\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}$$

Obtained by giving two different formulations for the principal specialisation

$$e^{-\alpha_0}\mapsto q,\quad e^{-\alpha_1}\mapsto q$$

of $e^{-\Lambda}\mathrm{ch}(L(\Lambda))$ where $L(\Lambda)$ is an irreducible highest weight $A_1^{(1)}$-module of level 3.

RHS: principal specialisation of the Weyl-Kac character formula

LHS: comes from the construction of a basis of V using vertex operators

Some other identities from representation theory

Studying other representations or other Lie algebras lead to new identities:

- Capparelli 1993: level 3 standard modules of $A_2^{(2)}$
- Nandi 2014: level 4 standard modules of $A_2^{(2)}$
- Meurman and Primc 1987-1999: higher levels of $A_1^{(1)}$
- Siladić 2002: twisted level 1 modules of $A_2^{(2)}$
- Primc 1999: $A_2^{(1)}$ and $A_1^{(1)}$ crystals
- Primc and Šikić 2016: level k standard modules of $C_n^{(1)}$

Capparelli's identity

From the study of level 3 standard modules of $A_2^{(2)}$:

Theorem (Capparelli (conj. 1992, proof 1994), Andrews 1992)

Let $C(n)$ *denote the number of partitions of* n *into distinct parts congruent to* $0, 2, 3, 4 \mod 6$*.*
Let $D(n)$ *denote the number of partitions* $\lambda_1 + \cdots + \lambda_s$ *of* n *such that* $\lambda_s \neq 1$ *and*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 2 & \textit{if } \lambda_i, \lambda_{i+1} \equiv 0 \mod 3 \textit{ or } \lambda_i + \lambda_{i+1} \equiv 0 \mod 6 \\ 4 & \textit{otherwise.} \end{cases}$$

Then for all n, $C(n) = D(n)$*.*

Example

The partitions counted by $C(9)$ are 9, $6+3$, and $4+3+2$.
The partitions counted by $D(9)$ are 9, $7+2$ and $6+3$.

The method of weighted words (Alladi–Andrews–Gordon)

- Consider partitions in coloured integers

$$1_a < 1_c < 1_d < 2_a < 2_c < 2_d < \cdots ,$$

satisfying the difference conditions

$$\lambda_i - \lambda_{i+1} \geq C(color(\lambda_i), color(\lambda_{i+1})),$$

where C is the following matrix

$$C = \begin{array}{c} \\ a \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} a & c & d \end{array} \\ \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \end{array}.$$

After performing the transformations

$$k_a \mapsto 3k-1, k_c \mapsto 3k, k_d \mapsto 3k+1.$$

these partitions satisfy the difference conditions of Capparelli's identity.

The method of weighted words (Alladi–Andrews–Gordon)

- Compute “directly” the generating function for $C(n; i, j, k)$, the number of partitions of n with i parts coloured a, j parts coloured c and k parts coloured d , satisfying the difference conditions from matrix D.

$$\sum_{i,j,k,n\geq 0} C(n; i, j, k) a^i c^j d^k q^n = \sum_{i,j\geq 0} \frac{a^i d^j q^{i^2+j^2} (-q; q)_{i+j} (-cq^{i+j+1}, q)_\infty}{(q^2; q^2)_i (q^2; q^2)_j}.$$

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- Using q-series identities, we show that this is a suitable infinite product if and only if $c = 1$, and in that case it equals

$$\frac{(-aq; q^2)_\infty (-dq; q^2)_\infty}{(q; q^2)_\infty}.$$

Non-dilated version (Alladi-Andrews-Gordon 1993)

Theorem (Non-dilated version of Capparelli's identity)

Let $C(n; k, m)$ denote the number of partitions satisfying the difference conditions of matrix C, with k parts coloured a and m parts coloured d.

$$\sum_{n,k,m \geq 0} C(n; k, \ell, m) q^n a^k d^m = \frac{(-aq; q^2)_\infty (-dq; q^2)_\infty}{(q; q^2)_\infty}.$$

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The dilation $q \to q^3, a \to aq^{-1}, d \to dq$ gives a refinement of Capparelli's identity.

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By using other dilations or changing the order on the integers, one can obtain infinitely many new partition identities.

Primc's identity from crystal bases of $A_1^{(1)}$ (1999)

Partitions in four colours a, b, c, d, with the order

$$1_a < 1_b < 1_c < 1_d < 2_a < 2_b < 2_c < 2_d < \cdots ,$$

and difference conditions

$$P = \begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \end{array}.$$

After performing the dilations

$$k_a \to 2k-1, k_b \to 2k, k_c \to 2k, k_d \to 2k+1,$$

the generating function for these partitions (not keeping track of the colours) becomes $\frac{1}{(q;q)_\infty}$.

Non-dilated version

Theorem (D.–Lovejoy 2017)

Let $P(n; k, \ell, m)$ denote the number of partitions satisfying the difference conditions of matrix P, with k parts coloured a, ℓ parts coloured c and m parts coloured d. Then

$$\sum_{n,k,\ell,m\geq 0} P(n; k, \ell, m) q^n a^k c^\ell d^m = \frac{(-aq; q^2)_\infty (-dq; q^2)_\infty}{(q; q)_\infty (cq; q^2)_\infty}.$$

Proved via a variant of the method of weighted words (D. 2016) using q-difference equations.

Another identity of Primc (1999)

Studying crystal bases of $A_2^{(1)}$, Primc proved that, after performing certain dilations (corresponding to the principal specialisation), the generating function for coloured partitions satisfying the difference conditions

$$\begin{array}{c|ccccccccc}
 & a_2b_0 & a_2b_1 & a_1b_0 & a_0b_0 & a_2b_2 & a_1b_1 & a_0b_1 & a_1b_2 & a_0b_2 \\
\hline
a_2b_0 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\
a_2b_1 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\
a_1b_0 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 2 \\
a_0b_0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
a_2b_2 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 2 & 2 \\
a_1b_1 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 2 \\
a_0b_1 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 2 \\
a_1b_2 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 2 & 2 \\
a_0b_2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 2
\end{array}$$

becomes

$$\frac{1}{(q;q)_\infty}.$$

An identity of Meurman and Primc (1999)

Under the same dilations, the generating function for coloured partitions satisfying the difference conditions

$$\begin{array}{cccccccccc}
 & a_2b_0 & a_2b_1 & a_1b_0 & a_2b_2 & a_1b_1 & a_0b_1 & a_1b_2 & a_0b_2 & \\
a_2b_0 & \left(\begin{array}{c} 2 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} 2 \\ 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{array} & \begin{array}{c} 2 \\ 1 \\ 2 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{array} & \begin{array}{c} 2 \\ 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{array} & \begin{array}{c} 2 \\ 1 \\ 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} & \begin{array}{c} 2 \\ 2 \\ 2 \\ 1 \\ 2 \\ 2 \\ 1 \\ 1 \end{array} & \begin{array}{c} 2 \\ 2 \\ 2 \\ 2 \\ 1 \\ 1 \\ 2 \\ 1 \end{array} & \left.\begin{array}{c} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{array}\right)
\end{array}$$

$$\begin{array}{c|cccccccc}
 & a_2b_0 & a_2b_1 & a_1b_0 & a_2b_2 & a_1b_1 & a_0b_1 & a_1b_2 & a_0b_2 \\
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a_1b_1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 \\
a_0b_1 & 0 & 1 & 0 & 1 & 1 & 2 & 1 & 2 \\
a_1b_2 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 \\
a_0b_2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2
\end{array}$$

and avoiding the patterns $(k+1)_{a_1b_0} + k_{a_2b_2} + k_{a_2b_0}$ and $(k+1)_{a_0b_2} + (k+1)_{a_2b_2} + k_{a_0b_1}$ is

$$\frac{(q^3;q^3)_\infty}{(q;q)_\infty}.$$

Outline

Generalised difference conditions

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of symbols.
We use them to define sets of colours:

- free colours: $\{a_i b_i : i \in \mathbb{N}\}$,
- bound colours: $\{a_i b_k : i \neq k, i, k \in \mathbb{N}\}$.

Definition

For all $i, k, i', k' \in \mathbb{N}$, we define the minimal difference Δ between a part coloured $a_i b_k$ and a part coloured $a_{i'} b_{k'}$ in the following way:

$$\Delta(a_i b_k, a_{i'} b_{k'}) = \chi(i \geq i') - \chi(i = k = i') + \chi(k \leq k') - \chi(k = i' = k'),$$

where $\chi(\mathit{prop})$ equals 1 if *prop* is true and 0 otherwise.

For every positive integer n, let $\mathcal{P}_n$ denote the set of partitions with colours $\{a_i b_k : 0 \leq i, k \leq n-1\}$, satisfying the difference conditions Δ.

Generalised difference conditions

$$\Delta(a_i b_k, a_{i'} b_{k'}) = \chi(i \geq i') - \chi(i = k = i') + \chi(k \leq k') - \chi(k = i' = k').$$

- $\mathcal{P}_2$: Primc partitions

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 2 & 1 & 2 & 2 \\ b & 1 & 0 & 1 & 1 \\ c & 0 & 1 & 0 & 2 \\ d & 0 & 1 & 0 & 2 \end{array},$$

  with the correspondence

$$a = a_1 b_0, \qquad b = a_0 b_0, \qquad c = a_1 b_1, \qquad d = a_0 b_1.$$

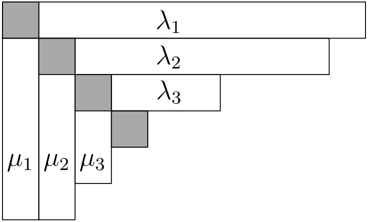
- $\mathcal{P}_3$: 9-coloured partitions satisfying the difference conditions of Primc's 9×9 matrix.

Frobenius partitions

A Frobenius partition is a two-rowed array

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ \mu_1 & \mu_2 & \cdots & \mu_s \end{pmatrix},$$

where $\lambda := \lambda_1 + \lambda_2 + \cdots + \lambda_s$ and $\mu := \mu_1 + \mu_2 + \cdots + \mu_s$ are two partitions into s distinct non-negative parts.
Its weight is $s + \sum_{i=1}^{s} \lambda_i + \sum_{i=1}^{s} \mu_i$.



Frobenius partitions are in bijection with classical partitions ($s =$ size of Durfee square).

Coloured Frobenius partitions

A n^2*-coloured Frobenius partition* is a Frobenius partition

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ \mu_1 & \mu_2 & \cdots & \mu_s \end{pmatrix},$$

where $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_s$ is a partition into s distinct non-negative parts coloured with some a_i, $i \in \{0, \ldots, n-1\}$, with the order

$$0_{a_{n-1}} < 0_{a_{n-2}} < \cdots < 0_{a_0} < 1_{a_{n-1}} < 1_{a_{n-2}} < \cdots < 1_{a_0} < \cdots ,$$

and $\mu = \mu_1 + \mu_2 + \cdots + \mu_s$ is a partition into s distinct non-negative parts coloured with some b_i, $i \in \{0, \ldots, n-1\}$, with the order

$$0_{b_0} < 0_{b_1} < \cdots < 0_{b_{n-1}} < 1_{b_0} < 1_{b_1} < \cdots < 1_{b_{n-1}} < \cdots .$$

Let $\mathcal{F}_n$ denote the set of n^2-coloured Frobenius partitions.

Coloured Frobenius partitions

The *colour sequence* of

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ \mu_1 & \mu_2 & \cdots & \mu_s \end{pmatrix}$$

is $c(\lambda_1)c(\mu_1), \ldots, c(\lambda_s)c(\mu_s)$.

Let $F_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1})$ denote the number of n^2-coloured Frobenius partitions of m, where for $i \in \{0, \ldots, n-1\}$, the symbol a_i (resp. b_i) appears u_i (resp. v_i) times in its colour sequence. Then

$$\sum_{m,u_0,\ldots,u_{n-1},v_0,\ldots,v_{n-1} \geq 0} F_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1}) q^m a_0^{u_0} \cdots a_{n-1}^{u_{n-1}} b_1^{v_1} \cdots b_{n-1}^{v_{n-1}}$$
$$= [x^0] \prod_{i=0}^{n-1} (-xa_iq; q)_\infty (-x^{-1}b_i; q)_\infty .$$

Reduced colour sequences

Given a sequence $C = c_1, \dots, c_s$ of colours taken from $\{a_i b_k : i, k \in \mathbb{N}\}$, do the following operations as long as possible:

- if there is some i such that $c_i = a_k b_\ell$ and $c_{i+1} = a_\ell b_\ell$, then remove c_{i+1} from the colour sequence,
- if there is some i such that $c_i = a_k b_k$ and $c_{i+1} = a_k b_\ell$, then remove c_i from the colour sequence.

The resulting colour sequence is called the *reduction* of C, denoted $\mathrm{red}(C)$. A colour sequence that cannot be reduced is a *reduced colour sequence*.

Example

The reduction of

$$a_1 b_1, a_1 b_2, a_2 b_2, a_3 b_3, a_3 b_1, a_1 b_3, a_3 b_3, a_3 b_3, a_3 b_2, a_1 b_1$$

is

$$a_1 b_2, a_3 b_1, a_1 b_3, a_3 b_2, a_1 b_1.$$

Link between Primc generalised partitions and coloured Frobenius partitions

Let $\lambda = \lambda_1 + \cdots + \lambda_s$ be a partition such that $c(\lambda_1) = c_1, \ldots, c(\lambda_s) = c_s$. The *kernel* of λ, denoted by $\ker(\lambda)$, is the reduced colour sequence $\mathrm{red}(c_1, \ldots, c_s)$.

Theorem (D.–Konan (2019))

Let n be a positive integer and m a non-negative integer. Let $S = c_1, \ldots, c_s$ *be a reduced colour sequence. Then*

$$\sum_{\substack{\lambda \in \mathcal{P}_n: \\ \ker(\lambda) = S}} q^{|\lambda|} = \sum_{\substack{F \in \mathcal{F}_n: \\ \ker(F) = S}} q^{|F|}.$$

Proof: combinatorial reasoning on reduced colour sequences + q-series manipulations.

Generalisation of Primc's identity

Set for all i, $a_i = b_i^{-1}$.
Let $P_n(m; u_0, \dots, u_{n-1}; v_0, \dots, v_{n-1})$ denote the number of n^2-coloured Primc partitions of m, where for $i \in \{0, \dots, n-1\}$, the symbol a_i (resp. b_i) appears u_i (resp. v_i) times in its colour sequence.

Theorem (D.–Konan (2019))

For every positive integer n, we have

$$\sum_{m,u_0,\dots,u_{n-1},v_0,\dots,v_{n-1} \geq 0} P_n(m; u_0, \dots, u_{n-1}; v_0, \dots, v_{n-1}) q^m b_0^{v_0-u_0} \cdots b_{n-1}^{v_{n-1}-u_{n-1}}$$

$$= \sum_{m,u_0,\dots,u_{n-1},v_0,\dots,v_{n-1} \geq 0} F_n(m; u_0, \dots, u_{n-1}; v_0, \dots, v_{n-1}) q^m b_0^{v_0-u_0} \cdots b_{n-1}^{v_{n-1}-u_{n-1}}$$

$$= [x^0] \prod_{i=0}^{n-1} (-b_i^{-1} x q; q)_\infty (-b_i x^{-1}; q)_\infty .$$

Principal specialisation

In his paper, Primc used the principal specialisation:

$$\begin{cases} q & \mapsto \quad q^n \\ b_i & \mapsto \quad q^i \quad \text{for all } i \in \{0, \dots, n-1\}. \end{cases}$$

Corollary (D.–Konan (2019))

Let n be a positive integer. By doing the dilations above, the generating function for the Primc generalised partitions in $\mathcal{P}_n$ becomes:

$$[x^0] \prod_{i=0}^{n-1} (-q^{n-i}x; q^n)_\infty (-q^i x^{-1}; q^n)_\infty = [x^0](-qx; q)_\infty (-x^{-1}; q)_\infty$$

$$= \frac{1}{(q; q)_\infty}.$$

The cases $n = 2$ and $n = 3$ recover Primc's original results.

Asymptotics

Theorem (Andrews (1984))

Let $F_n(m)$ be the number of n^2-coloured Frobenius partitions of m. As $m \to \infty$,

$$F_n(m) \sim \frac{1}{4m\sqrt{3}} \exp\left(\pi\sqrt{\frac{2nm}{3}}\right).$$

Recall that the asymptotics for the number of partitions of n was

$$p(m) \sim \frac{1}{4m\sqrt{3}} \exp\left(\pi\sqrt{\frac{2m}{3}}\right).$$

Expression as a sum of infinite products

Theorem (D.–Konan (2019))

For every positive integer n, we have

$$[x^0]\prod_{i=0}^{n-1}(-b_i^{-1}xq;q)_\infty(-b_ix^{-1};q)_\infty$$
$$=\frac{1}{(q;q)_\infty}\left(\prod_{i=1}^{n-1}\frac{\left(q^{i(i+1)};q^{i(i+1)}\right)_\infty}{(q;q)_\infty}\right)\sum_{\substack{r_1,\dots,r_{n-1}:\\0\leq r_j\leq j-1\\r_n=0}}\prod_{i=1}^{n-1}b_i^{-r_i+r_{i+1}}q^{r_i(r_i-r_{i+1})}$$
$$\times\left(-\left(\prod_{\ell=0}^{i-1}b_\ell b_i^{-1}\right)q^{\frac{i(i+1)}{2}+(i+1)r_i-ir_{i+1}};q^{i(i+1)}\right)_\infty$$
$$\times\left(-\left(\prod_{\ell=0}^{i-1}b_i b_\ell^{-1}\right)q^{\frac{i(i+1)}{2}-(i+1)r_i+ir_{i+1}};q^{i(i+1)}\right)_\infty.$$

Outline

1. Partition identities coming from representation theory
2. Infinite family of identities generalising Primc's identity
3. Representation theoretic consequences
4. Connection with Capparelli's identity

Crystals: "combinatorial representations" of Lie algebras

Crystal for the vector representation of the affine Lie algebra $A_{n-1}^{(1)}$:

$$\begin{tikzcd}
\mathcal{B}: & \boxed{0} \arrow[r, "1"] & \boxed{1} \arrow[r, "2"] & \cdots \arrow[r, "n-2"] & \boxed{n-2} \arrow[r, "n-1"] & \boxed{n-1} \arrow[llll, bend left, "0"]
\end{tikzcd}$$

If $b_1 \xrightarrow{i} b_2$, we write $\tilde{f}_i b_1 = b_2$, or equivalently $b_1 = \tilde{e}_i b_2$.
Let $\varphi_i(b)$ (resp. $\varepsilon_i(b)$) denote the length of the maximal chain of i-arrows emanating from (resp. arriving in) b.

The dual of $\mathcal{B}$:

$$\begin{tikzcd}
\mathcal{B}^\vee: & \boxed{0} \arrow[rrrr, bend right, "0"'] & \boxed{1} \arrow[l, "1"'] & \cdots \arrow[l, "2"'] & \boxed{n-2} \arrow[l, "n-2"'] & \boxed{n-1} \arrow[l, "n-1"']
\end{tikzcd}$$

We have $\tilde{f}_i b_1 = b_2$ in $\mathcal{B}$ if and only if $\tilde{e}_i b_1^\vee = b_2^\vee$.

Crystals: “combinatorial representations” of Lie algebras

If $\mathcal{B}_1$ is a crystal for the representation M_1 and $\mathcal{B}_2$ is a crystal for the representation M_2, then we can define a crystal $\mathcal{B}_1 \otimes \mathcal{B}_2$ with the following arrows:

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$
$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases}$$

and $\mathcal{B}_1 \otimes \mathcal{B}_2$ is a crystal for $M_1 \otimes M_2$.

Example: $A_1^{(1)}$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases}$$

$$\mathcal{B} : \quad \begin{tikzcd} \boxed{0} \arrow[r, "1"] & \boxed{1} \arrow[l, bend left, "0"] \end{tikzcd}$$

$$\mathcal{B} \otimes \mathcal{B}^{\vee} : \quad \begin{tikzcd} \boxed{0} \otimes \boxed{1} \arrow[r, "1"] & \boxed{1} \otimes \boxed{1} \arrow[d, "1"] \\ \boxed{0} \otimes \boxed{0} \arrow[u, "0"] & \boxed{1} \otimes \boxed{0} \arrow[l, "0"] \end{tikzcd}$$

Energy functions

Definition

An *energy function* on $\mathcal{B} \otimes \mathcal{B}$ is a map $H : \mathcal{B} \otimes \mathcal{B} \to \mathbb{Z}$ satisfying for all i,

$$H\left(\tilde{e}_i(b_1 \otimes b_2)\right) = \begin{cases} H(b_1 \otimes b_2) & \text{if} \quad i \neq 0, \\ H(b_1 \otimes b_2) + 1 & \text{if} \quad i = 0 \text{ and } \varphi_0(b_1) \geq \varepsilon_0(b_2) \\ H(b_1 \otimes b_2) - 1 & \text{if} \quad i = 0 \text{ and } \varphi_0(b_1) < \varepsilon_0(b_2). \end{cases}$$

By definition, the value of $H(b_1 \otimes b_2)$ determines the values $H(b_1' \otimes b_2')$ of all the vertices $b_1' \otimes b_2'$ which are in the same connected component as $b_1 \otimes b_2$.

The matrix P of Primc gives the energy functions in $(\mathcal{B} \otimes \mathcal{B}^\vee) \otimes (\mathcal{B} \otimes \mathcal{B}^\vee)$ for $A_1^{(1)}$. Can this be generalised?

Our difference conditions as energy functions

Theorem (D.–Konan (2019))

Let n be a positive integer, and let $\mathcal{B}$ denote the crystal of the vector representation of $A_{n-1}^{(1)}$. The crystal $\mathbb{B} = \mathcal{B} \otimes \mathcal{B}^{\vee}$ is a perfect crystal of level 1. Furthermore, the energy function on $\mathbb{B} \otimes \mathbb{B}$ such that $H((v_0 \otimes v_0^{\vee}) \otimes (v_0 \otimes v_0^{\vee})) = 0$ satisfies for all $k, \ell, k', \ell' \in \{0, \dots, n-1\}$,

$$H((v_{\ell'} \otimes v_{k'}^{\vee}) \otimes (v_{\ell} \otimes v_{k}^{\vee})) = \Delta(a_k b_\ell; a_{k'} b_{\ell'}),$$

where Δ is the minimal difference for the Primc generalised partitions.

Previous result of Benkart, Frenkel, Kang, and Lee (2006) : value of the energy on each connected component of $\mathbb{B} \otimes \mathbb{B}$ after removal of 0-arrows.

A character formula with obviously positive coefficients

The $(KMN)^2$ character formula allows us to relate the character of the irreducible highest weight $A_{n-1}^{(1)}$-modules of level 1 with the generating function for Primc generalised partitions.

Theorem (D.–Konan (2019))

Let $\Lambda_0, \dots, \Lambda_{n-1}$ *be the fundamental weights of* $A_{n-1}^{(1)}$. *For all* $\ell \in \{0, \dots, n-1\}$, *we have*

$$e^{-\Lambda_\ell} \mathrm{ch}(L(\Lambda_\ell)) = \left(\prod_{i=1}^{n-1} \frac{\left(e^{-i(i+1)\delta}; e^{-i(i+1)\delta}\right)_\infty}{(e^{-\delta}; e^{-\delta})_\infty} \right) \sum_{\substack{r_1, \dots, r_{n-1} \\ r_0 = r_n = 0 \\ 0 \leq r_j \leq j-1}} e^{-r_l \delta} \prod_{i=1}^{n-1} e^{r_i \alpha_i} e^{r_i (r_{i+1} - r_i)\delta}$$

$$\times \left(-e^{(ir_{i+1} - (i+1)r_i - \frac{i(i+1)}{2} - \ell \chi(i \geq l > 0))\delta + \sum_{j=1}^{i} j\alpha_j}; e^{-i(i+1)\delta} \right)_\infty$$

$$\times \left(-e^{((i+1)r_i - ir_{i+1} - \frac{i(i+1)}{2} + \ell \chi(i \geq l > 0))\delta - \sum_{j=1}^{i} j\alpha_j}; e^{-i(i+1)\delta} \right)_\infty .$$

A character formula with obviously positive coefficients

Theorem (Kac–Peterson (1984))

Let $\Lambda_0, \ldots, \Lambda_{n-1}$ *be the fundamental weights of* $A_{n-1}^{(1)}$. *For all* $\ell \in \{0, \ldots, n-1\}$, *we have*

$$e^{-\Lambda_\ell} \mathrm{ch}(L(\Lambda_\ell)) = \frac{1}{(e^{-\delta}; e^{-\delta})_\infty^{n-1}} \sum_{\substack{s_1, \ldots, s_{n-1} \in \mathbb{Z} \\ s_0 = s_n = 0}} e^{-s_\ell \delta} \prod_{i=1}^{n-1} e^{s_i \alpha_i} e^{s_i (s_{i+1} - s_i) \delta}.$$

This formula can be easily recovered from the generating function for coloured Frobenius partitions. Thus, our partition identity gives a combinatorial connection between the $(\mathrm{KMN})^2$ character formula and the Kac-Peterson character formula for $A_{n-1}^{(1)}$.

Outline

1. Partition identities coming from representation theory

2. Infinite family of identities generalising Primc's identity

3. Representation theoretic consequences

4. Connection with Capparelli's identity

Original connection between Capparelli and Primc

Capparelli	**Primc**
$\begin{array}{cc} & \begin{matrix} a & c & d \end{matrix} \\ \begin{matrix} a \\ c \\ d \end{matrix} & \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \end{array}$	$\begin{array}{cc} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \end{array}$
$\dfrac{(-aq;q^2)_\infty(-dq;q^2)_\infty}{(q;q^2)_\infty}$	$\dfrac{(-aq;q^2)_\infty(-dq;q^2)_\infty}{(q;q)_\infty(cq;q^2)_\infty}$

Original bijection

Theorem (D. (2018))

Let $\mathcal{CC}_2$ denote the set of partition pairs (λ, μ) where λ is a coloured Capparelli partition and μ is a classical partition coloured b.
Let $\mathcal{P}_2$ denote the set of 4-coloured Primc partitions.
There is a bijection between $\mathcal{CC}_2$ and $\mathcal{P}_2$ which preserves the total weight, the number of parts, the size of the parts, and the number of appearances of colours a and d.

Can this also be generalised?

A first family of difference conditions

Definition

For all $i, k, i', k' \in \mathbb{N}$, let us define the minimal difference $\delta(a_i b_k, a_{i'} b_{k'})$ between a part coloured $a_i b_k$ and a part coloured $a_{i'} b_{k'}$ in the following way:

$$\begin{aligned} \delta(a_k b_k, a_k b_k) &= 1 \text{ for all } k > 0, \\ \delta(a_k b_k, a_k b_\ell) &= 1 \text{ for all } \ell < k, \\ \delta(a_\ell b_k, a_k b_k) &= 1 \text{ for all } \ell < k, \\ \delta(a_i b_k, a_{i'} b_{k'}) &= \Delta(a_i b_k, a_{i'} b_{k'}) \text{ in all the other cases.} \end{aligned}$$

For every positive integer n, let $\mathcal{C}_n$ be the set of partitions with colours $\{a_i b_k : 0 \leq i, k \leq n-1, (i,k) \neq (0,0)\}$, satisfying the difference conditions δ and some additional forbidden patterns of length 3 (except when $n = 2$).

$n = 2$: Capparelli

A second family of difference conditions

Definition

For all $i, k, i', k' \in \mathbb{N}$, define

$$\begin{aligned} \delta'(a_k b_k, a_k b_k) &= 1 \text{ for all } k \in \mathbb{N}^*, \\ \delta'(a_k b_k, a_\ell b_{k-1}) &= 1 \text{ for all } \ell \geq k \geq 1, \\ \delta'(a_{k-1} b_\ell, a_k b_k) &= 1 \text{ for all } \ell \geq k \geq 1, \\ \delta'(a_i b_k, a_{i'} b_{k'}) &= \Delta(a_i b_k, a_{i'} b_{k'}) \text{ in all the other cases.} \end{aligned}$$

For every positive integer n, let $\mathcal{C}'_n$ be the set of partitions with colours $\{a_i b_k : 0 \leq i, k \leq n-1, (i,k) \neq (0,0)\}$, satisfying the difference conditions δ' and some additional forbidden patterns of length 3 (except when $n = 2$).

$n = 2$: Capparelli
$n = 3$: Meurman–Primc's 8×8 matrix + forbidden patterns

Generalised bijection

Theorem (D.-Konan (2019))

For every positive integer n, let $\mathcal{CC}_n$ *(resp.* $\mathcal{CC}'_n$*) denote partition pairs* (λ, μ)*, where* $\lambda \in \mathcal{C}_n$ *(resp.* $\mathcal{C}'_n$*) and* μ *is a partition where all parts have colour* $a_0 b_0$*.*

There is a bijection between:

- *coloured partitions in* $\mathcal{P}_n$*,*
- *coloured partition pairs in* $\mathcal{CC}_n$*,*
- *coloured partition pairs in* $\mathcal{CC}'_n$*.*

This bijection preserves the total weight, the number of parts, the size of the parts, and the number of appearances of each bound colour.

$\rightarrow$ After multiplication by $(q;q)_\infty$, all the formulas for the generating functions of Primc partitions are also true for these two generalisations of Capparelli.

$n = 2$: bijection between Capparelli and Primc

Free colours: b and c.

Important remark:

$$P = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 2 & 1 & 2 & 2 \\ b & 1 & 0 & 1 & 1 \\ c & 0 & 1 & 0 & 2 \\ d & 0 & 1 & 0 & 2 \end{array}.$$

The values in column and row b in matrix P mean that if there is a part k_b in the partition, then it can repeat but the number k cannot appear in any other colour.

$n = 2$: bijection between Capparelli and Primc

Let $(\lambda, \mu) \in \mathcal{CC}_2$. The partition λ satisfies the difference conditions

$$C = \begin{array}{c} \\ a \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} a & c & d \end{array} \\ \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \end{array},$$

and μ is a partition coloured c.

Example

$$\lambda = 8_d + 8_a + 6_c + 5_c + 3_d + 1_a,$$
$$\mu = 8_b + 8_b + 7_b + 5_b + 3_b + 2_b + 2_b + 1_b + 1_b.$$

Step 1: For all j, if there are some parts of size j in μ but none in λ, then move these parts from μ to λ, according to the order:

$$1_a < 1_b < 1_c < 1_d < 2_a < 2_b < 2_c < 2_d < \cdots$$

Call λ_1 and μ_1 the resulting partitions.

$$C = \begin{array}{c|ccc} & a & c & d \\ \hline a & 2 & 2 & 2 \\ c & 1 & 1 & 2 \\ d & 0 & 1 & 2 \end{array} \longrightarrow C_1 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 2 & 1 & 2 & 2 \\ b & 1 & 0 & 1 & 1 \\ c & 1 & 1 & 1 & 2 \\ d & 0 & 1 & 1 & 2 \end{array}.$$

λ_1 satisfies the difference conditions of C_1,
μ_1 is a partition coloured b containing only parts of sizes that also appear in λ_1 but in a colour different from b.

Example

$$\lambda_1 = 8_d + 8_a + 7_b + 6_c + 5_c + 3_d + 2_b + 2_b + 1_a,$$
$$\mu_1 = 8_b + 8_b + 5_b + 3_b + 1_b + 1_b.$$

Step 2**:** For all j, if there are some parts j_b in μ_1, and j_c appears in λ_1 (by C_1, it cannot repeat nor appear in another colour), then transform those j_b's into j_c's and move them from μ_1 to λ_1.
Call λ_2 and μ_2 the resulting partitions.

$$C_1 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 2 & 1 & 2 & 2 \\ b & 1 & 0 & 1 & 1 \\ c & 1 & 1 & 1 & 2 \\ d & 0 & 1 & 1 & 2 \end{array} \quad \longrightarrow \quad C_2 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 2 & 1 & 2 & 2 \\ b & 1 & 0 & 1 & 1 \\ c & 1 & 1 & 0 & 2 \\ d & 0 & 1 & 1 & 2 \end{array},$$

λ_2 satisfies the difference conditions in the matrix C_2,
μ_2 is a partition coloured b containing only parts of sizes that also appear in λ_2 with colour a or d.

Example

$$\lambda_1 = 8_d + 8_a + 7_b + 6_c + 5_c + 5_c + 3_d + 2_b + 2_b + 1_a,$$
$$\mu_1 = 8_b + 8_b + 3_b + 1_b + 1_b.$$

Step 3: For all j, if there are some parts j_b in μ_2, then j appears in λ_2 in colour a or d, but not c. Transform those j_b's into j_c's and insert them inside λ_2.
Call λ_3 the resulting partition.

$$C_2 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 2 & 1 & 2 & 2 \\ b & 1 & 0 & 1 & 1 \\ c & 1 & 1 & 0 & 2 \\ d & 0 & 1 & 1 & 2 \end{array} \longrightarrow P = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 2 & 1 & 2 & 2 \\ b & 1 & 0 & 1 & 1 \\ c & 0 & 1 & 0 & 2 \\ d & 0 & 1 & 0 & 2 \end{array}.$$

The partition λ_3 satisfies the difference conditions P of Primc's identity.

Example

$\lambda_3 = 8_d + 8_c + 8_c + 8_a + 7_b + 6_c + 5_c + 5_c + 3_d + 3_c + 2_b + 2_b + 1_c + 1_c + 1_a.$

All the steps are reversible.

Thank you for your attention!