Variations on the Index Coding Problem: Caching and Pliable Index Coding

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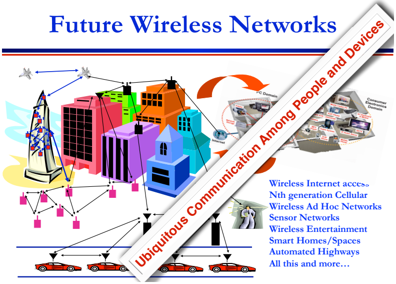
Outline of our Work

1. Index Coding:
   - Strict improvement over the composite coding inner bound of [Arbabjolfaei et al. (Jul. 2013)]. To be presented at [(Wan et al., Feb 2017, ITA 2017)].
2. Caching, under the constraint of uncoded cache placement:
   - Outer bounds by using "IC acyclic outer bound" of [Arbabjolfaei et al. (Jul. 2013)].
   - Optimality of [Maddah-Ali and Niesen (May 2014),Maddah-Ali and Niesen (August 2015)] for the case of less users than files. Appeared [(Wan et al., Jul. 2016, ISIT 2016),(Wan et al., Sep. 2016, ITW 2016)].
   - Use inner bound in point 1 as alternative to the one in [Yu et al. (Sep. 2016)] for the case of more users than files so as to matches our outer bounds. To be presented at [(Wan et al., Feb 2017, ITA 2017)].
   - Novel delivery in the finite file size regime [(Wan et al., May 2017, ICC 2017)].
3. Pliable Index Coding:
   - Refined analysis of the fraction of users satisfied by a single transmission from [Brahma and Fragouli (Nov 2015)]. Appeared [(Liu and Tuninetti, Sep. 2016, ITW 2016)].
   - Optimality of the linear schemes in [Brahma and Fragouli (Nov 2015)] when users have side information sets of equal cardinality. Again use IC acyclic outer bound [work in progress].

Motivation, Network Coding (NC), and Index Coding

- Communications in complex wireless networks.
- Challenges: noise, multi-hop (relaying), multi-cast (compound setting), generalized feedback (overheard information interference or "side information"?), etc.
- Simplification: noiseless links without "multiaccess" but still with "broadcast".

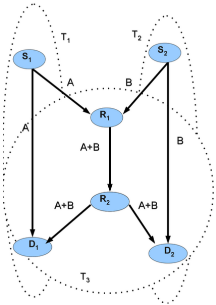


My introduction slide for ECE 437 Wireless Communications.

From S. Goldmsmith's slides.

Network Coding (NC)

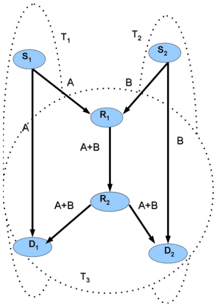
- A network is a directed graph with capacitated edges connecting nodes.
- Source nodes have independent messages destined to subset of destination nodes; wlog multiple unicast.
- Links are noiseless, with broadcast but no multiaccess interference.
- A node can implement *any* function of its received messages.



Example of a NC (Ahlswede, Cai, Li, Yeung [TIT2000]). If relay sends first A and then B, two transmission satisfies both receivers. If instead it sends A+B, one transmission satisfies both receivers, which results in 50% bandwidth savings!

Network Coding (NC)

- NC is **hard** problem.
- Non-linear strategies needed in general; related to entropic region / non-Shannon inequalities.
- NB: an *interference* channel with *relays*, i.e., a combination of two problems believed to be difficult in isolation. We hope to make progress because of its deterministic nature.



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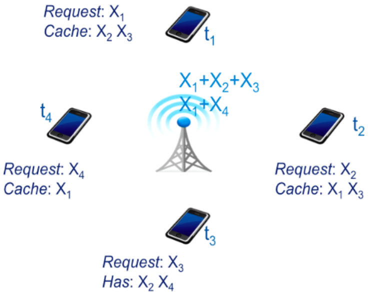
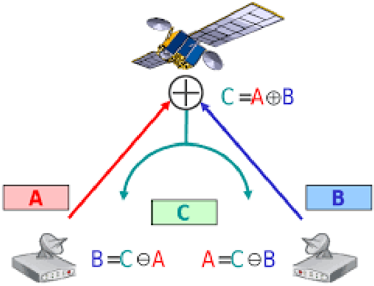
- Special case of NC: one transmitter, no relays, and multiple unicast.
- The TX / server / base station has msgs $\mathcal{X} := \{x_1, ..., x_N\}$.
- RX_j / client_j / user_j requires x_j and knows $\mathcal{S}_j \subseteq \mathcal{X} \backslash \{j\}$.
- Goal: satisfy all clients with the minimum number of transmissions from TX.



Example of a IC. If the TX serves the requested messages one by one, 4 transmissions are necessary. If instead it uses coded transmission, 2 transmissions suffice. 50% bandwidth savings!

Index Coding (IC)

- IC recently attracted LOTS of attention due to its theoretical significance and applications in wireless ad-hoc networks.
- IC is **equivalent** to NC but simpler to describe.
- IC solved for no more than five messages, where linear codes suffice; some 'structured' classes too.
- General inner (graph theoretic quantities or random coding) and outer (cut-set, non-Shannon inequalities) bounds have exponential complexity.



Satellite transmission (Birk and Kol [INFOCOM1998]).

Index Coding

Index Coding: Model

- N (not necessarily unicast) independent messages.
- K (non cooperative) receivers.
- Message $W_j \in [1 : 2^{nR_j}]$, uniform and independent.
- Codeword $X^n(W_1, ..., W_N) \in [1 : 2^n]$, received by all receivers.
- RX_j wants $W_{\mathcal{D}_j} := (W_i : i \in \mathcal{D}_j)$ and has $W_{\mathcal{S}_j} := (W_i : i \in \mathcal{S}_j)$.
- RX_j estimates $\widehat{W_{\mathcal{D}_j}}(X^n, W_{\mathcal{S}_j})$. Assume $\mathcal{S}_j \neq [1 : N], \mathcal{D}_j \neq \emptyset, \ \mathcal{D}_j \cap \mathcal{S}_j = \emptyset$ for all $j \in [1 : K]$.
- Find largest rate region for $(R_1, \ldots, R_N)$ such that probability of error vanishes for each receiver as $n \to \infty$.
- Open problem in general.
- Often one is interested in smallest $1/R_{\text{sym}}$: $(R_1, \ldots, R_N) = (R_{\text{sym}}, \ldots, R_{\text{sym}})$ is achievable.

Index Coding: Example

- Multiple unicast index coding problem with $K = 6$ users and

$$\begin{aligned} \mathcal{D}_1 &= \{A\}, & \mathcal{S}_1 &= \{B, C\}, \\ \mathcal{D}_2 &= \{B\}, & \mathcal{S}_2 &= \{D, E\}, \\ \mathcal{D}_3 &= \{C\}, & \mathcal{S}_3 &= \{A, B\}, \\ \mathcal{D}_4 &= \{D\}, & \mathcal{S}_4 &= \{B, E\}, \\ \mathcal{D}_5 &= \{E\}, & \mathcal{S}_5 &= \{A, F\}, \\ \mathcal{D}_6 &= \{F\}, & \mathcal{S}_6 &= \{C, D\}, \end{aligned}$$

here, for better readability, messages are indicated with the letters A through F.

- Messages are iid binary vectors of length b bits. TX sends

$$\begin{aligned} X^{3b} = (&A \oplus B \oplus C, \\ &B \oplus D \oplus E, \\ &C \oplus D \oplus F). \end{aligned}$$

- RX_5 sums the components in X^{3b} to get $A \oplus E \oplus F$; then, he recovers E.
- Other RX's can directly 'read' the desired message from one component of X^{3b}.
- Rate $R_{\text{sym}} = 1/3$ is achievable; or, $1/R_{\text{sym}} = 3$ transmissions suffice to satisfy all users.

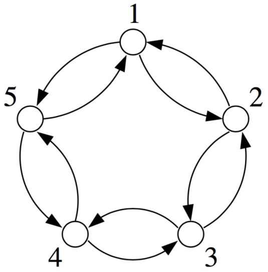
Index Coding: Outer Bound 1

- RX_j desires $\mathcal{D}_j$ and has $\mathcal{S}_j$.
- Theorem: Outer bound

$$\sum_{\ell \in \mathcal{D}_j} R_\ell \le T_{\mathcal{S}_j} - T_{\mathcal{D}_j \cup \mathcal{S}_j}$$

for some monotonic submodular function $T_{\{\cdot\}}$ such that $T_\emptyset \le 1$. Proof: (Fano)

$$\sum_{\ell \in \mathcal{D}_j} R_\ell \le \frac{1}{n} I\left(W_{\mathcal{D}_j}; X^n | W_{\mathcal{S}_j}\right)$$
$$= \frac{1}{n} H\left(X^n | W_{\mathcal{S}_j}\right)$$
$$- \frac{1}{n} H\left(X^n | W_{\mathcal{D}_j \cup \mathcal{S}_j}\right)$$



Index Coding Problem Example.

$R_1 \le T_{2,5} - T_{2,5,1}$,

$R_2 \le T_{1,3} - T_{1,3,2}$,

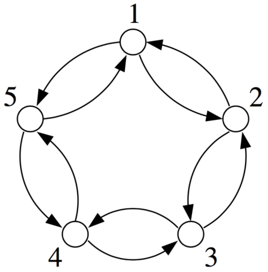
etc.

The optimize over all possible submodular function $T_{\{\cdot\}}$.

Index Coding: Outer Bound 2

- Unicast case: RX_j desires $\mathcal{D}_j = \{j\}$ and has $\mathcal{S}_j$.
- Directed graph representation of side information sets.
- Corollary [**Acyclic Outer Bound**]: (looser bound but easier to evaluate)

$$\sum_{\substack{\ell \in \mathcal{J}: \\ \mathcal{J} \text{ is subgraph} \\ \mathcal{J} \text{ not a d-cyclic}}} R_\ell \le \max H(X) = 1.$$



Index Coding Problem Example.

Corollary: $R_1 + R_3 \le 1$, etc. $(2 \le 1/R_{\text{sym}})$.

Theorem: $R_1 + R_2 + R_3 + R_4 + R_5 \le 2$. $(2.5 \le 1/R_{\text{sym}})$.

Index Coding: Example

- Multiple unicast index coding problem with $K = 6$ users and

$$\mathcal{D}_1 = \{A\}, \quad \mathcal{S}_1 = \{B, C\},$$
$$\mathcal{D}_2 = \{B\}, \quad \mathcal{S}_2 = \{D, E\},$$
$$\mathcal{D}_3 = \{C\}, \quad \mathcal{S}_3 = \{A, B\},$$
$$\mathcal{D}_4 = \{D\}, \quad \mathcal{S}_4 = \{B, E\},$$
$$\mathcal{D}_5 = \{E\}, \quad \mathcal{S}_5 = \{A, F\},$$
$$\mathcal{D}_6 = \{F\}, \quad \mathcal{S}_6 = \{C, D\},$$

  here, for better readability, messages are indicated with the letters A through F.

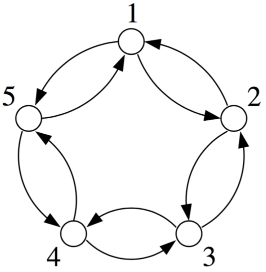
- Messages A, D, F form an acyclic subgraph. [I could not draw a non-messy figure!]
- Also: let users 1 and 4 cooperate == super-user with $\mathcal{D} = \{A, D\}$ and $\mathcal{S} = \{B, C, E\}$.
  After the super-user recovers D, it can mimic user 6, and thus recover F. Thus

$$R_1 + R_4 + R_6 \leq \frac{1}{n} I(A, D, F; X^n | B, C, E) \leq 1,$$

  or $R_{\text{sym}} \leq 1/3$.

Index Coding: Composite Coding

- Two-step encoding:
  1) Generate a ‘composite index’ $X^n_{\mathcal{P}}(W_{\mathcal{P}}) \in [1 : 2^{nS_{\mathcal{P}}}]$, $\forall \mathcal{P} \subseteq [1 : N]$.
  2) Bin all composite indices in $X^n \in [1 : 2^n]$.
- Two-step decoding:
  1) RX_j recovers all composite indices in X^n given the message side information $W_{\mathcal{S}_j}$.
  2) RX_j recovers the messages in $\mathcal{D}_j \subseteq \mathcal{K}_j \subseteq [1 : N]\backslash\mathcal{S}_j$, only those in $\mathcal{D}_j$ uniquely.



Index Coding Problem Example.

E1: (W_1, W_2) to index $X_{1,2}$ at rate $S_{1,2}$;

(W_2, W_3) to index $X_{2,3}$ at rate $S_{2,3}$; etc.

E2: $X^n(X_{1,2}, X_{2,3}, X_{3,4}, X_{4,5}, X_{5,1})$ at rate $\max H(X) = 1$.

D1: $S_{1,2} + S_{2,3} + S_{3,4} + S_{4,5} + S_{5,1} \le 1$. Same for every RX.

D2: $R_1 \le S_{1,2} + S_{1,5}$; $R_2 \le S_{1,2} + S_{2,3}$; etc.

FourierMotzkin elimination of the composite index rates ...

Index Coding: Composite Coding

- Achievable region: Composite Coding.
  A non-negative rate tuple $\mathbf{R} := (R_1, \ldots, R_N)$ is achievable if

$$\mathbf{R} \in \bigcap_{j \in [1:K]} \bigcup_{\mathcal{K}_j : \mathcal{D}_j \subseteq \mathcal{K}_j \subseteq [1:N] \setminus \mathcal{S}_j} \mathscr{R}(\mathcal{K}_j | \mathcal{S}_j, \mathcal{D}_j),$$

$$\mathscr{R}(\mathcal{K} | \mathcal{S}, \mathcal{D}) := \bigcap_{\mathcal{J} : \mathcal{J} \subseteq \mathcal{K}, \mathcal{D} \cap \mathcal{J} \neq \emptyset} \left\{ \sum_{i \in \mathcal{J}} R_i < v_{\mathcal{J}} \right\},$$

$$v_{\mathcal{J}} := \sum_{\mathcal{P} : \mathcal{P} \subseteq \mathcal{S} \cup \mathcal{K}, \mathcal{P} \cap \mathcal{J} \neq \emptyset} S_{\mathcal{P}},$$

$$\sum_{\mathcal{J} : \mathcal{J} \in [1:N], \mathcal{J} \not\subseteq \mathcal{S}_j} S_{\mathcal{J}} \leq 1, \quad \forall j \in [1:K].$$

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- Composite coding allows for $R_{\text{sym}} \leq 1/4$.
- Proof: no matter the choice of decoding sets $(\mathcal{K}_j, j \in [1:6]) : |\mathcal{K}_j| \leq 4$ the symmetric rate is cannot exceed $R_{\text{sym}} \leq 1/4$.
- Strictly suboptimal compared to the scheme introduced previously.

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- A non-negative rate tuple $\mathbf{R} := (R_1, \ldots, R_N)$ is achievable if

$$\mathbf{R} \in \bigcap_{j\in[1:K]} \bigcup_{\mathcal{K}_j:\mathcal{D}_j\subseteq\mathcal{K}_j\subseteq[1:N]\setminus\mathcal{S}_j} \mathscr{R}(\mathcal{K}_j|\mathcal{S}_j, \mathcal{D}_j),$$

$$\mathscr{R}(\mathcal{K}|\mathcal{S}, \mathcal{D}) := \bigcap_{\mathcal{J}:\mathcal{J}\subseteq\mathcal{K}, \mathcal{D}\cap\mathcal{J}\neq\emptyset} \left\{ \sum_{i\in\mathcal{J}} R_i < v_{\mathcal{J}} \right\},$$

$$v_{\mathcal{J}} := I\Big((U_i : i \in \mathcal{J}) \ ; \ (X_{\mathcal{P}} : \mathcal{P} \subseteq [1:N]) \Big| (U_i : i \in \mathcal{S}_j \cup \mathcal{K}_j \setminus \mathcal{J}) \Big),$$

$$H\Big((X_{\mathcal{P}} : \mathcal{P} \subseteq [1:N]) \Big| (U_i : i \in \mathcal{S}_j) \Big) \leq 1, \quad \forall j \in [1:K].$$

for some $P_{U_1,\ldots,U_N} = \prod_{i\in[1:N]} P_{U_i}$ and functions $X_{\mathcal{P}}\big((U_i : i \in \mathcal{P})\big)$.

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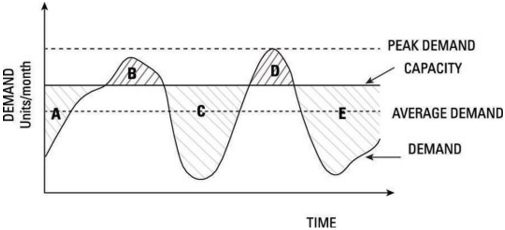
- New inner bound $R_{\text{sym}} \leq 1/3$.
- Proof: first achieveble scheme.
- Strict improvement over composite coding.

Caching

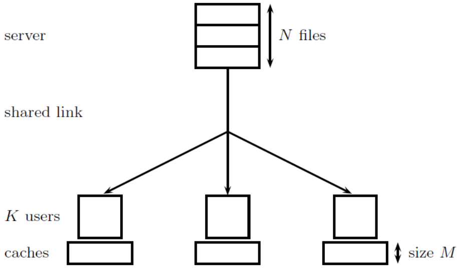
Caching: Motivation



- High temporal variability of network traffic leads to high traffic in peak-traffic times and low traffic in peak-off times.
- Idea: Caching (prefetching) can help to smooth the traffic in peak-traffic times by storing part of the contents in users' local memories during peak-off periods.

Caching: Problem Formulation

- Placement: Cache content creation, without knowledge of later demands. Limit M.
- Delivery: Broadcast packets based on demands and cache contents. Limit R.
- Goal: Minimize the broadcast rate R for the worst case demands.
- Connection to IC: can choose the best side side information for the worst case multicast demands.



Caching system. [Maddah-Ali and Niesen 2016 ITSoc Paper Award].

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Centralized vs Decentralized placement

- Centralized: Users in placement and delivery phases are the same.
- Decentralized: Otherwise.

Uncoded vs Coded placement

- Uncoded placement: Each user directly stores M out of N files.
- Coded placement: Otherwise.

Maddah-Ali and Niesen (MAN) Coded Caching Schemes

- Centralized: Combinatorial uncoded cache placement, coded delivery. Cut-set outer bound. Optimality within factor ca 4.
- Decentralized: i.i.d. bit storage, coded delivery. Centralized outer bound.

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Caching: cMAN and dMAN

- Let $M = t\frac{N}{K}$, $t \in [0:K]$. Files $F_1, \ldots, F_N$ of size B bits.
- Placement phase: each file is split into $\binom{K}{t}$ non-overlapping sub-files of equal size $\frac{B}{\binom{K}{t}}$ bits. The sub-files of F_i are denoted by $F_{i,\mathcal{W}}$ for $\mathcal{W} \subseteq [1:K]$ where $|\mathcal{W}| = t$.
- User k fills his cache as

$$Z_k = \Big(F_{i,\mathcal{W}} : k \in \mathcal{W} \subseteq [1:K], |\mathcal{W}| = t,\ i \in [1:N] \Big).$$

- Delivery phase: the server transmits

$$X_{\mathbf{Z},\mathbf{d}} = \Big(\oplus_{s \in \mathcal{S}} F_{d_s, \mathcal{S} \setminus \{s\}} : \mathcal{S} \subseteq [1:K], |\mathcal{S}| = t+1 \Big),$$

which requires broadcasting $B\binom{K}{t+1}/\binom{K}{t}$ bits.

Caching: cMAN and dMAN

- cMAN [Maddah-Ali and Niesen (May 2014)]

$$R_{\text{cMAN}}[t] := \binom{K}{t+1} / \binom{K}{t}.$$

Removing the redundant transmissions [Yu et al. (Sep. 2016)]

$$R_{\text{c,uncoded placement}}[t] := \left(\binom{K}{t+1} - \binom{K-\min(K,N)}{t+1}\right) / \binom{K}{t}.$$

- dMAN [Maddah-Ali and Niesen (August 2015)]

$$R_{\text{dMAN}}(M) := \left(\frac{N}{M} - 1\right)\left[1 - \left(1 - \frac{M}{N}\right)^{K}\right].$$

Removing the redundant transmissions [Yu et al. (Sep. 2016)]

$$R_{\text{d,uncoded placement}}(M) := \left(\frac{N}{M} - 1\right)\left[1 - \left(1 - \frac{M}{N}\right)^{\min(K,N)}\right].$$

Caching: Our Contributions

- Leveraged the connection between IC and caching (for uncoded placement).
- Used "IC acyclic outer bound" and showed optimality of MAN for uncoded placement for $N \geq K$, or for $N = 2$. Note: Result holds when the total cache size is fixed or/and the total file size is fixed.
- Used new IC inner bound to show optimality of MAN for uncoded placement for $N < K$, as alternate proof to [Yu et al. (Sep. 2016)].
- Consequence: Improvement over MAN can be obtained only by **coded cache placement**.
- Novel delivery scheme for dMAN placement in the finite file size regime [ICC2017].

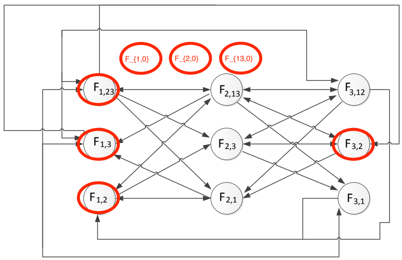
Caching: Outer Bound

- After the users' demands are revealed in a caching scheme with uncoded cache placement, the delivery phase is equivalent to a general multicast index coding problem.
- We used "IC acyclic outer bound" and leveraged the intrinsic symmetries of the caching problem
- Let $\mathcal{N}(\mathbf{d})$ be the set of demanded files. For each $i \in \mathcal{N}(\mathbf{d})$ and for each $\mathcal{W} \subseteq [1:K]$, the sub-file $F_{i,\mathcal{W}}$ (containing the bits of file F_i within the cache of the users indexed by $\mathcal{W}$) is an independent message in the index coding problem with user set $[1:K]$.
- IC with $M = tM/N, t \in [0:K]$, $N = |\mathcal{N}(\mathbf{d})|\binom{K}{t}$ for centralized caching or $N = |\mathcal{N}(\mathbf{d})|(2^t - 1)$ for decentralized caching, and

$$\mathcal{D}_k = \left(F_{d_k,\mathcal{W}} : \mathcal{W} \subseteq [1:K], k \notin \mathcal{W}\right),$$
$$\mathcal{S}_k = \left(F_{i,\mathcal{W}} : \mathcal{W} \subseteq [1:K], i \in \mathcal{N}(\mathbf{d}), k \in \mathcal{W}\right).$$

Caching: Example $N = K = 3$

- Each file is divided into $2^K = 8$ disjoint parts, denoted as $F_{i,\mathcal{W}}$.
- For each demand vector $\mathbf{d}$ we generate an index coding problem with $K2^{K-1} = 12$ independent messages.
- In the constructed graph we want to find sets of nodes not forming a cycle.
- $F_{1,\emptyset}, F_{2,\emptyset}, F_{3,\emptyset}$ are always in such sets $\mathcal{J}$ the acyclic outer bound.



Directed graph representing the equivalent IC. Demand $\mathbf{d} = (1, 3, 2)$; set not containing a cycle is $(F_{1,\emptyset}, F_{1,2}, F_{1,3}, F_{1,23}, F_{3,\emptyset}, F_{3,2}, F_{2,\emptyset})$.

Caching: General Case

- In general, we can find a bound for all possible pairs $(\mathbf{d}, \mathbf{u}) \in$ all permutations of the integers $[1 : K]$.

$$R_{\mathrm{c,u}} \geq \sum_{i \in [0:K]} \frac{\binom{K}{i+1} - \binom{K-\min(K,N)}{i+1}}{\binom{K}{i}} \frac{x_i}{NB}, \quad (1)$$

$$x_t := \sum_{j \in [1:N]} \sum_{\mathcal{W} \subseteq [1:K]:|\mathcal{W}|=t} |F_{j,\mathcal{W}}|, \quad t \in [0:K], \quad (2)$$

where x_t in (2) represents the total number of bits known by t users, and where the constraints on the file size and on the cache size imply

$$x_0 + x_1 + \cdots + x_K = NB, \quad (3)$$

$$0x_0 + 1x_1 + 2x_2 + \cdots + ix_i + \cdots + Kx_K \leq \frac{KM}{N} NB. \quad (4)$$

- Fourier-Motzkin elimination $(x_0, \ldots, x_K)$.

Caching: Main Result: Converse

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Theorem 1

- *Outer bound for centralized caching systems: lower convex envelope*

$$(M, R) = \left(t\frac{N}{K}, \frac{\binom{K}{t+1} - \binom{K-\min(K,N)}{t+1}}{\binom{K}{t}} \right), \quad t \in [0 : K].$$

- *Outer bound for decentralized caching systems:*

$$R = \frac{1 - \frac{M}{N}}{\frac{M}{N}} \left[1 - \left(1 - \frac{M}{N} \right)^{\min(K,N)} \right].$$

- *Further results: optimality for* $N = 2$.

Theorem 2

- *Inner bound for centralized caching systems: our new IC inner bound matched applied to centralized caching systems attains the outer bound.*
- *Inner bound for decentralized caching systems: our new IC inner bound matched applied to decentralized caching systems attains the outer bound.*

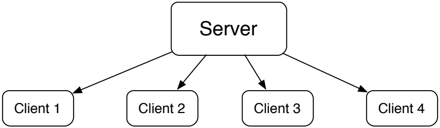
Pliable Index Coding (PICOD)

PICOD: Motivation

- Today's communication networks increasingly deliver *type of content* rather than specific messages. Example: content distribution networks, advertisement networks, social networks, etc.
- PICOD clients are flexible/pliable and happy to receive any one message they do not already have (for instance, any search results they have not already received, etc.).
- Model: Same as IC except that client$_j$ is satisfied by decoding successfully *any* message not in its side information.
- Connection to IC: *choose optimal demands; multi-casting allowed.* Still side information sets fixed.

IC vs PICOD

$$\mathcal{M} = \{b_1, b_2, b_3, b_4\}$$

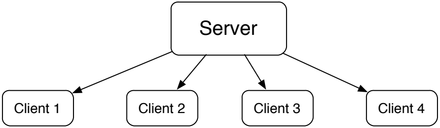


$\mathcal{I}_1 = \{b_2, b_4\}$ $\mathcal{I}_2 = \{b_1, b_3\}$ $\mathcal{I}_3 = \{b_2, b_4\}$ $\mathcal{I}_4 = \{b_1, b_3\}$

$\mathcal{R}_1 = \{b_1\}$ $\mathcal{R}_2 = \{b_2\}$ $\mathcal{R}_3 = \{b_3\}$ $\mathcal{R}_4 = \{b_4\}$

IC optimal transmission: $t_1 = b_1 + b_2, \quad t_2 = b_3 + b_4$.

$$\mathcal{M} = \{b_1, b_2, b_3, b_4\}$$



$\mathcal{I}_1 = \{b_2, b_4\}$ $\mathcal{I}_2 = \{b_1, b_3\}$ $\mathcal{I}_3 = \{b_2, b_4\}$ $\mathcal{I}_4 = \{b_1, b_3\}$

$\mathcal{R}_1 = \mathcal{I}_1^c$ $\mathcal{R}_2 = \mathcal{I}_2^c$ $\mathcal{R}_3 = \mathcal{I}_3^c$ $\mathcal{R}_4 = \mathcal{I}_4^c$

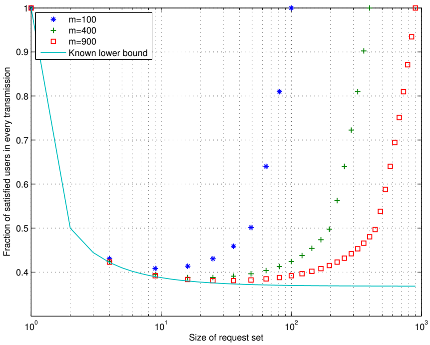
PICOD optimal transmission: $t_1 = b_1 + b_2$. 50% bandwidth saving! ($\mathcal{I}$ side information set, $\mathcal{R}$ possible request set.)

PICOD: Past Work

- m files, n users, $\mathcal{R}$ possible request set, $\mathcal{S}$ side information set.
- PICOD is NP-hard, as general IC.
- Sufficiency of $O(\log n)$ transmissions for $|\mathcal{S}_j| = \text{constant}, \forall j$.
- Sufficiency of $O(\min\{(\log m)(1 + \log^+(\frac{n}{\log m})), m, n\})$ transmissions.
- There exists PICOD's requiring $\Omega(\log n)$ transmissions.
- Achievability: Greedy algorithm with binary linear codes to satisfy the largest fraction of yet-to-be-satisfied clients.

PICOD: Our Work

- Refined analysis of the fraction of clients that can be satisfied by one transmission.
- Deterministic (not probabilistic) approach.
- Strictly better than the known lower bound.
- Capture effect of size of side information set (not seen in IC).
- No converse bounds.



Fraction of satisfied users with a single transmission vs. size of side information set [ITW2016].

PICOD: Main Result

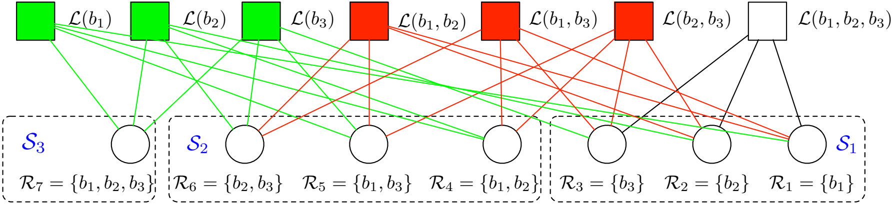
Theorem 3

For a PICOD instance with m messages and where the request set of all n clients has cardinality d, the fraction of unsatisfied clients that can be satisfied by one transmission is lower bounded by
$\max\left\{\left(1-\frac{d}{m}\right)^{\frac{m}{d}-1} e\left(1-\frac{1}{d}\right)^{d-1}, \frac{d}{m}\right\}$*, which is at least* $1/e$.

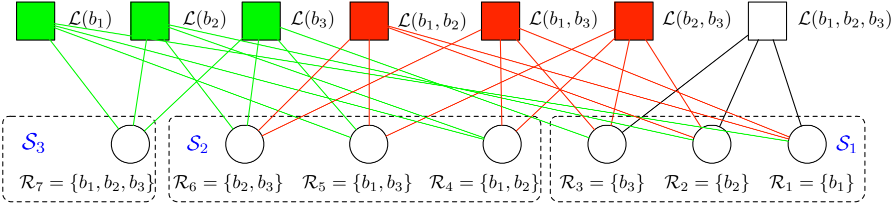
- Refined analysis of the fraction of clients that can be satisfied by one transmission.
- Deterministic approach.

Maximum Number of Satisfied Clients by one Transmission

- m is # of messages; $k \in [1:m]$ is # of messages in linear combinations; $j \in [1:\binom{m}{k}]$ for each k; $D_{kj}, k \in [1:m], j \in [1:\binom{m}{k}]$ number of satisfied clients with one transmission; Group S_d contains all clients with $|\mathcal{D}_i| = d$. Let $C_i = |\mathcal{S}_i|$.



Main Result

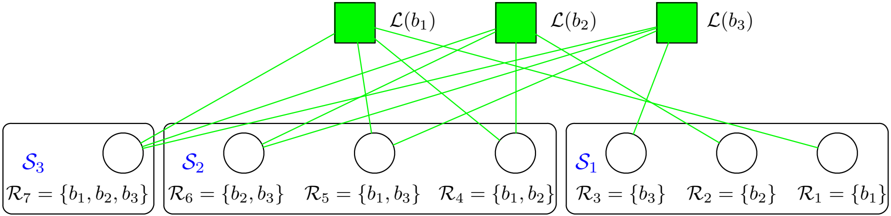


$$\max_{kj} D_{kj} \geq \max_k \frac{D_{k1} + \cdots + D_{k\binom{m}{k}}}{\binom{m}{k}}$$

$$= \max_k \frac{\sum_{i=1}^{m} \binom{i}{1}\binom{m-i}{k-1} C_i}{\binom{m}{k}}$$

$$:= \max_k \sum_{i=1}^{m} R_{i,m,k} C_i$$

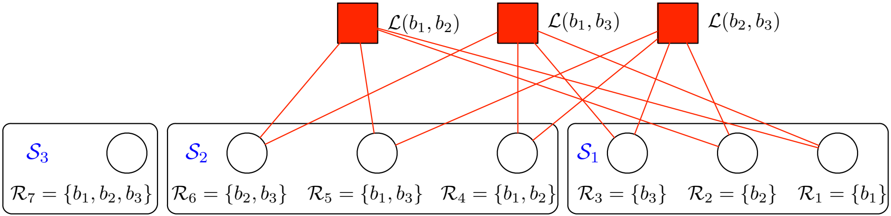
Number of Satisfied Clients by $k = 1$



- Maximum number of client satisfied by one transmission

$$\frac{\sum_{i=1}^{3} \binom{i}{1}\binom{3-i}{1-1} C_i}{\binom{3}{1}} = \frac{C_1 + 2C_2 + 3C_3}{3} = \frac{12}{3} = 4.$$

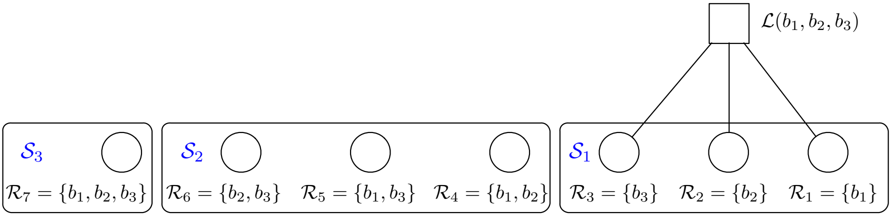
Number of Satisfied Clients by $k = 2$



- Maximum number of client satisfied by one transmission

$$\frac{\sum_{i=1}^{3} \binom{i}{1}\binom{3-i}{2-1} C_i}{\binom{3}{2}} = \frac{2C_1 + 2C_2}{3} = \frac{12}{3} = 4.$$

Number of Satisfied Clients by $k = 3$



- Maximum number of client satisfied by one transmission

$$\frac{\sum_{i=1}^{3} \binom{i}{1}\binom{3-i}{3-1} C_i}{\binom{3}{3}} = \frac{C_1}{1} = 3.$$

We now consider the case where all clients have the same size d for the request set, i.e., $C_d = n, C_i = 0, \forall i \neq d$.

- To derive the maximum fraction of clients that can be satisfied by one transmission, we solve

$$\max_k R_{d,m,k} = \max_k \frac{\binom{d}{1}\binom{m-d}{k-1}}{\binom{m}{k}}.$$

- Note that $R_{d,m,1} = d/m$, i.e., d/m is the fraction of clients that can be satisfied with one transmission.
- For $k \geq 2$, use Stirling's approximation:

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}$$

and some algebra.

- Finally we can have

$$\max_k R_{d,m,k} > \left(1 - \frac{d}{m}\right)^{\frac{m}{d}-1} e \left(1 - \frac{1}{d}\right)^{d-1}.$$

- Combined with the result for $k = 1$ we have

$$\begin{aligned}\max_k R_{d,m,k} &\geq \max\left\{\left(1 - \frac{d}{m}\right)^{\frac{m}{d}-1} e \left(1 - \frac{1}{d}\right)^{d-1}, \frac{d}{m}\right\} \\ &\geq \left(1 - \frac{1}{\sqrt{m}}\right)^{2\sqrt{m}-2} e \\ &> 1/e.\end{aligned}$$

This proves the main result.

Application: PICOD instance with m messages and n clients, whose side information sets are generated i.i.d. according to a Bern(p) distribution.

- Not exactly the same request set size. However, by the law of large number, for sufficiently large m, almost all clients will be in the "typical client set" defined as

$$\mathcal{T}_{\epsilon}^{(m)} = \left\{ \text{client}_i, \left| \frac{|\mathcal{R}_i|}{m} - (1-p) \right| \leq \varepsilon(1-p) \right\}.$$

- Fraction of clients satisfied by single transmission at least

$$r_{m,p} = \max \left\{ 1-p, p^{\frac{p}{1-p}} \left(1 - \frac{1}{m(1-p)} \right)^{m(1-p)-1} e \right\} > 1/e$$

with high probability.

Computation Complexity

- In PICOD $1 \leq n \leq 2^m - 1$.
- Known algorithms have running time $O\left(mn^2\right)$ [1] and $O(m^2 n \log(n))$ [2].
- Our achievability has running time $O(\frac{2^m}{\sqrt{m}})$, which is independent of n.
- The two known algorithms are faster when n is small, but slower when n is exponential in m.

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Conclusions

- IC is a fundamental open problem in network information theory.
- It is deceivingly simple. It is representative of the general NC.
- It is the building block in several data delivery & management system problems.
- For IC and its variations, new directions needed for:
  - Inner bounds (right now mainly based on linear codes, or too complex / combinatorial).
  - Outer bounds (right now very few techniques but essentially all cut-set-based, and in general too complex / combinatorial).
- New ideas ...
- THANK YOU

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