Polymer partition functions and the geometric RSK correspondence

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Overview

RSK correspondence plays key role in integrable probability and algebraic combinatorics, connecting last passage percolation with Schur processes, eventually leading to KPZ universality class asymptotics such as in Baik-Deift-Johansson '99, Johansson '99.

Dauvergne-Orthmann-Virág '18 found that last passage times are invariant under replacing weights by their semi-discrete RSK images. Starting point for constructing Airy sheet from Airy line ensemble.

Part 1: Discrete geometric lifting

At the discrete geometric level, I will describe three different proofs:

- C '20 gave a proof based on Pitman transforms in spirit of DOV '18.
- Matveev showed me an inductive proof using Desnanot-Jacobi.
- Dauvergne pointed out to me that this in Noumi-Yamada '04.

Part 2: Continuum limits

I will discuss how for special weights, the RSK images enjoy Gibbs properties which are useful in passing the RSK-invariance to the continuum.

The Dauvergne-Orthmann-Virág '18 identity

- Functions $f_1,\dots,f_n:\mathbb{R}_{\geq 0}\to\mathbb{R}$ with $f_i(0)=0$; write $f=(f_1,\dots f_n)$
- Starting points $U=\big((u_1,n),\dots,(u_k,n)\big)$ with $u_1\leq\cdots\leq u_k$
- Ending points $V=\big((v_1,n),\dots,(v_k,n)\big)$ with $v_1\leq\cdots\leq v_k$
- Semi-discrete non-intersecting paths $\pi=(\pi_1,\dots,\pi_k)$ from $U\to V$
- Sum of increments of f along π, denoted $\int df\circ\pi$
- Last passage time from $U\to V$ through f

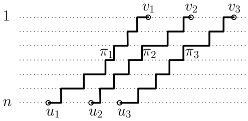
$$f[U\to V]:=\max_{\pi\in U\to V}\int df\circ\pi$$

- Semi-discrete RSK shape $\widetilde{f}(x):=\big(\widetilde{f}_1(x),\dots,\widetilde{f}_n(x)\big)$ where

$$\sum_{r=1}^{\ell}\widetilde{f}_r(x)=f\big[(0,n)^\ell,(x,1)^\ell\big],\qquad \text{for } 1\leq\ell\leq n.$$

Theorem (DOV '18)

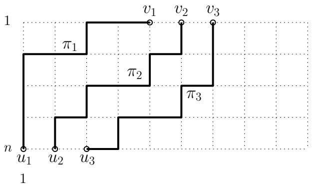
$$f[U\to V]=\widetilde{f}[U\to V]$$

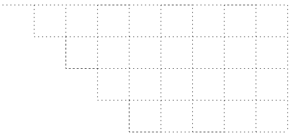


The discrete geometric setup

- Functions $D_i : \mathbb{Z}_{\geq 1} \to (0,\infty)$ with $D_i(0) = 1$; write $D = (D_1, \ldots, D_n)$.
- From D we get vertex weights $d_{x,m} = \frac{D_m(x)}{D_m(x-1)}$.
- Starting / ending points with $u_i, v_i \in \mathbb{Z}_{\geq 1}$.
- Non-intersecting paths $\pi = (\pi_1, \ldots, \pi_k)$ from $U \to V$.
- Partition function from $U \to V$ through D

$$D\big[U \to V\big] := \sum_{\pi \in U \to V} \prod_{(x,m) \in \pi} d_{x,m}.$$

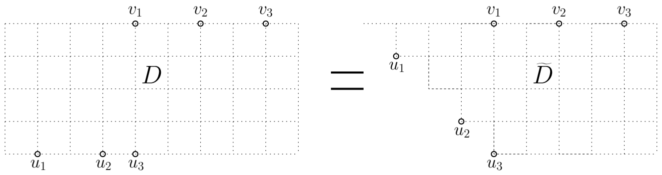




- Geometric RSK shape $\widetilde{D}(x) := \big(\widetilde{D}_1(x), \ldots, \widetilde{D}_n(x)\big)$ where

$$\prod_{r=1}^{\ell} \widetilde{D}_r(x) = D\big[(\llbracket 1, \ell \rrbracket, n), (\llbracket x - \ell + 1, x \rrbracket 1)\big], \qquad \text{for } 1 \leq \ell \leq n \wedge x.$$

The discrete geometric identity



Theorem (C ’20)

For all k and all starting and ending points,

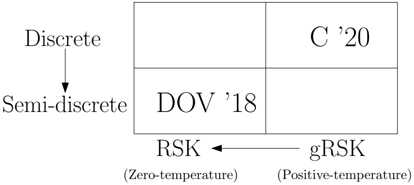
$$D\left[U \to V\right] = \widetilde{D}\left[\Uparrow U \to V\right]$$

where $\Uparrow U = \big((u_i, n \wedge u_i)\big)_{i=1}^{k}$.

When $u_i = i$ and $v_i = i + x$ for all i, this is true by definition.

We will describe three proof strategies and then explain one way this type of identity is useful when studying special solvable choices of weights.

But first, some degenerations



- Discrete RSK: Replacing $(+, \times)$ by $(\max, +)$ reduces gRSK to usual RSK and partition functions to last passage times. Under these replacements, the identity holds true precisely as written.
- Semi-discrete gRSK: If $f_1, \ldots, f_n : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with $f_i(0) = i$, and

$$f[U \to V] := \log\left(\int_{\pi \in U \to V} e^{\int df \circ \pi} \right), \qquad \sum_{r=1}^{\ell} \widetilde{f}_r(x) = f\left[(0,n)^{\ell}, (x,1)^{\ell}\right]$$

  then $f[U \to V] = \widetilde{f}[U \to V]$.
- These further reduce to the semi-discrete RSK identity of DOV '18

Pitman transforms and RSK insertion/bumping

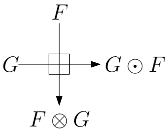
Greene '74 showed the insertion/bumping definition for RSK was equivalent to the last passage times definition given earlier.

For $F, G : \mathbb{Z}_{\geq r} \to \mathbb{R}$ (with $F(r-1) = G(r-1) = 0$ by convention) define

$$G \odot F : \mathbb{Z}_{\geq r} \to \mathbb{R} \qquad \text{maps} \quad x \mapsto F(x) + \sup_{m \in [\![r,x]\!]} \big(G(m) - F(m-1)\big),$$

$$F \otimes G : \mathbb{Z}_{\geq r+1} \to \mathbb{R} \qquad \text{maps} \quad x \mapsto G(x) - \sup_{m \in [\![r,x]\!]} \big(G(m) - F(m-1)\big).$$

We depict the Pitman transform (above) pictorially as



To insert $1^{f_1} 2^{f_2} \cdots$ into the word $1^{g_1} 2^{g_2} \cdots$ we form their partial sums $G = (g_1, g_1 + g_2, \ldots)$ and $F = (f_1, f_1 + f_2, \ldots)$ and apply the Pitman transform:

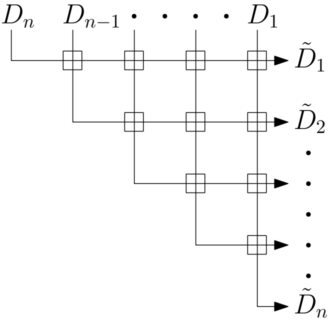
- $G \odot F$ gives us the partial sums of the new word.
- $F \otimes G$ gives us the partial sums of the bumped word.

RSK via Pitman transforms

Recall, the last passage time version of RSK is defined by

$$\sum_{r=1}^{\ell} \widetilde{D}_r(x) = D\big[(\llbracket 1,\ell \rrbracket, n), (\llbracket x-\ell+1, x \rrbracket 1)\big], \qquad D\big[U \to V\big] := \sup_{\pi \in U \to V} \sum_{(x,m)\in\pi} d_{x,m}.$$

We can compute $\widetilde{D}$ as follows:



This picture is essentially in O'Connell '02.

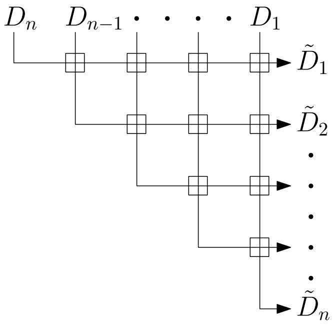
gRSK via geometric Pitman transforms

For $F, G : \mathbb{Z}_{\geq r} \to (0, \infty)$ (with $F(r-1) = G(r-1) = 1$ by convention) define

$$G \odot F : \mathbb{Z}_{\geq r} \to \mathbb{R} \qquad \text{maps} \quad x \mapsto F(x) \sum_{m=r}^{x} \frac{G(m)}{F(m-1)},$$

$$F \otimes G : \mathbb{Z}_{\geq r+1} \to \mathbb{R} \qquad \text{maps} \quad x \mapsto G(x) \left(\sum_{m=r}^{x} \frac{G(m)}{F(m-1)} \right)^{-1}.$$

Then the gRSK image $\widetilde{D}$ of D can be computed via Pitman transform as

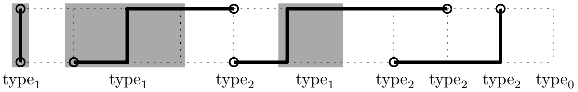


Proof 1 of $D[U \to V] = \widetilde{D}[\Uparrow U \to V]$ (C '20)

Recall $D = (D_1, \ldots, D_n)$ and $|U| = |V| = k$.

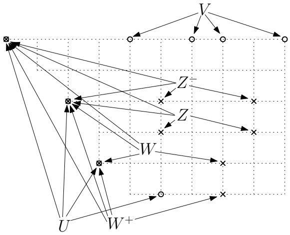
Step 1. The $n = 2$ and $k = 1$ case is easily checked by hand.

Step 2. The $n = 2$ and $k \geq 1$ case follows from Step 1 via the decomposition:



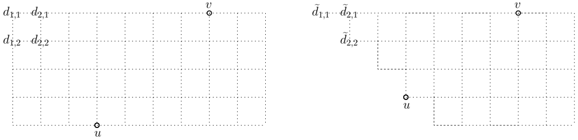
Step 3. The $n \geq 2$ and $k \geq 1$ case follows from Step 2 via the decomposition:

$$D[U \to V] = \sum_{W,Z} D[U \to W^+) D[W^+ \to W) D[W \to Z] D(Z \to Z^-] D[Z^- \to V]$$



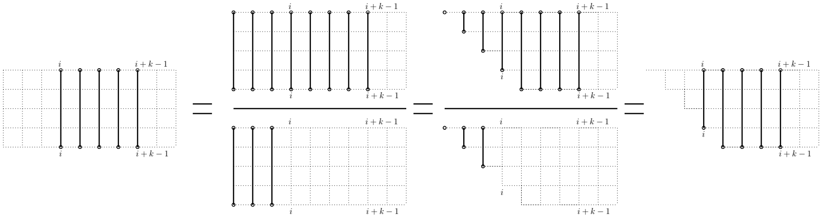
which isolates the dependence on D_m and D_{m+1} to the middle term.

Proof 2 of $D[U \to V] = \widetilde{D}[\Uparrow U \to V]$ (Matveev)



Define matrices $M_{u,v} = D[(u,n) \to (v,1)]$ and $\widetilde{M}_{u,v} = D[(u, n \wedge u) \to (v,1)]$.
By LGV $M_{U,V} := \det\left[M_{u_i,v_j}\right]_{i,j=1}^{k} = D[U \to V]$; likewise for $\widetilde{M}$.
We would like to show $M = \widetilde{M}$. We know

- $M_{[\![1,k]\!],[\![j,j+k-1]\!]} = \widetilde{M}_{[\![1,k]\!],[\![j,j+k-1]\!]}$ (immediate from the definition of $\widetilde{D}$),
- $M_{[\![i,i+k-1]\!],[\![j,j+k-1]\!]} > 0 \iff i \le j \iff \widetilde{M}_{[\![i,i+k-1]\!],[\![j,j+k-1]\!]} > 0$ (else $= 0$),
- $M_{[\![i,i+k-1]\!],[\![i,i+k-1]\!]} = \widetilde{M}_{[\![i,i+k-1]\!],[\![i,i+k-1]\!]}$



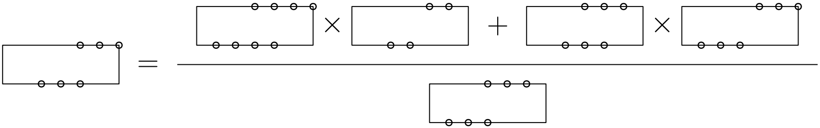
Proof 2 of $D[U \to V] = \widetilde{D}[\Uparrow U \to V]$ (Matveev)

Prove by induction that for all $i \geq j$ and all k,

$$M_{[\![i,i+k-1]\!],[\![j,j+k-1]\!]} = \widetilde{M}_{[\![i,i+k-1]\!],[\![j,j+k-1]\!]} > 0.$$

This is true for $i = j$ and all k, as well as for $i = 1$ and all j and k.

The induction is on $(i, j+k-1)$ in lexicographic order and relies on the Desnanot-Jacobi identity which implies that



The denominator is positive by induction, and all of the numerator terms on the RHS match between M and $\widetilde{M}$ by induction.

Proof 3 of $D[U \to V] = \widetilde{D}[\Uparrow U \to V]$ (Noumi-Yamada '04)

$$G \odot F : \mathbb{Z}_{\geq 1} \to \mathbb{R} \qquad \text{maps} \quad x \mapsto F(x) \sum_{m=1}^{x} \frac{G(m)}{F(m-1)},$$

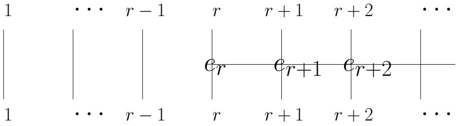
$$F \otimes G : \mathbb{Z}_{\geq 2} \to \mathbb{R} \qquad \text{maps} \quad x \mapsto G(x) \left(\sum_{m=1}^{x} \frac{G(m)}{F(m-1)} \right)^{-1}.$$

The geometric Pitman transform (above) satisfies the matrix equation:

$$H_1(G)H_1(F) = H_2(F \otimes G)H_1(G \odot F),$$

where, for $E : \mathbb{Z}_{\geq r} \to (0, \infty)$ with $E(r+i) = e_r e_{r+1} \cdots e_{r+i}$,

$H_r(E) =$



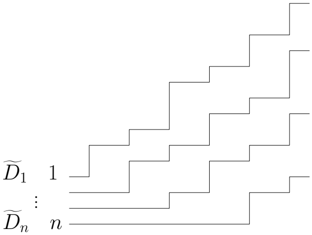
Using a more general version of this relation we see that

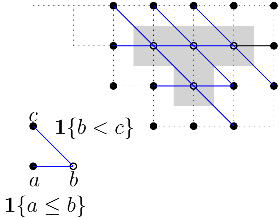
$$H_1(D_n)H_1(D_{n-1}) \cdots H_1(D_1) = H_n(\widetilde{D}_n)H_{n-1}(\widetilde{D}_{n-1}) \cdots H_1(\widetilde{D}_1)$$

from which the identity follows by LGV.

Non-intersecting Gibbsian line ensembles from RSK

For standard RSK O'Connell '02 showed that if $D = (D_1, \ldots, D_n)$ are n independent random walks with geo(p) distributed increments, then $\widetilde{D}$ is distributed as n such random walks conditioned to never intersect.

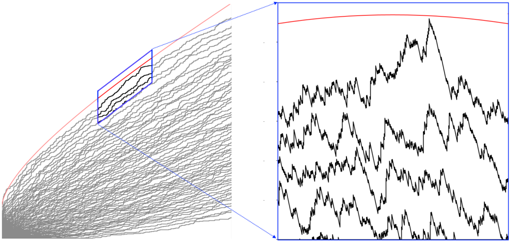




- Non-intersecting Gibbs property: Conditioned on the boundary, law of $\widetilde{D}$ in the gray area is uniform provided the blue relations are satisfied.
- $\widetilde{D}$ is a Schur process $\Longrightarrow$ finite dimensional distribution asymptotics.
- C-Hammond '14: Method based on Gibbs propoprty and one-point tightness to establish regularity/tightness of the entire line ensemble.

Airy line ensemble / sheet

Prähofer-Spohn '02, Johansson '03, C-Hammond '14: Large n limit yields the **Airy line ensemble** with non-intersecting Brownian Gibbs prop.



Graphic from Dauvergne-Nica-Virág '19

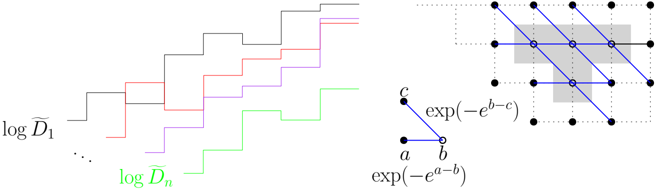
- Top curve (Airy$_2$ process) encodes limiting fluctuations of last passage times from fixed starting point to variable ending points.
- Using their identity + determinantal tools + Gibbs property, Dauvergne-Orthmann-Virág '18 show that the Airy line ensemble also contains the Airy sheet – the limiting fluctuations of last passage times for variable starting and ending points.

Soft Gibbsian line ensembles from gRSK

To show that the non-intersecting Gibbs property / Schur process arises from RSK with geometric inputs one can either use:

- Bijective nature of RSK,
- An intertwining of the tableaux and shape Markov kernels.

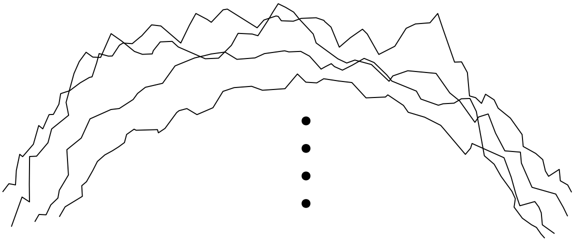
For gRSK with random $d_{x,m}$ with density on $(0,\infty)$ given by $y^{-\gamma-1}e^{-y^{-1}}/\Gamma(\gamma)$ $\widetilde{D}$ is a Whittaker process and enjoys the following soft Gibbs property



► Follows from C-O'Connell-Seppäläinen-Zygouras '14 (intertwining) and O'Connell-Seppäläinen-Zygouras '14 (|Jacobian of gRSK| = 1).

► Whittaker process is not determinantal so fewer tools are available; still Borodin-C-Remenik '13 prove one-point GUE Tracy-Widom limit.

KPZ line ensemble / sheet?



C-Hammond '16 construct KPZ line ensemble with Brownian version of soft Gibbs property. Relates to $n \to \infty$ gRSK limit O'Connell-Warren '16.

Challenge: Prove a limiting version of the invariance identity.

Implication: The KPZ line ensemble contains the KPZ sheet $h_t(x,y)$:

$$\partial_t h_t(x,y) = \partial_y^2 h_t(x,y) + \left(\partial_y h_t(x,y)\right)^2 + \xi(t,y), \qquad e^{h_0(x,y)} = \delta_{x=y}.$$

Take home messages

Generalities:

- There is value in finding discrete versions of a continuous property.
- There is value in finding continuous limits of a discrete property.
- Sometimes both of these are hard.

Specificities:

- The discrete gRSK invariance so far has three (related) proofs.
- For semi-discrete RSK, this is the starting point to construct the Airy sheet from the Airy line ensemble.
- An analogous construction for KPZ is an outstanding challenge.
- Gibbs properties control regularity of line ensembles.