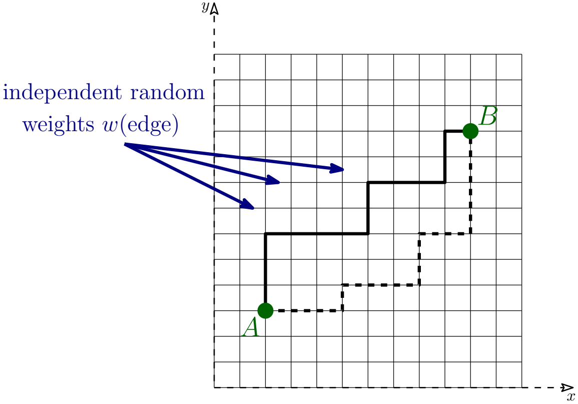
Shift invariance for six-vertex model and directed polymers

Vadim Gorin

UW Madison / MIT / IITP

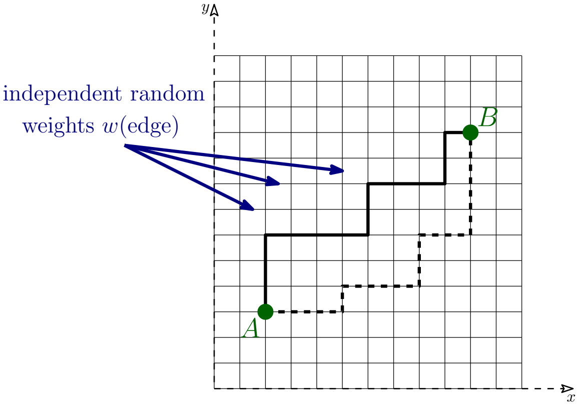
February 2020

Directed percolation



$$D(A,B) = \min_{A=\pi_0\to\pi_1\to\cdots\to\pi_N=B} \sum_{i=1}^{N} w(\pi_{i-1}\to\pi_i)$$

Directed percolation



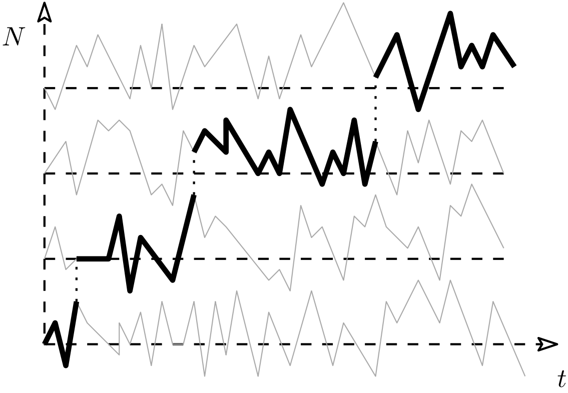
$$D(A,B) = \min_{A=\pi_0\to\pi_1\to\cdots\to\pi_N=B} \sum_{i=1}^{N} w(\pi_{i-1}\to\pi_i)$$

We proceed with a *particular* choice of weights — white noise.

Brownian directed percolation

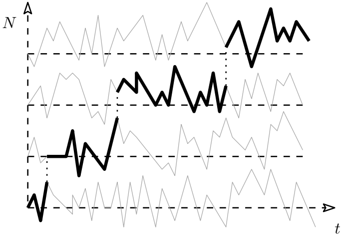
$$D((s,m);(t,n)) = \min_{s=t_0<t_1<\cdots<t_{n-m+1}=t} \sum_{i=0}^{n-m} \left[B_{i+m}(t_{i+1}) - B_{i+m}(t_i)\right]$$

with independent Brownian motions $B_i(\tau)$.



Aka **Brownian Last Passage Percolation** with min $\leftrightarrow$ max.

Brownian directed percolation: what's known?



$$D((s,m);(t,n))$$

$$\min_{\{t_i\}} \sum_{i=0}^{n-m} \left[B_{i+m}(t_{i+1}) - B_{i+m}(t_i)\right]$$

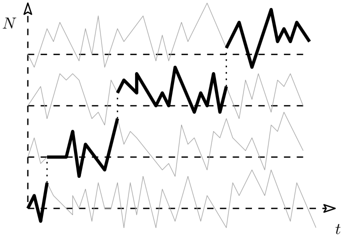
- **Explicit joint laws** $D((0,0);(t,n))$ via Fredholm determinants

  [Kuperberg-99], [Baryshnikov-01], [Borodin-Olshanski-04], [Ferrari-08], [Dotsenko-10], [Johansson-Rahman-19]

Important idea: Robinson–Schensted–Knuth correspondence maps the computation to Dyson Brownian Motion — evolution of eigenvalues of Hermitian matrix with Brownian entries.

Formulas are efficient for the asymptotic analysis

Brownian directed percolation: what's known?



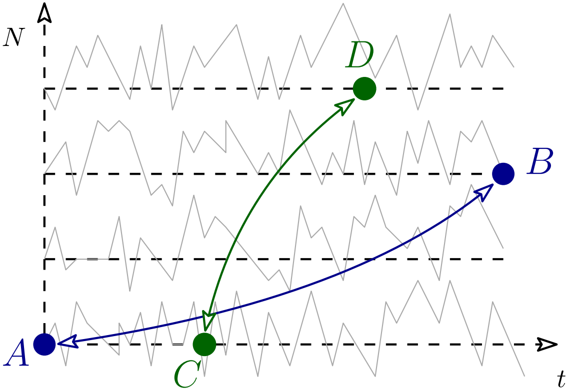
$$D((s,m);(t,n))$$

$$\min_{\{t_i\}} \sum_{i=0}^{n-m} \left[B_{i+m}(t_{i+1}) - B_{i+m}(t_i) \right]$$

- **Explicit joint laws** $D((0,0);(t,n))$ via Fredholm determinants
- **No exact formulas** for both endpoints (s,m); (t,n) varying.

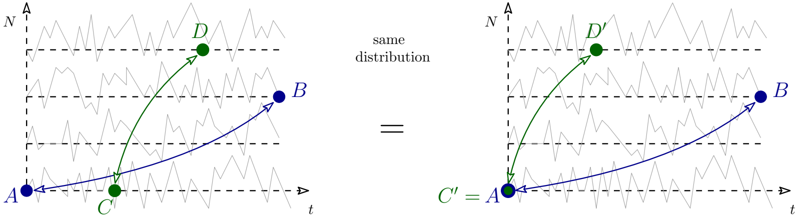
However, formula-less asymptotic analysis is possible. E.g. in [Balasz–Busani–Seppalainen-19], [Fan–Seppalainen-19], [Basu–Ganguly–Hammond-19], [Dauvergne–Ortman–Virag-18], [Basu–Sarkar–Sly-18], [Basu–Hoffman–Sly-18], [Hammond-17], [Matetski–Quastel–Remenik-17], [Georgiou–Rasoul-Agha–Seppalainen-16], . . . [Newman-94], . . .

Brownian directed percolation: simplest unknown



Question: What is the joint law for the (intersecting) **pair** of distances $D(A,B)$ and $D(C,D)$?

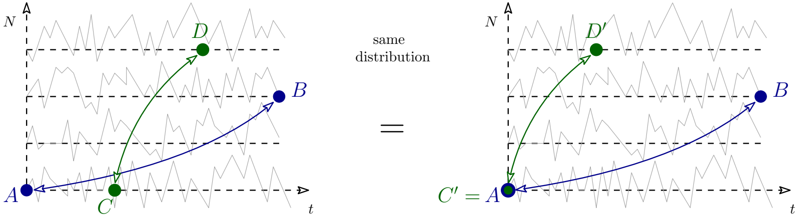
Brownian directed percolation: sample result



Theorem. [Borodin–Gorin–Wheeler-19] The law is **shift invariant**: If C is on the same horizontal as A, with abscissa **between** the ones of A and B, and if D is **above and to the left** from B, then

$$\big[D(A,B), D(C,D)\big] \stackrel{d}{=} \big[D(A,B), D(A, D+A-C)\big].$$

Brownian directed percolation: sample result

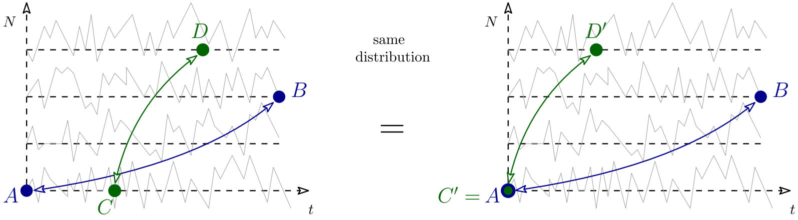


Theorem. [Borodin–Gorin–Wheeler-19] The law is **shift invariant**: If C is on the same horizontal as A, with abscissa **between** the ones of A and B, and if D is **above and to the left** from B, then

$$\left[D(A,B), D(C,D)\right] \stackrel{d}{=} \left[D(A,B), D(A, D+A-C)\right].$$

- Hence, the distribution is explicit.

Brownian directed percolation: sample result

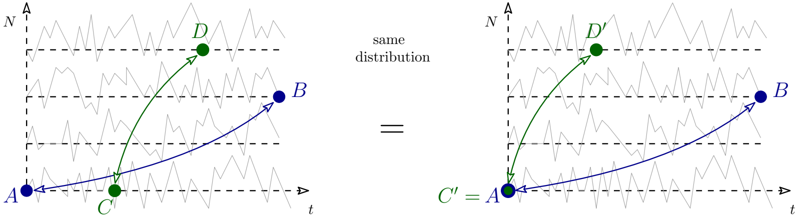


Theorem. [Borodin–Gorin–Wheeler-19] The law is **shift invariant**: If C is on the same horizontal as A, with abscissa **between** the ones of A and B, and if D is **above and to the left** from B, then

$$\left[D(A,B), D(C,D)\right] \stackrel{d}{=} \left[D(A,B), D(A, D+A-C)\right].$$

- Hence, the distribution is explicit.
- The **intersection condition** is crucial.

Brownian directed percolation: sample result

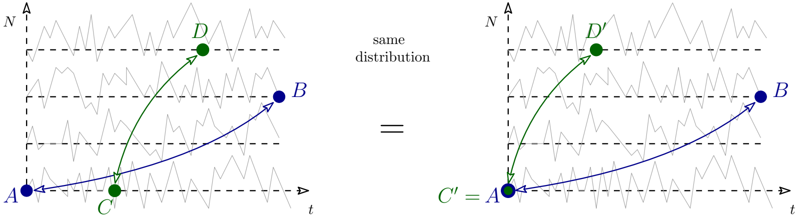


Theorem. [Borodin–Gorin–Wheeler-19] The law is **shift invariant**: If C is on the same horizontal as A, with abscissa **between** the ones of A and B, and if D is **above and to the left** from B, then

$$\big[D(A,B), D(C,D)\big] \stackrel{d}{=} \big[D(A,B), D(A, D+A-C)\big].$$

- Hence, the distribution is explicit.
- The **intersection condition** is crucial.
- No **elementary** proof.

Brownian directed percolation: sample result

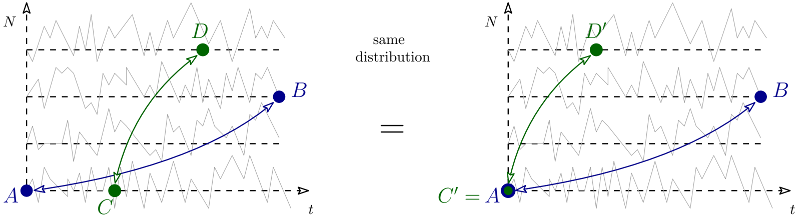


Theorem. [Borodin–Gorin–Wheeler-19] The law is **shift invariant**: If C is on the same horizontal as A, with abscissa **between** the ones of A and B, and if D is **above and to the left** from B, then

$$\big[D(A,B), D(C,D)\big] \overset{d}{=} \big[D(A,B), D(A, D+A-C)\big].$$

- Hence, the distribution is explicit.
- The **intersection condition** is crucial.
- No **elementary** proof.
- Similar property holds for multi-point distributions...

Brownian directed percolation: sample result

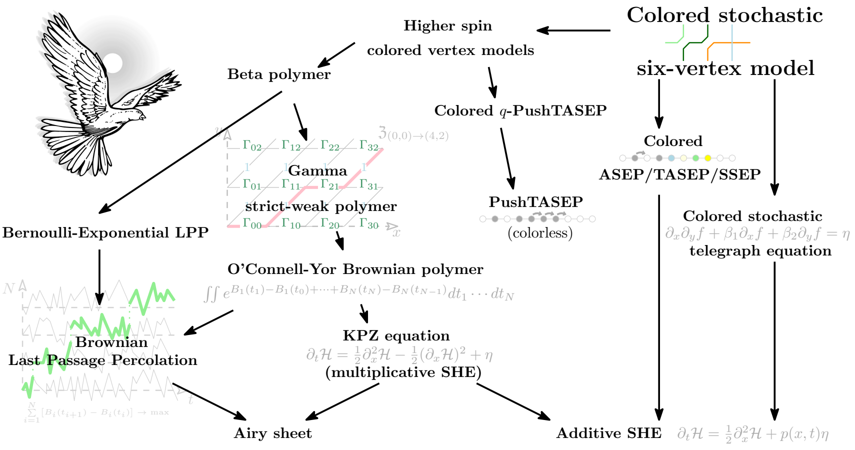


Theorem. [Borodin–Gorin–Wheeler-19] The law is **shift invariant**: If C is on the same horizontal as A, with abscissa **between** the ones of A and B, and if D is **above and to the left** from B, then

$$\left[D(A,B), D(C,D)\right] \stackrel{d}{=} \left[D(A,B), D(A, D+A-C)\right].$$

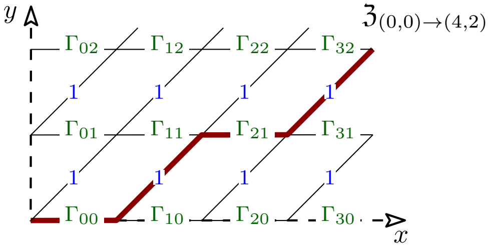
- Hence, the distribution is explicit.
- The **intersection condition** is crucial.
- No **elementary** proof.
- Similar property holds for multi-point distributions...
- ... and for many more models!

Zoo of integrable stochastic systems



Theorem. [BGW-19] **Shift invariance** holds for each system.

Gamma polymer and its shift invariance



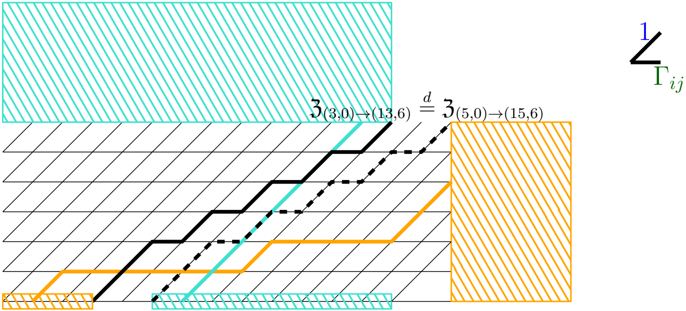
$\Gamma_{ij} \sim$ i.i.d. Gamma(κ)
of density

$$\frac{1}{\Gamma(\kappa)} z^{\kappa-1} e^{-z}, \quad z > 0.$$

$$\mathfrak{Z}_{(x',y')\to(x,y)} = \sum_{(x',y')=\pi_0\to\cdots\to\pi_{x+y-x'-y'}=(x,y)} \prod_{k=1}^{x+y-x'-y'} w(\pi_{k-1}\to\pi_k)$$

[Seppäläinen-10], [Corwin–Seppäläinen–Shen-14], [O'Connell–Ortmann-14]:
Integrability and limits.

Gamma polymer and its shift invariance



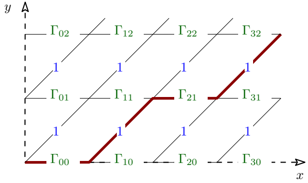
Theorem. [BGW-19] $\{k_i\}$, $\{\mathcal{U}_i\}$, $1 \le \iota \le n$, $\Delta > 0$. Set

$$k_i' = \begin{cases} k_i, & i \neq \iota, \\ k_\iota + \Delta, & i = \iota, \end{cases} \qquad \mathcal{U}_i' = \begin{cases} \mathcal{U}_i, & i \neq \iota \\ \mathcal{U}_\iota + (\Delta, 0), & i = \iota. \end{cases}$$

Under **intersection condition**, we have:

$$\left(\mathfrak{Z}_{(0,k_1)\to\mathcal{U}_1}, \dots, \mathfrak{Z}_{(0,k_n)\to\mathcal{U}_n}\right) \overset{d}{=} \left(\mathfrak{Z}_{(0,k_1')\to\mathcal{U}_1'}, \dots, \mathfrak{Z}_{(0,k_n')\to\mathcal{U}_n'}\right)$$

KPZ equation and its shift invariance



mesh size $\to 0$

2d white noise η

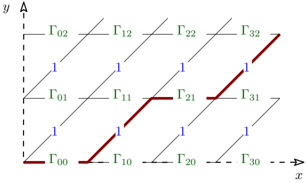
$$Z_t = \frac{1}{2} Z_{xx} - \eta Z$$

$H = -\ln(Z)$

$$H_t = \frac{1}{2} H_{xx} - \frac{1}{2} (H_x)^2 + \eta$$

Kardar-Parisi-Zhang SPDE

KPZ equation and its shift invariance



mesh size → 0

2d white noise η

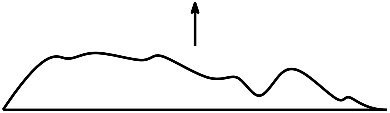
$$Z_t = \frac{1}{2} Z_{xx} - \eta Z$$

$H = -\ln(Z)$

$$H_t = \frac{1}{2} H_{xx} - \frac{1}{2} (H_x)^2 + \eta$$

Kardar-Parisi-Zhang SPDE

KPZ (1986): This equation is universal for **1d surface growth**

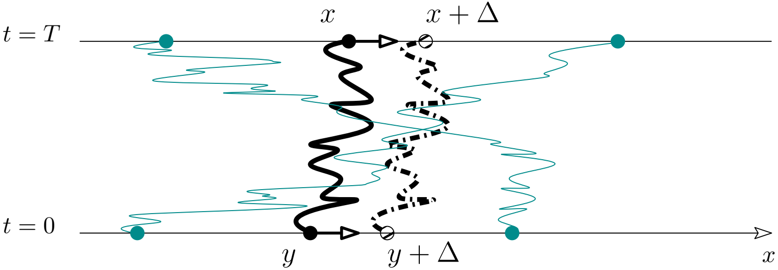


Rigorously found as a scaling limit in *numerous* stochastic systems.

KPZ equation and its shift invariance

$$Z_t = \frac{1}{2} Z_{xx} - \eta Z \quad \overset{H=-\ln(Z)}{\longleftrightarrow} \quad H_t = \frac{1}{2} H_{xx} - \frac{1}{2}(H_x)^2 + \eta$$

Solutions with $Z(0,x) = \delta_{x=y}$ **coupled** through $2d$ white noise η.



Theorem. [BGW-19] For $\Delta > 0$: $\quad Z_T^{(y)}(x) \overset{d}{=} Z_T^{(y+\Delta)}(x+\Delta)$,

jointly with any $Z_T^{(y')}(x')$ for $\begin{bmatrix} y' < y,\, x' > x + \Delta, \quad \textit{or} \\ y' > y + \Delta,\, x' < x. \end{bmatrix}$

Airy sheet and its shift invariance

Last Passage Percolation

$D((sL + L^{2/3}y, \alpha sL); (tL + L^{2/3}x, \alpha tL))$

$L \to \infty$

$H_t = \frac{1}{2} H_{xx} - \frac{1}{2} (H_x)^2 + \eta$

KPZ equation

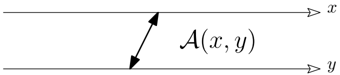
$T \to \infty$

Space-time Airy sheet / Directed landscape

$\mathcal{AL}(x, s; y, t)$

$s = 0$ $t = 1$

Airy sheet



Conjectures:

- $\mathcal{AL}(\cdot)$ should be a limit of LPP with general weights.
- $\mathcal{AL}(\cdot)$ should govern large-time behavior of KPZ.

  Because $H \approx \ln \left(\int_{\text{Brownian } \gamma} \exp(\int \text{white noise along } \gamma) \right)$.

Airy sheet and its shift invariance

Last Passage Percolation

$D((sL + L^{2/3}y, \alpha sL); (tL + L^{2/3}x, \alpha tL))$

$L \to \infty$

$H_t = \frac{1}{2}H_{xx} - \frac{1}{2}(H_x)^2 + \eta$

KPZ equation

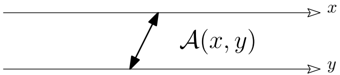
$T \to \infty$

Space-time Airy sheet / Directed landscape

$\mathcal{AL}(x, s; y, t)$

$s = 0$ $t = 1$

Airy sheet



Theorems:

- [Davergne–Ortmann–Virag-18]: Brownian LPP $\to \mathcal{AL}(\cdot)$
- Numerous partial results for LPP with integrable weights.
- [Amir–Corwin–Quastel-11][Sasomoto–Spohn–10][Calabrese–Le Dousal–Rosso-10][Dotsenko-10]
  1-point KPZ $\to \mathcal{A}(0, 0) =$ Tracy–Widom distribution

Airy sheet and its shift invariance

Last Passage Percolation

$D((sL + L^{2/3}y, \alpha sL); (tL + L^{2/3}x, \alpha tL))$

$L \to \infty$

$H_t = \frac{1}{2}H_{xx} - \frac{1}{2}(H_x)^2 + \eta$

KPZ equation

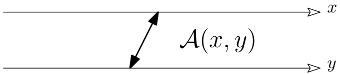
$T \to \infty$

Space-time Airy sheet / Directed landscape

$\mathcal{AL}(x, s; y, t)$

$s = 0$ $t = 1$

Airy sheet



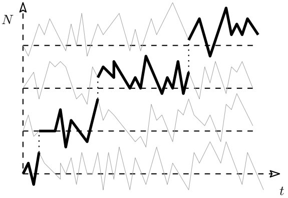
Theorem. [BGW-19] For $\Delta > 0$: $\mathcal{A}(x, y) \overset{d}{=} \mathcal{A}(x + \Delta, y + \Delta)$,

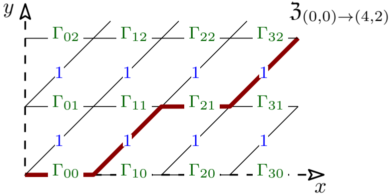
jointly with any $\mathcal{A}(x', y')$ for $\begin{cases} y' < y,\ x' > x + \Delta, & \text{or} \\ y' > y + \Delta,\ x' < x. \end{cases}$

$\mathcal{A}(0, \cdot)$ was known, but no distributions beyond it.

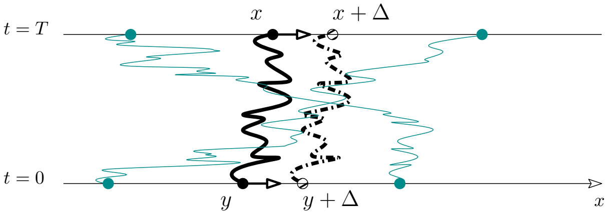
Shift invariance: How general is it?

- For **special integrable** weights in LPP and directed polymers.



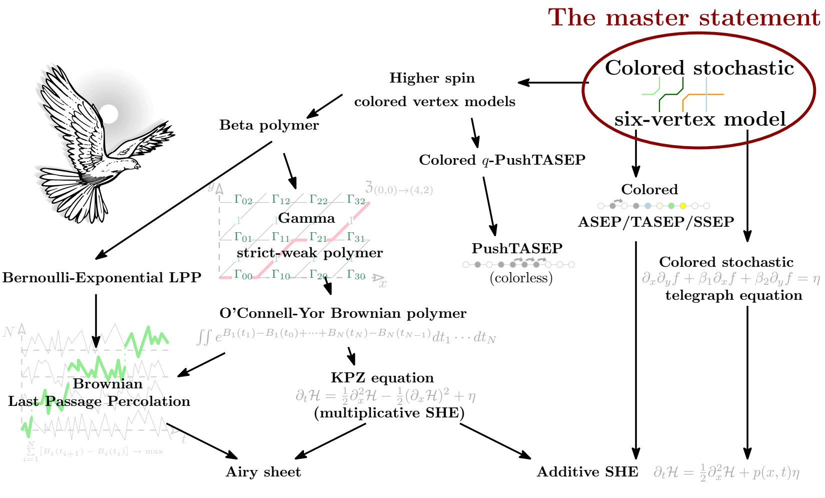


- For **universal limiting objects**: KPZ-equation, Airy sheet.



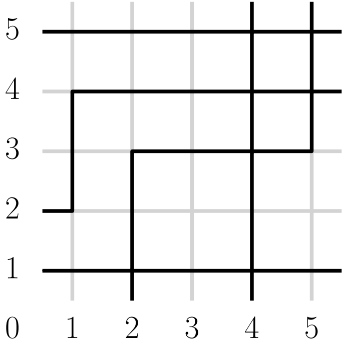
What **other weights** lead to shift invariance? We do not know.

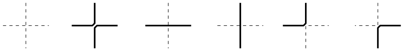
We proceed to another world



(still inside Integrable Probability)

Stochastic colored six-vertex model

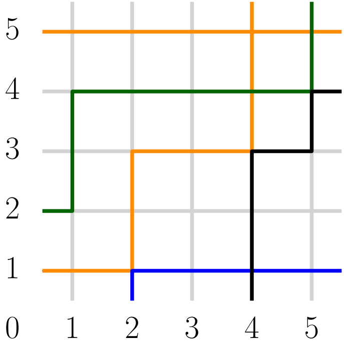


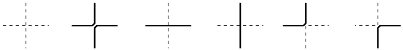


Six-vertex model configurations: paths on the square grid.

Treat it as white/black model.

Stochastic colored six-vertex model



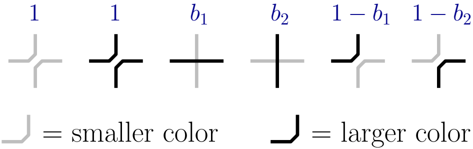


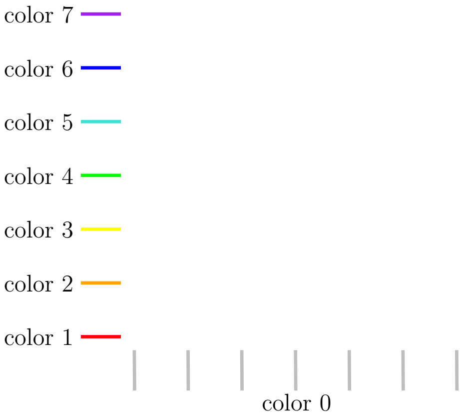
Six-vertex model configurations: paths on the square grid.

Treat it as white/black model.

Add **colors.**

Stochastic colored six-vertex model

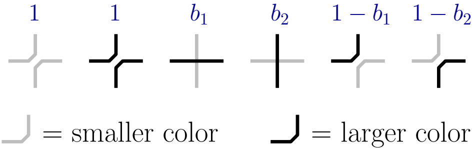


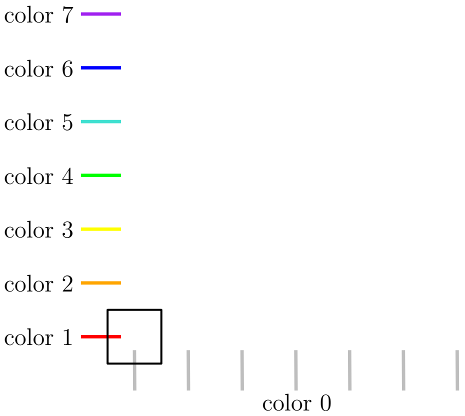


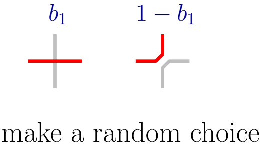
Domain wall

boundary condition

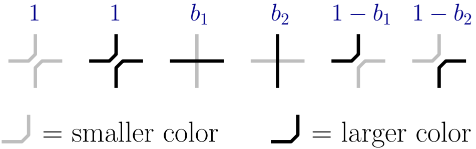
Stochastic colored six-vertex model

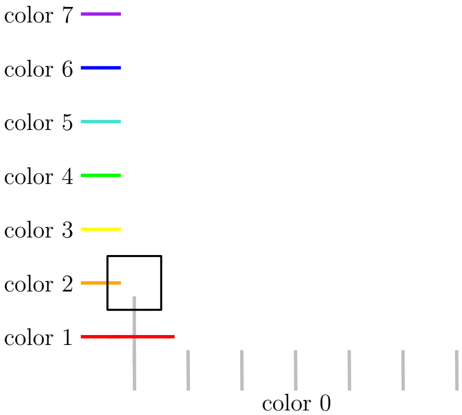


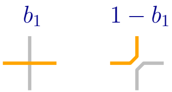




Stochastic colored six-vertex model

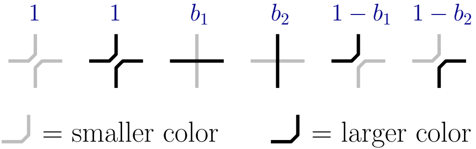


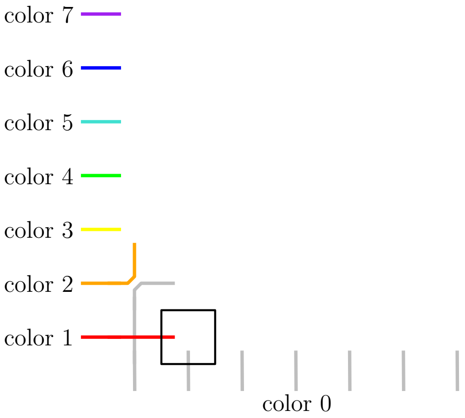


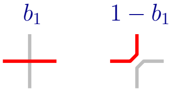


make a random choice

Stochastic colored six-vertex model

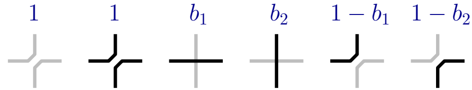




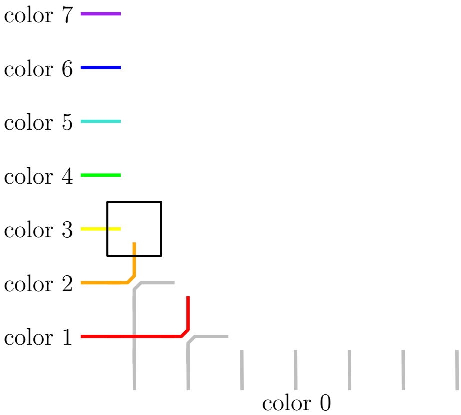


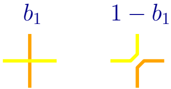
make a random choice

Stochastic colored six-vertex model



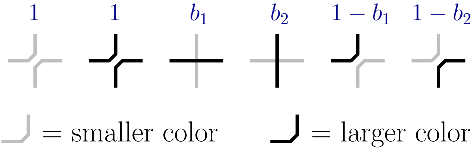
= smaller color   = larger color

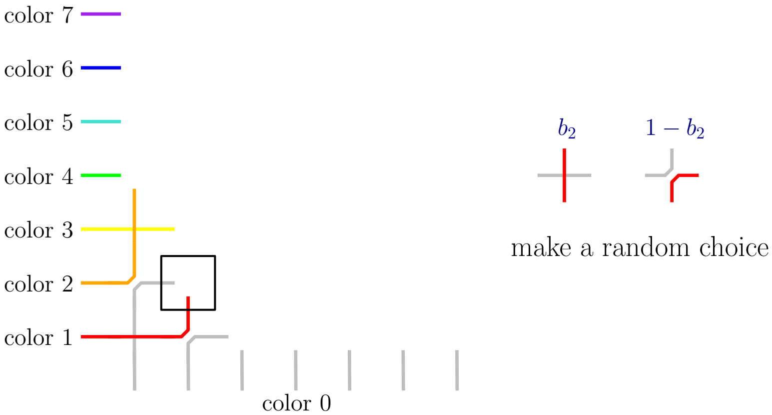




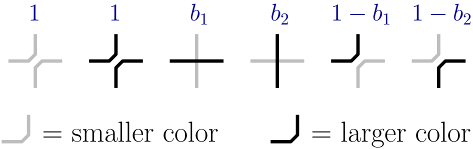
make a random choice

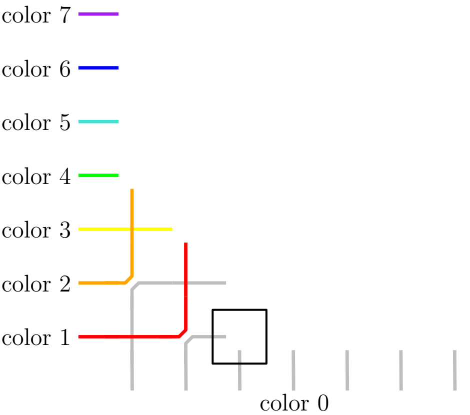
Stochastic colored six-vertex model

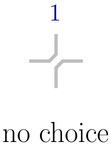




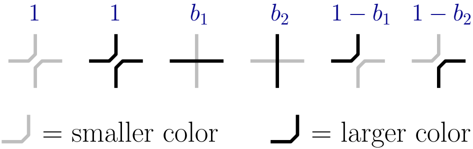
Stochastic colored six-vertex model

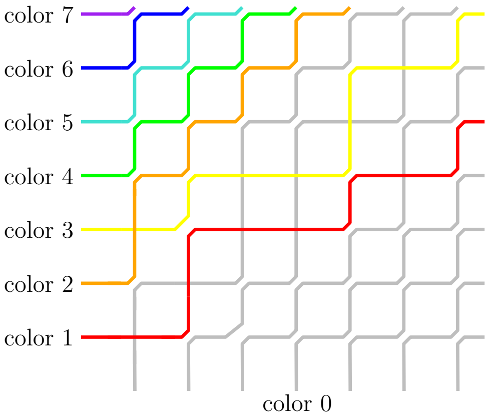






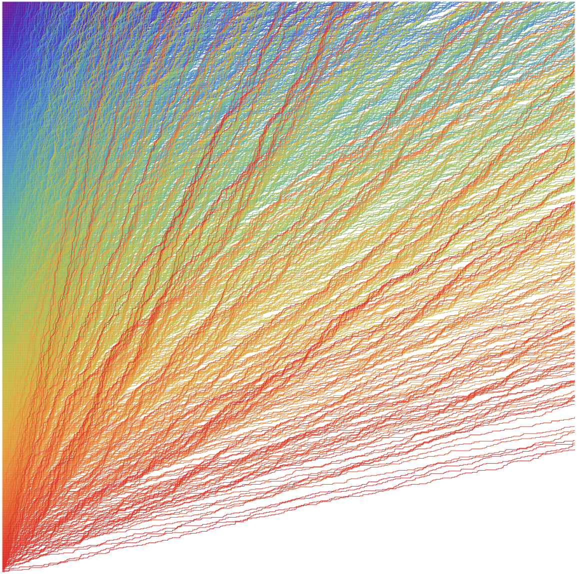
Stochastic colored six-vertex model





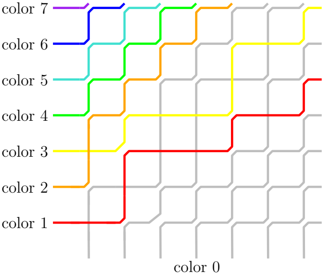
contininue until you fill the entire quadrant

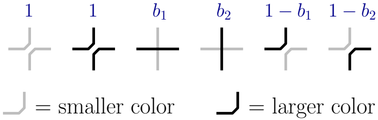
Stochastic colored six-vertex model



Simulation by Leonid Petrov

Stochastic colored six-vertex model

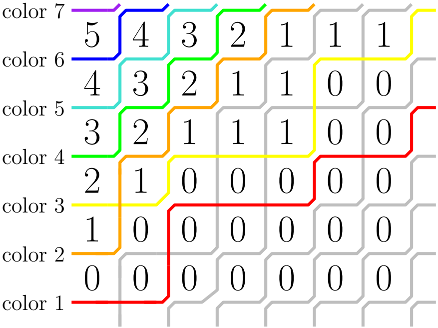




Relevant history:

- [Pauling-1935] Six-vertex (aka square ice) model.
- [Lieb-1967] Bethe ansatz analysis. Residual entropy of ice.
- [Bazhanov–1985], [Jimbo–1986] Colored vertex models
- [Gwa–Spohn-1992] Stochastic version. KPZ class indications.
- [Borodin–Corwin–Gorin-2014] Adaptation of integrable probability methods. Confirmation of KPZ class predictions.
- [Kuniba–Mangazeev–Maruyama–Okado-2016] Stochastic higher rank colored models.

Stochastic colored six-vertex model

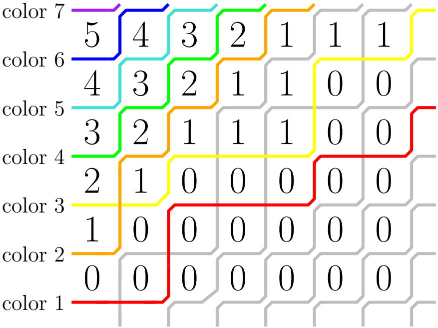


Height function $\mathcal{H}^{\geqslant i}(x, y)$

Counts the number of paths of colors $\geqslant i$ to the right/below (x, y).

$\longleftarrow$ $\mathcal{H}^{\geqslant 2}(x, y)$ shown

Stochastic colored six-vertex model



Height function $\mathcal{H}^{\geqslant i}(x, y)$

Counts the number of paths of colors $\geqslant i$ to the right/below (x, y).

$\longleftarrow \mathcal{H}^{\geqslant 2}(x, y)$ shown

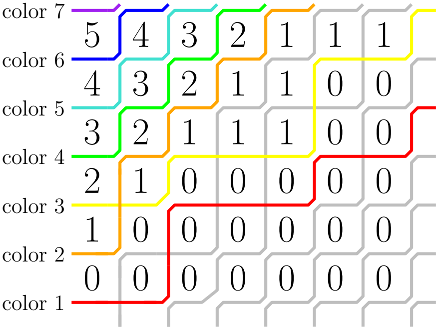
Theorem. [BGW-19] $\{k_i\}$, $\{\mathcal{U}_i\}$, $1 \leq \iota \leq n$, $\Delta > 0$. Set

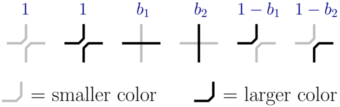
$$k'_i = \begin{cases} k_i, & i \neq \iota, \\ k_\iota + \Delta, & i = \iota, \end{cases} \qquad \mathcal{U}'_i = \begin{cases} \mathcal{U}_i, & i \neq \iota \\ \mathcal{U}_\iota + (0, \Delta), & i = \iota. \end{cases}$$

Under **intersection condition**, we have:

$$\left(\mathcal{H}^{\geqslant k_1}(\mathcal{U}_1), \ldots, \mathcal{H}^{\geqslant k_n}(\mathcal{U}_n)\right) \stackrel{d}{=} \left(\mathcal{H}^{\geqslant k'_1}(\mathcal{U}'_1), \ldots, \mathcal{H}^{\geqslant k'_n}(\mathcal{U}'_n)\right)$$

Stochastic colored six-vertex model





Theorem. [BGW-19]

$$\left(\mathcal{H}^{\geqslant k_1}(\mathcal{U}_1), \dots, \mathcal{H}^{\geqslant k_n}(\mathcal{U}_n)\right) \overset{d}{\|} \left(\mathcal{H}^{\geqslant k'_1}(\mathcal{U}_1), \dots, \mathcal{H}^{\geqslant k'_n}(\mathcal{U}_n)\right)$$

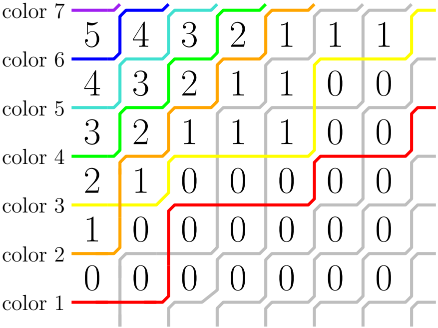
Elements of the proof:

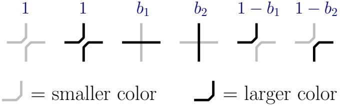
- Consider inhomogeneous model based on q, $\{v_x\}$, $\{u_y\}$:

$$b_1(x,y) = q\frac{u_y - v_x}{u_y - qv_x}, \qquad b_2(x,y) = \frac{u_y - v_x}{u_y - qv_x}.$$

- **Polynomial identity:** need to check at enough points.
- When $v_x = u_y$, behavior at (x,y)–vertex trivializes: only turns.
- In such situations **Yang-Baxter equation** relates the weighted sums to the ones available by induction assumption.

Stochastic colored six-vertex model





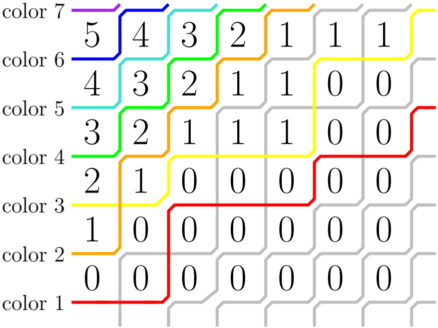
Theorem. [BGW-19]

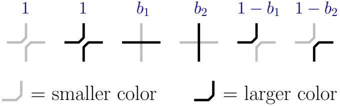
$$\left(\mathcal{H}^{\geqslant k_1}(\mathcal{U}_1), \ldots, \mathcal{H}^{\geqslant k_n}(\mathcal{U}_n)\right) \overset{d}{=} \left(\mathcal{H}^{\geqslant k'_1}(\mathcal{U}_1), \ldots, \mathcal{H}^{\geqslant k'_n}(\mathcal{U}_n)\right)$$

Our proof:

- **Lagrange interpolation** + **Yang-Baxter equation**.

Stochastic colored six-vertex model





Theorem. [BGW-19]

$$\left(\mathcal{H}^{\geqslant k_1}(\mathcal{U}_1), \ldots, \mathcal{H}^{\geqslant k_n}(\mathcal{U}_n)\right) \overset{d}{=} \left(\mathcal{H}^{\geqslant k'_1}(\mathcal{U}_1), \ldots, \mathcal{H}^{\geqslant k'_n}(\mathcal{U}_n)\right)$$

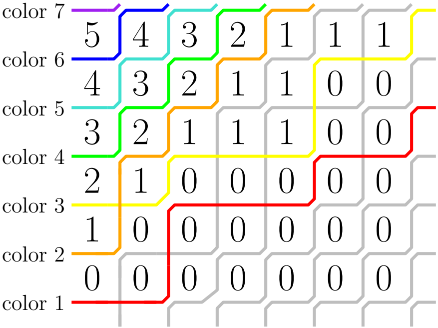
Our proof:

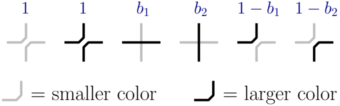
- **Lagrange interpolation** + **Yang-Baxter equation**.

Later alternative (still integrable!):

- (Davergne-20+) Last Passage Percolation case: symmetries of **Robinson–Schenstead–Knuth correspondence**
- (Galashin-20+) 6v case: **Hecke algebras** and two reflections.

Stochastic colored six-vertex model





Theorem. [BGW-19]

$$\left(\mathcal{H}^{\geqslant k_1}(\mathcal{U}_1), \ldots, \mathcal{H}^{\geqslant k_n}(\mathcal{U}_n)\right) \overset{d}{=} \left(\mathcal{H}^{\geqslant k_1'}(\mathcal{U}_1), \ldots, \mathcal{H}^{\geqslant k_n'}(\mathcal{U}_n)\right)$$

Our proof:

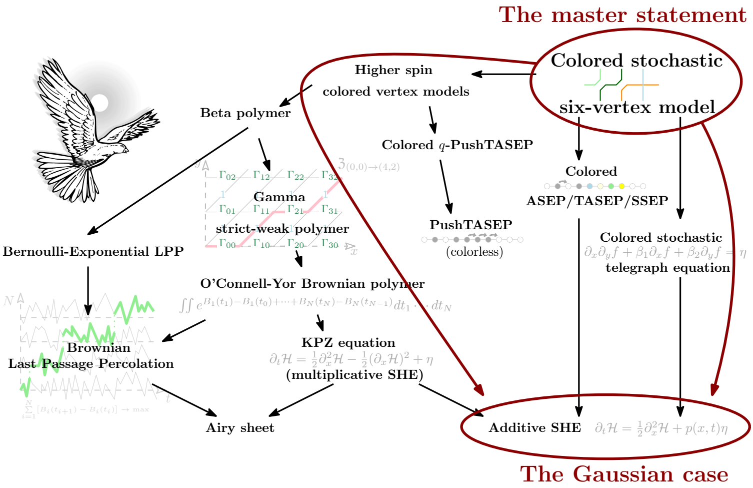
- **Lagrange interpolation** + **Yang-Baxter equation**.

Later alternative (still integrable!):

- (Davergne-20+) Last Passage Percolation case: symmetries of **Robinson–Schenstead–Knuth correspondence**
- (Galashin-20+) 6v case: **Hecke algebras** and two reflections.

Is there anything **simpler or more conceptual?**

Is there a simpler proof?



In general, we do not know. But if one cares only about covariance...

Simplest case: additive SHE

$$\mathcal{H}_t = \tfrac{1}{2}\mathcal{H}_{xx} + p(x-y,t)\cdot\eta, \qquad \mathcal{H}(x,0)=0,$$

y is a parameter, η is the **2d white noise**, and

$$p(x-y,t) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(x-y)^2}{2t}\right).$$

Proposition. For $\Delta > 0$: $\mathcal{H}^{(y)}(x, T) \overset{d}{=} \mathcal{H}^{(y+\Delta)}(x+\Delta, T)$,

jointly with any $\mathcal{H}^{(y')}(x', T)$ for $\left[\begin{array}{l} y' < y,\ x' > x+\Delta, \quad \textit{or} \\ y' > y+\Delta,\ x' < x. \end{array}\right.$

Simplest case: additive SHE

$$\mathcal{H}_t = \tfrac{1}{2}\mathcal{H}_{xx} + p(x-y,t)\cdot\eta, \qquad \mathcal{H}(x,0)=0,$$

y is a parameter, η is the **2d white noise**, and

$$p(x-y,t) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(x-y)^2}{2t}\right).$$

Proposition. For $\Delta > 0$: $\mathcal{H}^{(y)}(x,T) \overset{d}{=} \mathcal{H}^{(y+\Delta)}(x+\Delta,T)$, **jointly** with any $\mathcal{H}^{(y')}(x',T)$ for $\left[\begin{array}{l} y'<y,\ x'>x+\Delta, \quad \textit{or} \\ y'>y+\Delta,\ x'<x. \end{array}\right.$

Proof. $\mathcal{H}$ is an explicit Gaussian:

$$\mathcal{H}^{(y)}(x,T) = \int_0^T\int_{-\infty}^{\infty} p(z-y,s)p(x-z,T-s)\eta(z,s)\,dz\,ds$$

Simplest case: additive SHE

$$\mathcal{H}_t = \tfrac{1}{2}\mathcal{H}_{xx} + p(x-y,t)\cdot\eta, \qquad \mathcal{H}(x,0)=0,$$

y is a parameter, η is the **2d white noise**, and

$$p(x-y,t) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(x-y)^2}{2t}\right).$$

Proposition. For $\Delta > 0$: $\mathcal{H}^{(y)}(x,T) \stackrel{d}{=} \mathcal{H}^{(y+\Delta)}(x+\Delta,T)$,
jointly with any $\mathcal{H}^{(y')}(x',T)$ for $\left[\begin{array}{l} y'<y,\ x'>x+\Delta, \quad \textit{or} \\ y'>y+\Delta,\ x'<x. \end{array}\right.$

Proof. $\mathcal{H}$ is an explicit Gaussian:

$$\mathcal{H}^{(y)}(x,T) = \int_0^T\int_{-\infty}^{\infty} p(z-y,s)p(x-z,T-s)\eta(z,s)\,dz\,ds$$

Hence,

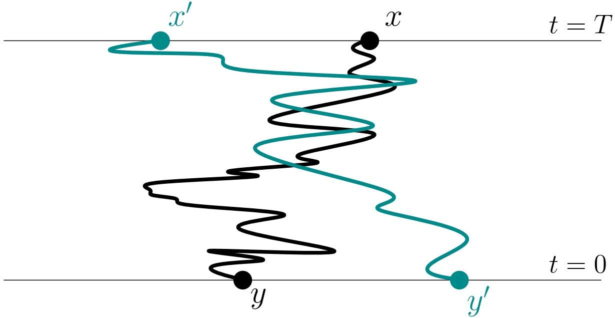
$$\mathbb{E}\mathcal{H}^{(y)}(x,T)\mathcal{H}^{(y')}(x',T) = \int_0^T\int_{-\infty}^{\infty} p(z-y,s)p(x-z,T-s)p(z-y',s)p(x'-z,T-s)\,dz\,ds$$

Simplest case: additive SHE

$$\mathcal{H}_t = \tfrac{1}{2}\mathcal{H}_{xx} + p(x-y,t)\cdot\eta, \qquad \mathcal{H}(x,0)=0,$$

Hence, $\mathbb{E}\mathcal{H}^{(y)}(x,T)\mathcal{H}^{(y')}(x',T) =$

expected **intersection local time** of two Brownian bridges

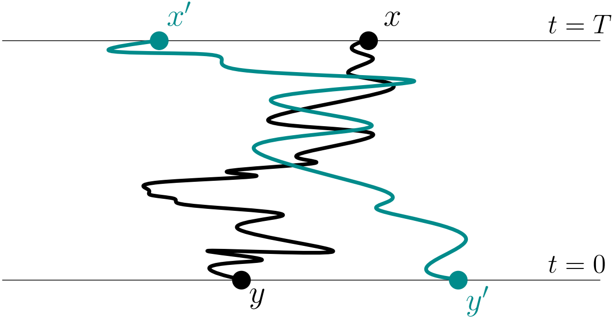


Simplest case: additive SHE

$$\mathcal{H}_t = \tfrac{1}{2}\mathcal{H}_{xx} + p(x-y,t)\cdot\eta, \qquad \mathcal{H}(x,0)=0,$$

Hence, $\mathbb{E}\mathcal{H}^{(y)}(x,T)\mathcal{H}^{(y')}(x',T) =$

expected **intersection local time** of two Brownian bridges



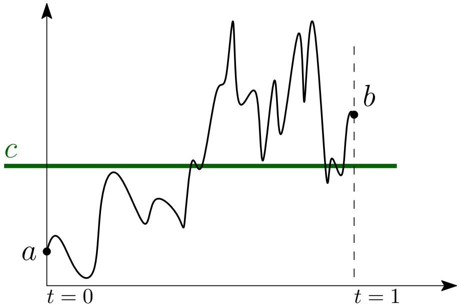
Shift invariance of additive SHE

$\Updownarrow$

$\mathbb{E}$[local time] invariant under $(x,y) \to (x+\Delta, y+\Delta)$

Simplest case: additive SHE

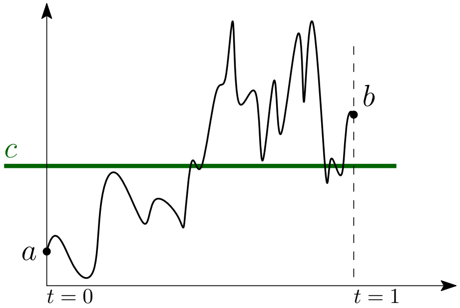
Eventually, shift invariance for additive SHE becomes:



Theorem. The expected local time the Brownian bridge from a to b spends at level c is independent of c as long as $a < c < b$.

Simplest case: additive SHE

Eventually, shift invariance for additive SHE becomes:



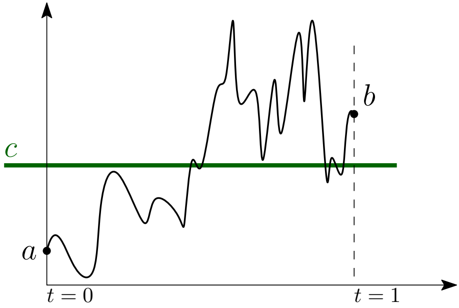
Theorem. The expected local time the Brownian bridge from a to b spends at level c is independent of c as long as $a < c < b$.

In fact, the law of the local time at c is known:

$$\sim (|c-a|+|c-b|+y) \exp\left(-\tfrac{1}{2}(|c-a|+|c-b|+y)^2\right) dy, \qquad y > 0.$$

[Ray-63], [Williams-74], [Biane-Yor-88], [Borodin-89], [Pitman-99]

Back to the six-vertex model

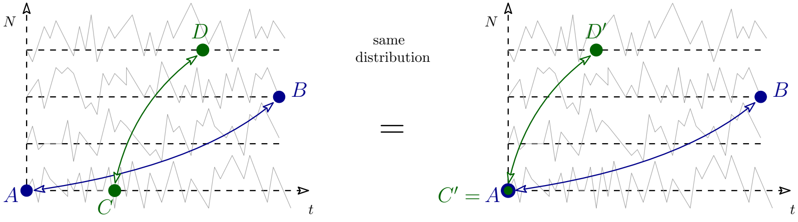


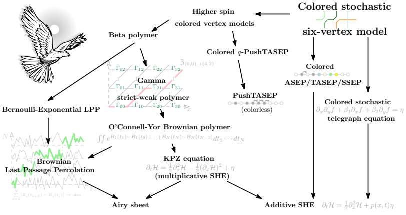
Theorem. The expected local time the Brownian bridge from a to b spends at level c is independent of c as long as $a < c < b$.

Theorem. [BGW-19] Discrete versions of this statement hold and generalize up to the level of the six-vertex model paths.

- Can be used to prove **shift invariance of covariances.**
- However, high enough in the hierarchy, the models are **non-gaussian**!

Summary of shift invariance





- Joint laws invariant with respect to partial shifts.
- Access to new distributions.
- Proof for covariance: shift invariance for local times.
- Proof for law: inhomogeneity / polynomiality / Yang-Baxter
- **Integrable Probability** ties all models together.