Schubert calculus and quantum integrability

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February 5, 2020

Australian Government
Australian Research Council

A. Knutson and PZJ, *Schubert puzzles and integrability I*, 1706.10019
II, III, IV, in preparation.
PZJ, *Honeycombs for Hall polynomials*, 1909.10720

What is Schubert calculus?

- Schubert calculus is a branch of enumerative geometry which is about answering questions such as "How many lines in 3-space intersect 4 given lines in general position?". Demo
- These questions reduce to calculations in (the cohomology ring of) the space of configurations, which in the example above is the Grassmannian of lines in projective 3-space $Gr(1, \mathbb{P}^3) \cong Gr(2, \mathbb{C}^4)$.
- More generally, the space of configurations can be a Grassmannian

$$Gr(k, \mathbb{C}^n) = \{\text{vector subspaces of dimension } k \text{ in } \mathbb{C}^n\}$$

or a partial flag variety

$$\mathcal{F}_d(\mathbb{C}^n) = \{0 \subseteq V_1 \subseteq \cdots \subseteq V_d \subseteq \mathbb{C}^n\}$$

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Cohomology rings

In practice, these cohomology rings are finitely generated commutative rings, i.e.,

$$H^*(X) \cong \mathbb{K}[t_1, \ldots, t_n]/\mathcal{I} \quad \leftarrow \quad \text{“onshell”}$$

They are finite-dimensional algebras over $\mathbb{K}$, with a preferred basis, the Schubert basis. These basis elements are usually represented by polynomials: Schubert and Grothendieck polynomials for singular cohomology and K-theory respectively. [Lascoux, Schützenberger].

We denote any of these polynomials by S^λ, where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a word in $\{0, 1, \ldots, d\}$. (for X=d-step flag variety)

The more traditional labelling using permutations $\pi \in \mathcal{S}_n$ can be recovered by considering the minimal permutation that sorts λ:

$$\omega_i = \lambda_{\pi(i)}, \ i = 1, \ldots, n \qquad \text{where } \omega = \text{sort}(\lambda)$$

Inversely, fixing X corresponds to fixing ω, i.e., the descent set of π.

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The Littlewood–Richardson coefficients

- We are interested in computing the structure constants of the commutative algebra $H^*(X)$:

$$S^\lambda S^\mu = \sum_\nu c_\nu^{\lambda\mu} S^\nu$$

- In many cases, such as X =partial flag variety, it can be shown that the "on-shell" condition is irrelevant and the question above is reduced to the product rule for the corresponding families of polynomials.
- For $X = Gr$, Schubert polynomials are Schur polynomials, and a combinatorial rule for the $c_\nu^{\lambda\mu}$ was first formulated by Littlewood and Richardson in 1934 (and proven by Schützenberger in the 1970s). In particular, $c_\nu^{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

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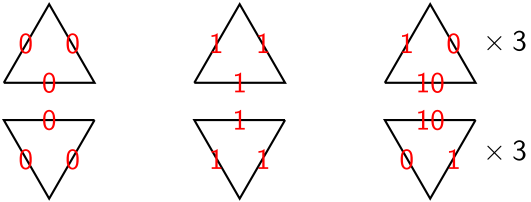
Puzzles

Here we are interested in another rule, introduced by Knutson and Tao:

$$c_{\nu}^{\lambda\mu} = \#\{\text{puzzles with sides } \lambda, \mu, \nu\}$$

A (nonequivariant) puzzle is an assignment of labels from the set $L_1 = \{0, 1, 10\}$ to each edge of a triangle inside the triangular lattice of the plane, such that the boundary edges are labelled with $\{0, 1\}$ only, where the admissible triangles are:



$H^*(Gr)$ [Knutson+Tao '00] Example

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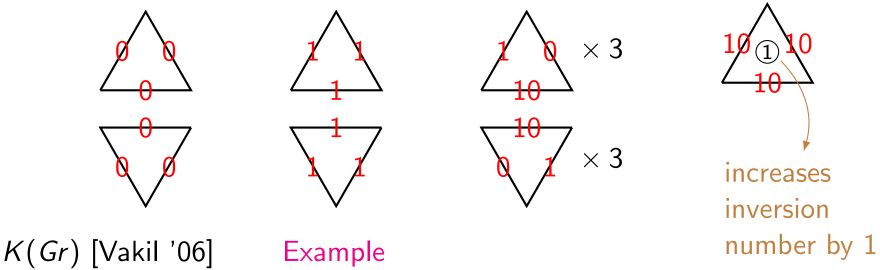

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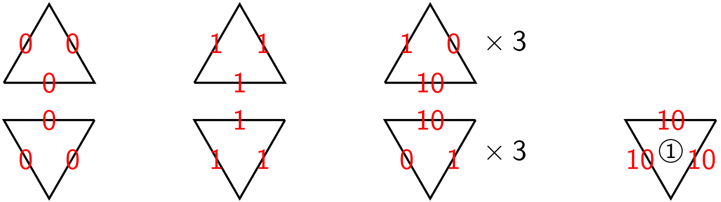

$K(Gr)$ dual [M. Wheeler, PZJ '16]

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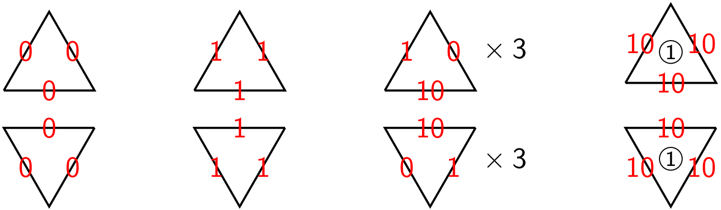

!!! [*II*, A. Knutson, PZJ '20?]

Where does quantum integrability come into play?

- It remains an open problem to provide a (positive, efficient) solution to the general problem of Schubert calculus, i.e., to give a combinatorial rule for the structure constants of the cohomology of flag varieties and related homogeneous spaces.
- My interest is in applying methods from quantum integrable systems / exactly solvable lattice models (QIS) to solve this problem.
- Thanks to the work of Nekrasov+Shatashvili, Maulik+Okounkov, Rimányi+Tarasov+Varchenko, etc, we know how to describe the cohomology of configuration spaces in terms of a QIS.
- This however does not help directly with the problem above. → Another idea is required to use quantum integrable methods for that.
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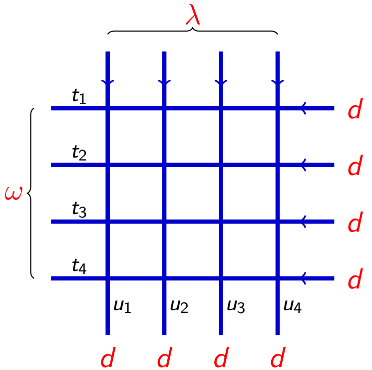
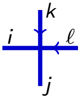
Schubert/Grothendieck polynomials as Bethe vectors

$S^\lambda =$



$\omega = \mathrm{sort}(\lambda)$

with implicit summation over each internal edge $= 0, 1, \ldots, d$.
The weight of a configuration is a product of vertex weights, itself given by the colored 5-vertex R-matrix, e.g., for Schubert,



$$= \begin{cases} 1 & \text{if } (k,\ell) = (i,j) \\ t_\bullet - u_\bullet & \text{if } (k,\ell) = (j,i),\ i < j \end{cases}$$

Bethe vectors cont'd

- Similarly, Grothendieck polynomials are given in terms of the trigonometric colored 5-vertex R-matrix.
- The colored 5-vertex R-matrix is a degeneration of the colored 6-vertex R-matrix:

$$R_{5v} = \lim_{q \to 0} R_{6v}$$

- The Bethe vectors for the six-vertex R-matrix also have an interesting geometric interpretation: Segre Schwartz–MacPherson classes / motivic Segre classes. [*II*, A. Knutson, PZJ '20?]
- These R-matrices are intertwiners for the quantized loop algebra $\mathcal{U}_q(\mathfrak{a}_d[z^\pm])$ based on the root system A_d, e.g., $A_1 = SL_2$ for Grassmannians.
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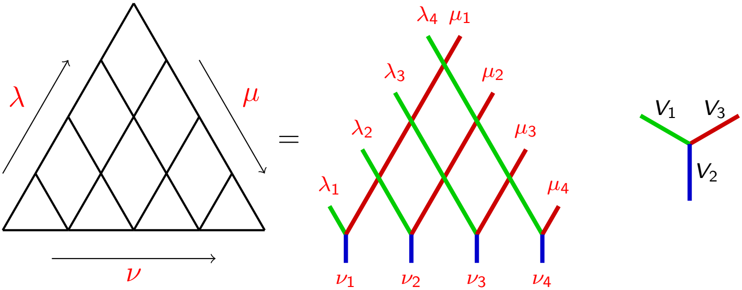
The doubling of the algebra

One can guess that there is a larger algebra $\mathcal{U} \supset \mathcal{U}_q(\mathfrak{a}_d[z^\pm])$ such that the puzzle itself can be interpreted as an intertwiner acting on a tensor product of representations of $\mathcal{U}$:



[A. Knutson, PZJ '17] For $d = 1, 2, 3, 4$ one should choose $\mathcal{U} = \mathcal{U}_q(\mathfrak{x}_{2d}[z^\pm])$ where $\mathfrak{x}_{2d} = \mathfrak{a}_2, \mathfrak{d}_4, \mathfrak{e}_6, \mathfrak{e}_8$. Furthermore, $\dim V_i = 3, 8, 27, 248 + 1$.

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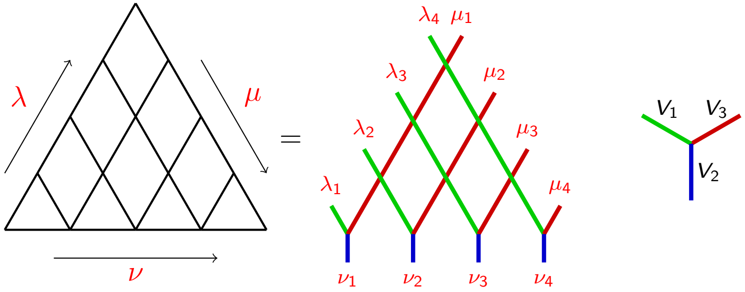

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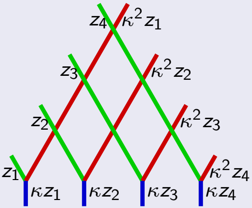
Multiplying motivic Segre classes

$d = 1, 2, 3, 4.$

Theorem (A. Knutson, PZJ '17–'20)

If $S^\lambda S^\mu = \sum_\nu c_\nu^{\lambda\mu} S^\nu$, then $c_\nu^{\lambda\mu}$ is given by the "puzzle"



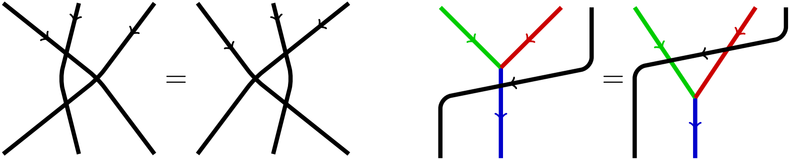
where each rhombus is an R-matrix of $\mathcal{U}_q(\mathfrak{x}_{2d}[z^\pm])$ and triangles are the intertwiner from $V_2(\kappa z) \to V_0(z) \otimes V_1(\kappa^2 z)$.

Here $\kappa = q^{2h/3}$, and $h = 3, 6, 12, 30$ is the Coxeter number of $\mathfrak{x}_{2d}$.

Remarks on proof

By now, we have two proofs of the results:

- In *I*, we proved this "combinatorially" by repeated application of the Yang–Baxter equation and related equations:



- In *II*, we give a geometric proof of this result in terms of certain Lagrangian correspondences between Nakajima quiver varieties. (with some restrictions)

Back to Schubert classes

In order to recover actual Schubert/Grothendieck polynomials, one should send the quantum parameter q to 0.

This leads to explicit combinatorial rules for $d \leq 4$ Schubert calculus, recovering the following known rules:

- $H_T(Gr)$ [Knutson Tao '03], $K(Gr)$ [Vakil '06], $K_T(Gr)$ [Pechenik Yong '15, Wheeler PZJ '16].
- $H_T(\mathcal{F}_2)$ [Buch Kresch Purbhoo Tamvakis '16].

Unfortunately the $d = 4$ rule is almost but not quite positive.

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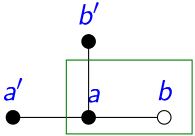

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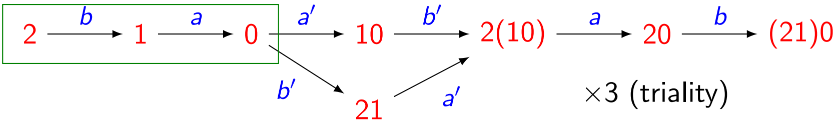

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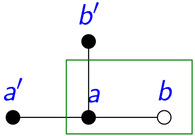

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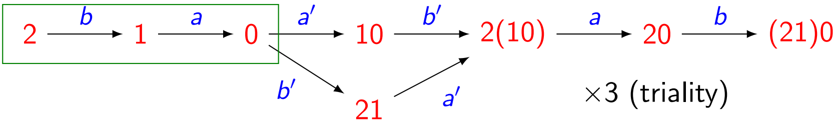
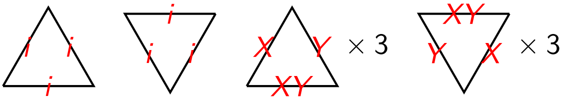


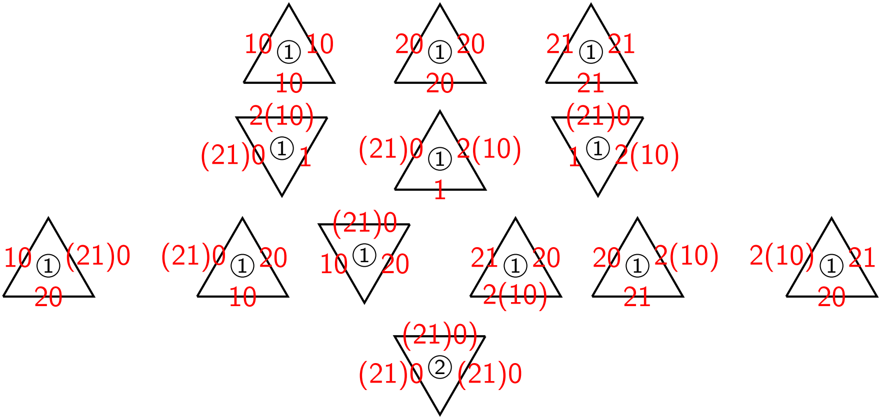
$\times 3$ (triality)

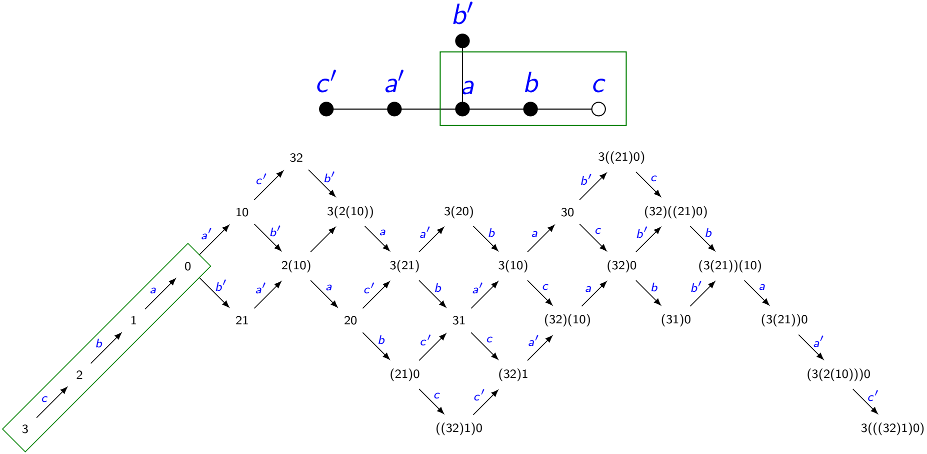
H-admissible triangles:



$K(\mathcal{F}_2)$

K-admissible triangles: H plus





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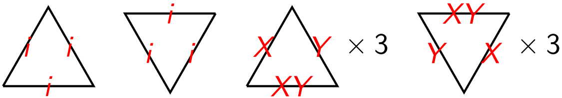
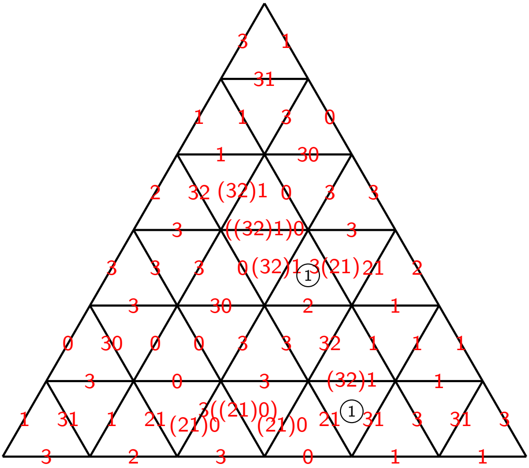


$d = 3$ example



3 1
31
1 1 3 0
1 30
2 32 (32)1 0 3 3
3 ((32)1)0 3
3 3 3 0 (32)1 1 3(21) 21 2
3 30 2 1
0 30 0 0 3 3 32 1 1 1
3 0 3 (32)1 1
1 31 1 21 (21)0 3((21)0) (21)0 21 1 31 3 31 3
3 2 3 0 1 1

Further result 1. Separated descents rule

The results above involved the series of algebras $\mathfrak{a}_2, \mathfrak{d}_4, \mathfrak{e}_6, \mathfrak{e}_8$. Are there Schubert calculus interpretations for the whole *ADE* series?

Answer for *A*:

Theorem (A. Knutson, PZJ '20?)

Consider puzzle pieces (and their 180 degree rotations):

with $j > i$,

,

.

Make size n puzzles with $1, \ldots, k$ *and* $n-k$ *blanks on NE side,* $k+1, \ldots, n$ *and* k *blanks on NW side. Then these compute the structure constants of* $H^*(\mathcal{F}(k, \ldots, n; \mathbb{C}^n)) \otimes H^*(\mathcal{F}(1, \ldots, k; \mathbb{C}^n)) \to H^*(\mathcal{F}(\mathbb{C}^n))$*, and with two more pieces we get the* K_T*-version.*

In other words, one can compute the product of two Schubert polynomials with permutations whose descent sets are on different sides of k.

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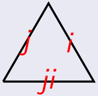
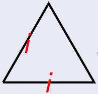
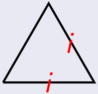


with $j > i$, 

, 

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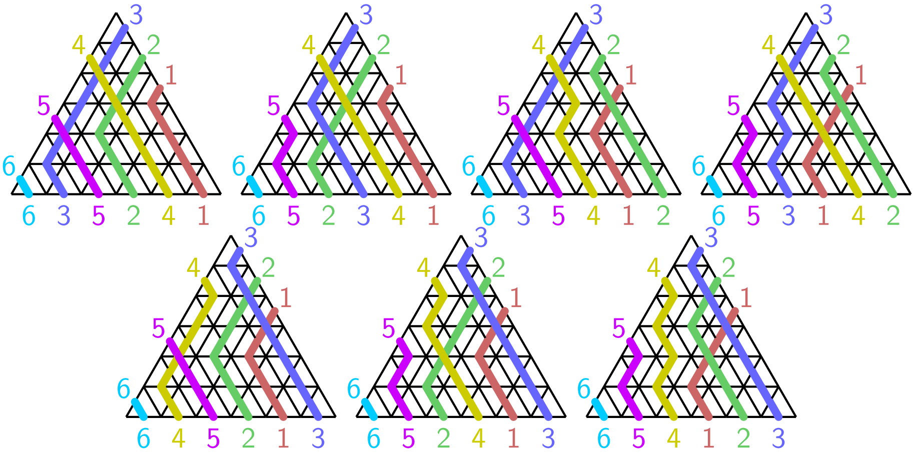
Separated descents example

$\pi = 246531$, $\rho = 321456$:



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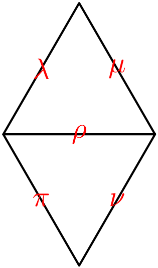

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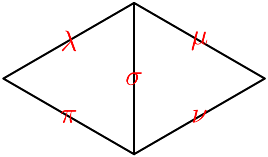

Is there a natural bijection?

In general, integrability/geometry provide a natural linear algebraic answer, which in the case of Schur polynomials turns out to be bijective!

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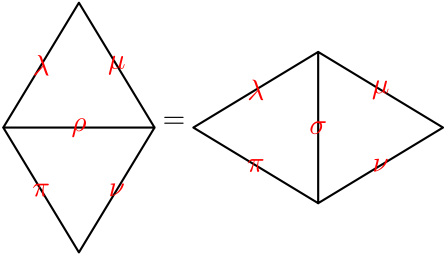

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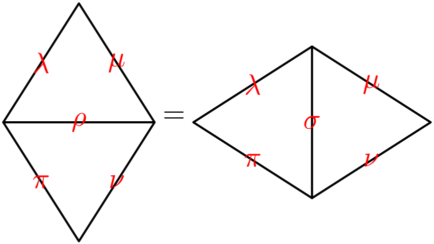

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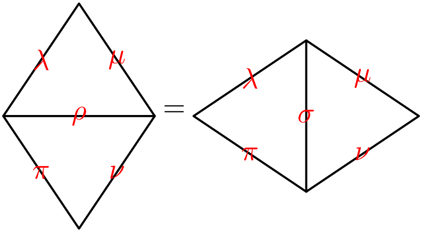

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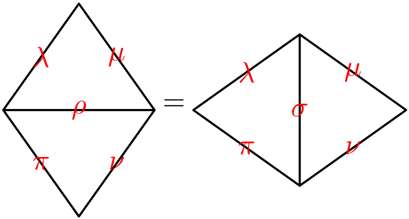

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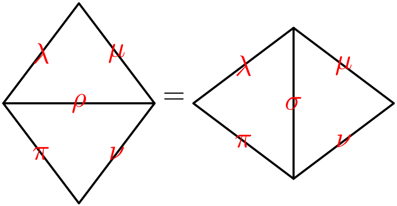

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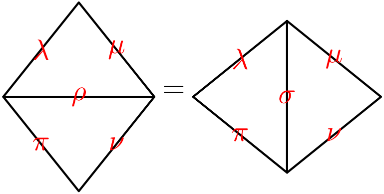

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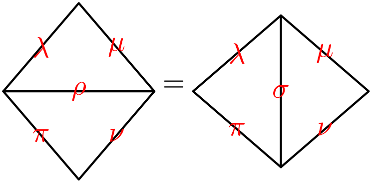

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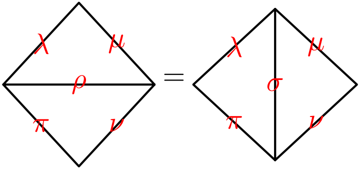

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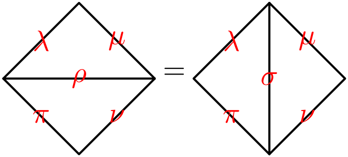

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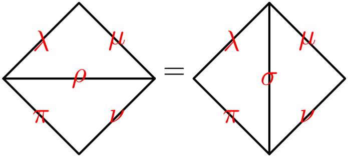

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Associativity and a table of algebras

$d \backslash d'$	1	2	3	4	5	6	>6
1	A_1	A_2	A_3	A_4	A_5	A_6	$A_{d'}$
2	A_2	D_4	E_6	E_8	$E_8^{(1,1)}$	K_{12}	$\cdots$
3	A_3	E_6	$E_7^{(1,1)}$	W_{12}	$\cdots$	$\cdots$	$\cdots$
4	A_4	E_8	W_{12}	$\ddots$	$\cdots$	$\cdots$	$\cdots$
5	A_5	$E_8^{(1,1)}$	$\vdots$	$\vdots$	$\ddots$	$\cdots$	$\cdots$
6	A_6	K_{12}	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$
>6	A_d	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Associativity and a table of algebras

quiver variety

product rule

associativity

$d \backslash d'$	1	2	3	4	5	6	> 6
1	A_1	A_2	A_3	A_4	A_5	A_6	$A_{d'}$
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3	A_3	E_6	$E_7^{(1,1)}$	W_{12}	$\cdots$	$\cdots$	$\cdots$
4	A_4	E_8	W_{12}	$\ddots$	$\cdots$	$\cdots$	$\cdots$
5	A_5	$E_8^{(1,1)}$	$\vdots$	$\vdots$	$\ddots$	$\cdots$	$\cdots$
6	A_6	K_{12}	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$
> 6	A_d	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Hall–Littlewood polynomials

- Hall–Littlewood polynomials $P_\lambda(t; x_1, \ldots, x_n)$ are a family of polynomials depending on one parameter t which interpolate between two bases: Schur polynomials ($t = 0$) and symmetrized monomials ($t = 1$).
- To express them as partition functions of a lattice model requires a trigonometric $\mathfrak{sl}_2$ model with infinite spin, where t plays the role of quantum parameter [Tsilevich '06, Korff '13, Borodin '14].
- The integrable model for their product rule is a $\mathfrak{sl}_3$ infinite spin (parabolic Verma module) model, best expressed in terms of honeycombs. [PZJ '19]

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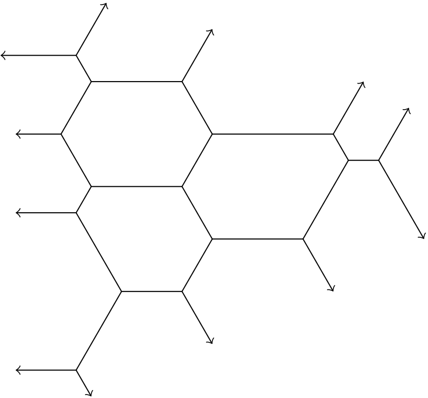
Honeycombs



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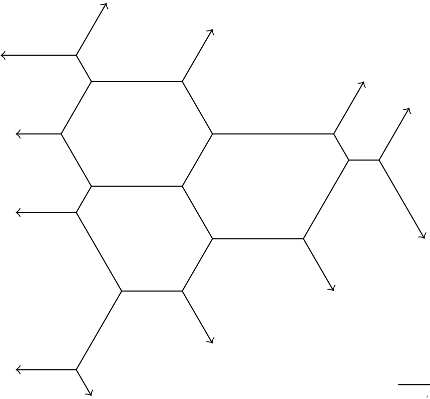

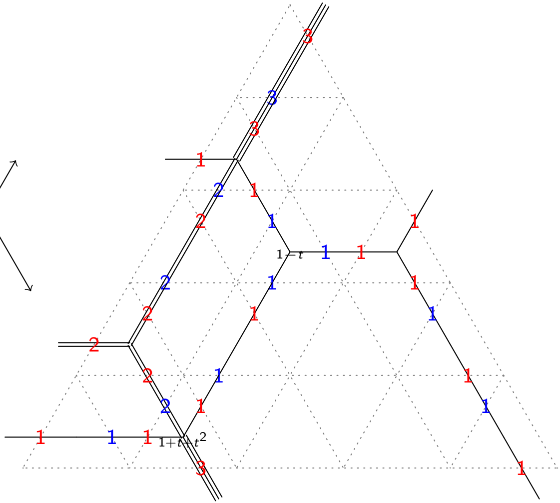

Honeycomb weights

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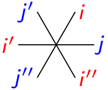
fug(

)

$$= \sum_{r=0}^{\min(i,i')} (-1)^r t^{j' r + r(r+1)/2} \frac{\varphi_{i+j-r}(t)}{\varphi_{i-r}(t)\varphi_r(t)\varphi_{i'-r}(t)}$$

where $\varphi_i(t) = \prod_{r=1}^{i} (1 - t^r)$.

- The simplest proof of the product rule is inductive, using the Pieri rule and associativity, following the corresponding proof of [Knutson, Tao, Woodward '04] for Schur polynomials.
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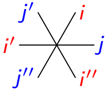

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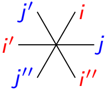

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