Understanding the asymptotics of the number of tableaux of skew shape through a variational principle

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Notations

$\lambda$: partition shape

$\lambda/\mu$: skew shape

$f^\lambda = \# \text{ SYT of shape } \lambda$

$f^{\lambda/\mu} = \# \text{ SYT of shape } \lambda/\mu$
Hook-length formula

Theorem (Frame-Robinson-Thrall 1954)

\[ f^\lambda = N! \prod_{(i,j) \in \lambda} \frac{1}{h(i,j)}, \]

\[ h(i, j) = \lambda_i - i - \lambda'_j - j + 1 \] is the hook-length of \((i, j)\).
Hook-length formula

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\[
\begin{array}{ccc}
& 4 & 3 & 1 \\
2 & & 1 & \\
\end{array}
\]

\[
f_{\begin{array}{ccc}
& 4 & 3 & 1 \\
2 & & 1 & \\
\end{array}} = \frac{5!}{1^2 \cdot 2 \cdot 3 \cdot 4} = 5
\]
$f^\lambda = \frac{|\lambda|!}{\prod_{u \in \lambda} h(u)}$

- asymptotics

$\frac{f^\lambda}{\mu} =$?
Naruse’s ”hook-length” formula for \( f^{\lambda/\mu} \)

\[
f^{\lambda/\mu} = N! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h(i,j)},
\]

where \( \mathcal{E}(\lambda/\mu) \) is the set of excited diagrams of \( \lambda/\mu \).
Excited diagrams of $\lambda/\mu$

An excited move on an excited cell $(i, j)$ in $S \subset \lambda$ replaces $(i, j)$ in $S$ by $(i + 1, j + 1)$.

Definition

**Excited diagrams** $\mathcal{E}(\lambda/\mu)$: diagrams obtained from $\mu$ by applying iteratively excited moves on excited cells.

Example

$\mathcal{E}(\text{left diagram})$: 

\[\text{Diagram transformations} \]

\[\text{Resulting diagram} \]
Theorem (Naruse 2014)

\[ f^{\lambda/\mu} = N! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h(i,j)}, \]

where \( \mathcal{E}(\lambda/\mu) \) is the set of excited diagrams of \( \lambda/\mu \).

Example

\[ \mathcal{E}(\begin{array}{cc}
\end{array}) = \left\{ \begin{array}{cc}
\end{array}, \begin{array}{cc}
\end{array} \right\} \]

\[ f^{\begin{array}{cc}
\end{array}} = 3! \cdot \left( \frac{1}{1 \cdot 2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 3} \right) = 3! \left( \frac{1}{4} + \frac{1}{12} \right) = 2. \]
Outline

- \( f^\lambda = \frac{N!}{\prod_{u \in \lambda} h(u)} \)
- asymptotics

- \( f^\lambda/\mu = N! \sum_{D \in \mathcal{E}(\lambda/\mu)} \ldots \)
- asymptotics?
Conjecture on the asymptotics of tableaux of skew shape

A sequence of partitions \( \{\lambda^{(N)}\} \) is strongly stable if it satisfies the following property:

\[
(\sqrt{N} - L)\psi < [\lambda^{(N)}] < (\sqrt{N} + L)\psi
\]

Conjecture (Morales, Panova and Pak)

For a strong stable shape \( \nu_N = \lambda^{(N)} / \mu^{(N)} \) converging to \( \psi / \phi \) and such that \( \text{area}(\psi / \phi) = 1 \):

\[
\frac{1}{N} \left( \log f^{\nu_N} - \frac{1}{2} N \log N \right) \to c(\psi / \phi)
\]
Rewriting Naruse’s formula

\[ f^{\lambda/\mu} = N! \sum_{D \in E(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h(i,j)} \]

\[ = N! \prod_{(i,j) \in \lambda} \frac{1}{h(i,j)} \sum_{D \in E(\lambda/\mu)} \prod_{(i,j) \in D} h(i,j) \]

Taking the logarithm we obtain:

\[ \log f^{\lambda/\mu} = \log \left( N! \prod_{(i,j) \in \lambda} \frac{1}{h(i,j)} \right) + \log \sum_{D \in E(\lambda/\mu)} \prod_{(i,j) \in D} h(i,j) \]

The asymptotics of the first term in the LHS are obtainable through the hook formula. Hence we are interested in

\[ \log \sum_{D \in E(\lambda/\mu)} \prod_{(i,j) \in D} h(i,j). \]
Simulations of the excited measure

The existence of a limit shape suggests that the "excited measure" concentrates towards an asymptotic profile.
The weight of a blue lozenge with coordinates \((i, j)\) is given by 
\[ \lambda_i - i + \lambda'_j - j + 1. \]
The boundary curve $\gamma_n$ of the corresponding region can be described in the following way:

1. The bottom and sides of the region are obtained by interpolating linearly between the points $(a, 0, d_1)$, $(a, 0, 0), (0, 0, 0), (0, b, 0)$ and $(0, b, d_k)$.

2. Starting from the point $(0, a, d_1)$ moves:
   - In the $xy$-plane, according to $\mu$ between any two inner corners of $\mu$.
   - Vertically by $d_i - d_{i+1}$ at each inner corner.
The variational principle for lozenge tilings

Let \( \{ \gamma_n \}_{n \in \mathbb{N}} \) be a sequence of curves such that \( \frac{1}{n} \gamma_n \) converges to a closed curve \( \gamma \) in \( \mathbb{R}^3 \) in the \( \ell_\infty \) norm.

**Theorem (Kenyon 2009)**

The number \( N_{\gamma_n} \) of lozenge tilings with boundary \( \gamma_n \) satisfies

\[
\lim_{n \to \infty} \frac{1}{n^2} \log N_{\gamma_n} \to \Phi(g_{\text{max}})
\]

where \( g_{\text{max}} \) is the only extension of \( \gamma \) which maximizes the integral

\[
\Phi(g) := \int_U \sigma(\nabla g) \, dx_1 \, dx_2
\]

and \( U \) is the region enclosed by the projection of \( \gamma \) in the \( xy \)-plane.
Important observations

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2. The VP implies that all the tilings look asymptotically the same with overwhelming probability.
3. Requires that all the discrete quantities converge to continuous objects after rescaling.
Rescaling the hook function

At this stage, the local hamiltonian function representing the hook

\[ \log h(x, y) = \log (\lambda_x - x + \lambda'_y - y + 1) \]

does not rescale with the system.
Rescaling the hook function

At this stage, the local hamiltonian function representing the hook

$$\log h(x, y) = \log \left( \lambda_x - x + \lambda'_y - y + 1 \right)$$

does not rescale with the system.

We rewrite it as:

$$\log h(x, y) = \log n + w_n(x, y)$$

with

$$w_n(x, y) = \log \left( \frac{1}{n} \left( \lambda_x - x + \lambda'_y - y + 1 \right) \right)$$. Now $w_n(nx, ny)$ has a well defined limit $\bar{h}_\psi$ when $n \to \infty$. 
Rescaling the hook function

We can now rewrite our asymptotics as:

\[
\log \sum_{T \in \mathcal{T}(\lambda/\mu)} \prod_{(x,y) \in \diamond} h(x, y) = \log \sum_{T \in \mathcal{T}(\lambda/\mu)} \prod_{(x,y) \in \diamond} h(x, y) \\
= \log \sum_{T \in \mathcal{T}(\lambda/\mu)} \prod_{(x,y) \in \diamond} e^{\log n + w_n(x, y)} \\
= |\mu_n| \log n + \log \sum_{T \in \mathcal{T}(\lambda/\mu)} \prod_{(x,y) \in \diamond} e^{w_n(x, y)}
\]

Under the hypothesis of strong stability the term $|\mu_n| \log n$ converges nicely. Hence we are solely interested in

\[
\log \sum_{T \in \mathcal{T}(\lambda/\mu)} \prod_{(x,y) \in \diamond} e^{w_n(x, y)}
\]
Proof of the weighted variational principle

The idea of the proof is similar to the original VP:

1. Undercount and overcount the weight of configurations which stays close to a piecewise affine asymptotic profile $f$.
2. Let the mesh of the affine interpolation go to 0 to obtain bounds for all asymptotic profiles.
3. Show that both bounds are asymptotically the same.
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2. Let the mesh of the affine interpolation go to 0 to obtain bounds for all asymptotic profiles.
3. Show that both bounds are asymptotically the same.

The expression of the functional in the integral arises when evaluating the weight of configurations with approximately linear slope.
Tiling weight of a macroscopic lozenge

The weight of tilings with slope \((s, t)\) on a lozenge of size \(m\) centered at \((na, nb)\) can be written as:

\[
\log \sum_{T \in \mathcal{T}_m(s,t)} \prod_{(x,y) \in \lozenge} e^{w_n(x,y)} = \log \sum_{T \in \mathcal{T}_m(s,t)} \prod_{(x,y) \in \lozenge} e^{\bar{h}_\psi(a,b) + o(n)}
\]

\[
= \log \sum_{T \in \mathcal{T}_m(s,t)} e^{(\bar{h}_\psi(a,b) + o(n))|\lozenge|}
\]

\[
= e^{m^2 \sigma(s,t) + O(m)} e^{(\bar{h}_\psi(a,b) + o(n))(1-s-t)m^2 + o(m^2)}
\]

\[
= e^{m^2(\sigma(s,t) + \bar{h}_\psi(a,b)(1-s-t)) + o(m^2)}
\]
Variational principle for lozenge tilings with local weights

Let \( \{ \gamma_n \}_{n \in \mathbb{N}} \) be a sequence of curves such that \( \frac{1}{n} \gamma_n \) converges to a closed curve \( \gamma \) in \( \mathbb{R}^3 \) in the \( \ell_\infty \) norm and let \( \{ w_n \}_{n \in \mathbb{N}} \) be a sequence of weight functions such that
\[
\lim_{n \to \infty} \sup_{(x,y) \in U} \| w_n(nx, ny) - \rho(x, y) \|_\infty = 0.
\]

**Theorem (Morales, Pak, T.)**

The number \( N_{\gamma_n} \) of lozenge tilings with boundary \( \gamma_n \) satisfies
\[
\lim_{n \to \infty} \frac{1}{n^2} \log W_{\gamma_n} \to \Phi(g_{\text{max}})
\]
where \( g_{\text{max}} \) is the only extension of \( \gamma \) in \( P_{1,1,1} \) which maximizes the integral
\[
\Phi(g) := \int_U \sigma(\nabla g) + L_\rho(x_1, x_2, \nabla g) dx_1 dx_2
\]
and \( L_\rho(x_1, x_2, \nabla g) = (\rho_1, \rho_2, \rho_3) \cdot (\nabla g_1, \nabla g_2, 1 - \nabla g_1 - \nabla g_2) \)
Excited measure vs uniform measure for lozenge tilings
The conjecture about strongly stable shapes is true. The constant $c(\psi/\phi)$ is given by

$$c(\psi/\phi) := k(\psi) + \Psi(f_{\text{max}}),$$

where

$$k(\psi) = \int_{C(\psi)} \tilde{h}_\psi(x, y) dx dy,$$

$$\Psi(f_{\text{max}}) = \max_{f \in P_{1,1,1}} \int_{C(\phi)} \left( \sigma(\nabla f) + (1 - \partial_x f - \partial_y f) \tilde{h}_\psi(x, y) \right) dx dy,$$
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Under the "hook measure", the excited lozenge concentrate around the unique asymptotic profile $f_{\text{max}}$. 


