From SLLN to MFG and HFT

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Outline

Starting from SLLN

1. Mean field games
2. High frequency tradings

(f)SLLN

- SLLN: If $\{X_i\}_{i=1}^N$ are i. i. d. random variables, with $\mu = E[X_i]$, then

$$\frac{X_1 + X_2 + \cdots X_N}{N} \to \mu, \qquad N \to \infty$$

- (f)SLLN: the functional form of SLLN, for instance if random variables are replaced by stochastic processes
- Building blocks for some recent progresses in financial mathematics
  - Mean Field Games
  - High Frequency trading models

Part I: Mean Field Games (MFG)

MFGs are

- originated from physics on weakly interacting particles;
- theoretical works pioneered by Lasry and Lions (2007) and Huang, Malhamé and Caines (2006);
- stochastic strategic decision games with very large population of small interacting individuals;
- about small interacting individuals, with each player choosing optimal strategy in view of the macroscopic information (mean field).

Main idea of MFG

- Take an N-player game;
- When N is large, (f)SLLN kicks in, consider the "aggregated" version of the N-player game;
- The aggregated version, MFG, becomes an "approximation" of the N-player game.

N-player game

$$\inf_{\alpha^i \in \mathcal{A}} E\{\int_0^T f^i(t, X_t^1, \cdots, X_t^N, \alpha_t^i)dt\}$$

$$subject \quad to \quad dX_t^i = b^i(t, X_t^1, \cdots, X_t^N, \alpha_t^i)dt + \sigma dW_t^i$$

$$\text{and} \quad X_0^i = x^i$$

- X_t^i is the state of player i at time t;
- α_t^i is the action/control of player i at time t, in an appropriate control set $\mathcal{A}$;
- f^i is the running cost for player i;
- g^i is the terminal cost for player i;
- b^i is the drift term for player i;
- σ is a volatility term for player i;
- W_t^i are i.i.d. standard Brownian motions.

From N-player game to MFG

Consider

Aggregation

$$\inf_{\alpha^i \in \mathcal{A}} E\{\int_0^T \frac{1}{N}\sum_{i=1}^N f^i(t, X_t^1, \cdots, X_t^N, \alpha_t^i)dt\}$$

$$s.t. \quad dX_t^i = \frac{1}{N}\sum_{i=1}^N b^i(t, X_t^1, \cdots, X_t^N, \alpha_t^i)dt + \sigma dW_t^i$$

$$\text{and} \quad X_0^i = x^i$$

As $N \to \infty$, consider the mean information μ_t as an unknown external signal, instead of $X_t^1, \cdots, X_t^N$

MFG

$$\inf_{\alpha \in \mathcal{A}} E[\int_0^T f(t, X_t^i, \mu_t, \alpha_t) dt]$$

$$\text{such that} \quad dX_t^i = b(t, X_t^i, \mu_t, \alpha_t) dt + \sigma dW_t^i \quad \text{and} \quad X_0^i = x^i$$

Assumptions (symmetry)

Players are indistinguisheable: they are rational, identical, and interchangeable

Toy model (Guéant, Lasry, and Lions (2009))

Question: deciding the starting time of a meeting

- Identical $N = 10K$ agents distributed on the negative half-line according to the distribution m_0 ($m_0(x) = 0$ for $x \geq 0$);
- A meeting scheduled to hold at $x = 0$ at time t_0;
- Actual meeting starting time T depending on the arrivals of participants (e.g. the meeting starts when 90% agents arrive at 0);
- X_t^i position of agent i at time t;
- α_t^i agent i's action at time t;

Toy model

Assume

- $dX_t^i = \alpha_t^i dt + \sigma dW_t^i$, with W_t^i i. i. d. standard Brownian motions;
- τ_i the time at which agent i would like to arrive;
- Real arrival time is $\tilde{\tau}^i = \tau^i + \sigma\epsilon^i$ where $\epsilon^i \sim N(0,1)$ i. i. d. with

$$\tilde{\tau}^i = \inf\{t > 0, X_t^i = 0\};$$

- Agent i's cost function $c^i(t_0, T, \tilde{\tau}^i, \alpha^i)$, such as $(T - \tilde{\tau}^i)^+, (\tilde{\tau}^i - t_0)^+, (\tilde{\tau}^i - T)^+$.

Toy model: MFG method when N big

- First, Fix T

$$
\begin{aligned}
u = \min_{\alpha} E[c^i(t_0, T, \tilde{\tau}^i, \alpha^i)] \\
s.t. \quad dX_t^i = \alpha_t^i dt + \sigma dW_t^i, \quad X_0^i = x_0 \\
\tilde{\tau}^i = \min\{s : X_s^i = 0\};
\end{aligned}
$$

  Denote α^* be an optimal control of this problem

- Based on α^*, get distribution of arrivals μ (all agents are identical). From μ, update T'. Repeat this process until a fixed point.

$$T \to \alpha^* \to T' \to \alpha^{*\prime} \to \cdots \to \text{fixed point}$$

PDE/control approach of general MFGs

- (i) Fix a deterministic function $t \in [0, T] \to \mu_t \in \mathcal{P}(\mathbb{R}^d)$
- (ii) Solve the stochastic control problem

$$\inf_{\alpha \in \mathcal{A}} \int_0^T f(t, X_t, \mu_t, \alpha_t) dt$$
$$s.t. \quad dX_t = b(t, X_t, \mu_t, \alpha_t) dt + \sigma dW_t \quad \text{and} \quad X_0 = x$$

- (iii) Update the function $t \in [0, T] \to \mu'_t \in \mathcal{P}(\mathbb{R}^d)$ so that $\mathcal{P}_{X_t} = \mu'_t$
- (iv) Repeat (ii) and (iii). If there exists a fixed point solution μ_t and α_t, then it is a solution for this model.

Three main approaches

- PDE/control approach:
  backward HJB equation + forward Kolmogorov equation
  Lions and Lasry (2007), Huang, Malhame and Caines (2006), Lions, Lasry and Guant (2009)
- Probabilistic approach: FBSDEs
  Buckdahn, Li and Peng (2009), Carmona and Delarue (2013)
- Stochastic McKean-Vlasov and DPP
  Pham and Wei (2016)

Main results for general MFGs

Under proper technical conditions,

Theorem

The MFG admits a unique optimal control.

Theorem

The value function of MFG approximate the value function of the N-player game, with an error of $O(\frac{1}{\sqrt{N}})$.

Growing literatures on MFGs (partial list)

- MFGs with common noise
  Sun (2006), Carmona, Fouque, and Sun (2013), Garnier, Papanicolaou and Yang (2012), Carmona, Delarue and Lacker (2016), Nutz (2016),
- MFGs with partial observations
  Buckdahn, Li, Ma (2015), Buckdahn, Ma, Zhang (2016)
- MFG with singular controls
  Zhang (2012), Hu, Oksendal and Sulem (2014), Fu and Horst (2016), G. Lee (2016)
- MFG for HFT
  Jaimungal and Nourian (2015), Lachapelle, Lasry, Lehalle, and Lions (2016)
- MFG for queuing system
  Manjrekar, Ramaswamy, and Shakkottai (2014), Wiecek, Altman, and Ghosh (2015), Bayraktar, Budhiraja, and Cohen (2016)
- MFG for energy
  Chan and Sircar (2016)

Exercise #1, Finite Fuel Follower Problem

Single player, dynamics of its position given by

$$X_t = x + B_t + \xi_t,$$

- ξ_t a controlled càdlàg process, with finite variation such that

$$\xi_t = \xi_t^+ - \xi_t^-$$

- ξ_t^+ and ξ_t^- non-decreasing càdlàg processes
- $\check{\xi}_t = \xi_t^+ + \xi_t^-$ accumulative fuel consumption

The objective is to minimize over all appropriate admissible controls

$$E \int_0^\infty e^{-\alpha t}\{d\check{\xi}_t + h(X_t)dt\}$$

Solution

- h convex, $h(-x) = h(x)$, $h''(x)$ decreasing and $0 < k < h''(x) < K$
- controls could be either finite variation or with bounded velocity
- the optimal control is "bang-bang" type

Benes, Shepp and Witsenhausen (1980) and Karatzas (1983)

N-player game

N identical players

$$X_t^i = x^i + B_t^i + \xi_t^i,$$

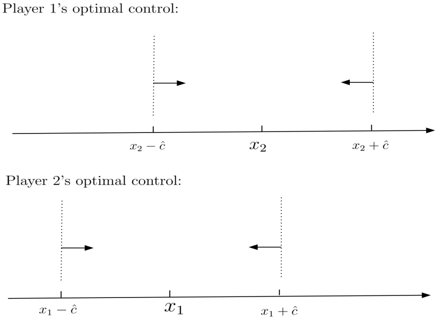
Each players i optimizes

$$\mathbb{E} \int_0^\infty e^{-\alpha t} \{ d\check{\xi}_t^{\,i} + h(X_t - \mu_t^N) dt \}$$

with $\mu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_t^i}$.

Solution of MFG: $N = 2$

- Solving HJB system
- the optimal control is bang-bang type for each player, centered around the other player's position



Solution of MFG: $N \to \infty$

- Assuming a stationary mean information
- the solution is almost the same as the single player game
- the optimal control is bang-bang type, centered around μ_0
- intuition: too much symmetry in the problem, the MFG game becomes "anticipative" and less insightful

G. Xu (2017)

Exercise #2: Irreversible problem

Single player

$$\sup_{L_t, M_t} E\left[\int_0^\infty e^{-rt}[\Pi(K_t)dt - pd\xi_t^+ + (1-\lambda)pd\xi_t^-]\right]$$

subject to

$$dK_t = K_t(\delta dt + \sigma dW_t) + d\xi_t^+ - d\xi_t^-, K_0 = k$$

Model setup

- K_t the capacity of the company
- ξ_t^+ and ξ_t^- nondecreasing of finite variation, $\mathcal{F}_t^k$ progressively measurable, cádlág processes, with $\xi_0^+ = \xi_0^- = 0$
- Π Lipschitz continuous, nondecreasing, bounded and concave over k and satisfies $\lim_{k \downarrow 0} \frac{\Pi}{k} = \infty, \sup_{k>0}[\Pi - kz] < \infty$. For instance, $\Pi = K_t^\alpha$
- $\delta, \sigma, r, p, \lambda \in (0, 1)$ nonnegative constants

Explicit solution

If $\Pi(K_t) = K_t^{\alpha}$ with $\alpha \in (0, 1)$, the optimal strategy is characterized by (k_b, k_s), so that

- neither increasing nor reducing capacity when it is in the region (k_b, k_s);
- increasing capital when it is below than k_b in order to reach the threshold k_b; and
- reducing capital when it is above k_s in order to attain the level k_s.
- The region $(0, k_b)$ is called the expansion region, and (k_s, ∞) the contraction region.

G. Pham (2005)

MFGs

$$\sup_{\xi^{i+},\xi^{i-}} E\left[\int_s^T e^{-rt}[\Pi(K_t^i, \mu_t)dt - pd\xi_t^{i+} + (1-\lambda)pd\xi_t^{i-}]\right]$$

subject to

$$dK_t^i = bK_t^i dt + \sigma K_t^i dW_t^i + d\xi_t^{i+} - d\xi_t^{i-}, \quad K_{s-}^i = k$$

Model setup

- $\Pi(k, \mu) = \mu k^{\alpha}$
- ξ_t^{i+}, ξ_t^{i-} are $\mathcal{F}_t$- progressively measurable, cádlág, of bounded velocity
- $\mu_t = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} K_t^i$
- $p > 0, r > 0, \lambda \in (0, 1)$ are constants.

Explicit solutions

- Kolmogorov forward equation translates into solving a piece-wise linear weakly reflected diffusion
- Value function and optimal control, and the fixed point μ^* explicitly solved
- For the fixed point μ^* $k_b^* = {\mu^*}^{\frac{1}{1-\alpha}} * C_1$ where C_1 is independent of μ^*. Similarly, $k_s^* = {\mu^*}^{\frac{1}{1-\alpha}} * C_2$ where C_2 is independent of μ^*
- The region of $(0, k_b^*)$ is of expansion and the region of (k_s^*, ∞) is of contraction

Comparison with and without MFGs

- In the case of μ^* sufficiently large (Good game)
  - $k_b < k_b^*$: everyone works harder
  - $k_s^* - k_b^* > k_s - k_b$: everyone benefit from other people's hard work
- In the case of μ^* sufficiently small (Bad game)
  - $k_b > k_b^*$: everyone works less hard
  - $k_s^* - k_b^* < k_s - k_b$: everyone gets hurt in the game

G. Joon (2017)

Why MFG?

- Young and one of the fastest growing research areas
- With some fun and challenging mathematical problems:
    - CLTM
    - large deviation
    - conditional SLLN
    - MFG vs. single player game
- With wide range of applications, economics, finance, queuing network, engieering

Some References with MFGs

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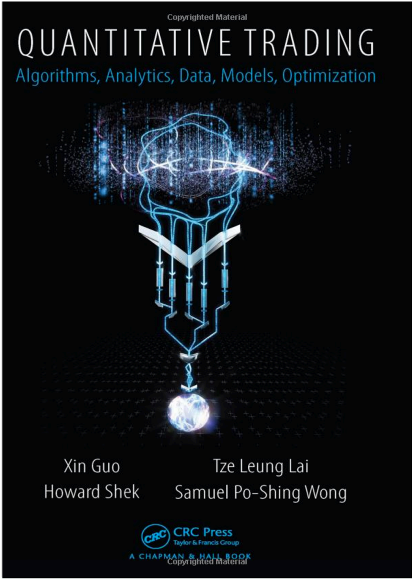
Part II: High Frequency Tradings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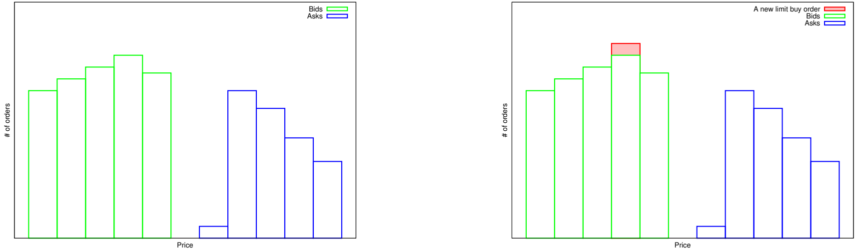
HFTs and Limit order book (LOB)

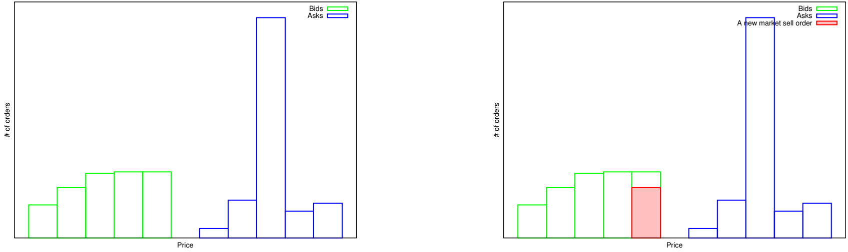
- The collection of outstanding buy and sell orders with both prices and quantities on the market
- Six types of orders which may change the state of the limit order book
  - Limit bid/ask orders: orders to buy/sell at a specified price, added to the queue and executed in order of arrival until canceled
  - Market buy/sell orders: buy/sell orders executed immediately at the best available price
  - Cancelations: any unexecuted limit orders could be canceled by their owners (without penalty)
  - (Best) bid/ask price: the highest/lowest price in limit buy/sell orders
  - Regulatory guidelines generally require exchanges to honor price-time priority: best price + FIFO

A limit buy order



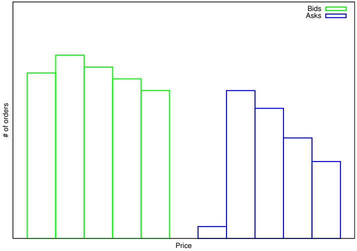
$\Longrightarrow$

A market sell order



$\Longrightarrow$

A cancelation of sell order



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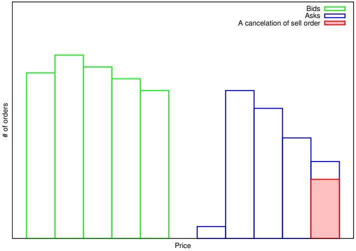

Trade off between market and limit orders

When using limit orders:

- Earn a rebate or discount for providing liquidity
- However no guarantee of execution: *execution risk*

When using market orders

- Have to pay the spread and fees
- Yet immediate execution guaranteed

In essence: trade-off of paying the spread and fees vs execution/inventory risk.

Why HFT

- Related to queuing theory (Harrison, Rieman, Williams, Whitt ...)
- LOBs (Cont and Larrard (2015) vs. classical queues with reneging (Ward and Glynn (2003, 2005))
- Classical queuing concerns on status/stability of the system, while algorithm trading focuses on the individual trade
- Different characteristics, such as cancellations

Cancelations account for over 80 percent of all orders

"A single mysterious computer program that placed orders—and then subsequently canceled them—made up 4 percent of all quote traffic in the U.S. stock market for the week (of October 5th, 2012).... The program placed orders in 25-millisecond bursts involving about 500 stocks and the algorithm never executed a single trade." (CNBC news)

The value of individual order positions

- Key factors for solving market making or algorithmic trading problems.
- "Fighting" for good order positions is one of the motivations for huge expenditure for high frequency traders: a better order position means less waiting time and higher probability of execution
- Numerical study shows that for some stocks the "value of order positions" has the same order of magnitude of half spread in US equity market (Moellami and Yuan (2015))

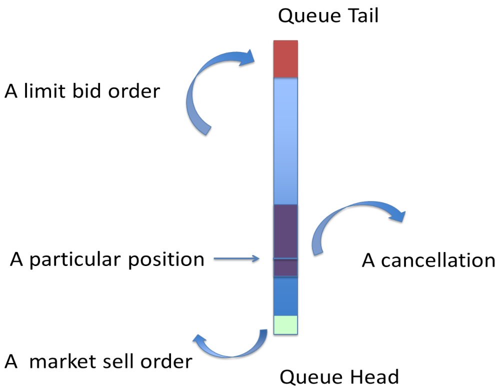

Figure: Orders happened in the best bid queue

Model assumptions and notations

- Consider the best bid and best ask
- Six order types: best bid orders (1), market orders at the best bid (2), cancellation at the best bid(3), best ask orders (4), market orders at the best ask (5), and cancellation at the best ask (6).
- $\mathbf{N} = (N(t), t \geq 0)$ the order arrival process, with the inter-arrival times $\{D_i\}_{i \geq 1}$, and $N(t) = \max\left\{m : \sum_{i=1}^{m} D_i \leq t\right\}$.
- $\{\overrightarrow{V}_i = (V_i^j, 1 \leq j \leq 6)\}_{i \geq 1}$: For each i, the component V_i^j represents the size of i-th order from the j-th type. For example, V_i^2 is the size of the *market order* at the best bid for the $i-$th order.
- No simultaneous arrivals of different orders. For example, $\overrightarrow{V}_5 = (0, 0, 0, 4, 0, 0)$ means the fifth order is a best ask order of size 4.

Assumptions

- (I). $\{D_i\}_{i\geq 1}$ is a stationary array of random variables with

$$\frac{D_1 + D_2 + \ldots + D_i}{i} \to \frac{1}{\lambda} \qquad \text{in probability.}$$

  Here λ is a positive constant.
- (II). $\{\overrightarrow{V}_i\}_{i\geq 1}$ is a stationary array of random vectors with

$$\frac{\overrightarrow{V}_1 + \overrightarrow{V}_2 + \ldots + \overrightarrow{V}_i}{i} \to \overrightarrow{\bar{V}} \qquad \text{in probability.}$$

  Here $\overrightarrow{\bar{V}} = (\bar{V}^j > 0, 1 \leq j \leq 6)$ is a constant vector.
- (III) Cancelations are uniformly distributed on every queue.

Fluid limit for order positions and related queues

Define scaled queue lengths $\mathbf{Q}_n^b$, $\mathbf{Q}_n^a$, and the scaled order position $\mathbf{Z}_n$ by

$$Q_n^b(t) = Q_n^b(0) + C_n^1(t) - C_n^2(t) - C_n^3(t)$$

$$Q_n^a(t) = Q_n^a(0) + C_n^4(t) - C_n^5(t) - C_n^6(t),$$

$$dZ_n(t) = -dC_n^2(t) - \frac{Z_n(t-)}{Q_n^b(t-)} dC_n^3(t).$$

where the scaled net order flow process

$$\overrightarrow{C}_n(t) = \frac{1}{n} \sum_{i=1}^{N(nt)} \overrightarrow{V}_i = \left(\frac{1}{n} \sum_{i=1}^{N(nt)} V_i^j, 1 \leq j \leq 6 \right),$$

Theorem

Under Assumptions (I), (II), and (III) and assume $(Q_n^b(0), Q_n^a(0), Z_n(0)) \Rightarrow (q^b, q^a, z)$. Then for $t < \tau$,

$$(\tilde{\mathbf{Q}}_n^b, \tilde{\mathbf{Q}}_n^a, \tilde{\mathbf{Z}}_n) \Rightarrow (\mathbf{Q}^b, \mathbf{Q}^a, \mathbf{Z}) \qquad \text{in} \quad (D^3[0,\infty), J_1),$$

where $(\mathbf{Q}^b, \mathbf{Q}^a, \mathbf{Z})$ is given by

$$Q^b(t) = q^b - \lambda v^b t,$$

$$Q^a(t) = q^a - \lambda v^a t.$$

$$\frac{dZ(t)}{dt} = -\lambda \left(\bar{V}^2 + \bar{V}^3 \frac{Z(t-)}{Q^b(t-)} \right), \qquad Z(0) = z.$$

Here $\tau = \tau^a \wedge \tau^z$ with $\tau^a = \frac{q^a}{\lambda v^a}$

$$\tau^z = \begin{cases} \left(\dfrac{(1+c)z}{a} + b\right)^{c/(c+1)} b^{1/(c+1)} c^{-1} - b/c & c \notin \{-1, 0\}, \\ b(1 - e^{-\frac{z}{ab}}) & c = -1, \\ b \ln\left(\dfrac{z}{ab} + 1\right) & c = 0. \end{cases}$$

Here

$$a = \lambda \bar{V}^2, \quad b = q^b/(\lambda \bar{V}^3), \quad c = (\bar{V}^1 - \bar{V}^2 - \bar{V}^3)/\bar{V}^3,$$

$$v^b = -(\bar{V}^1 - \bar{V}^2 - \bar{V}^3), \quad v^a = -(\bar{V}^4 - \bar{V}^5 - \bar{V}^6).$$

G. Ruan, and Zhu (2016)

Why HFT

- Many mathematical and statistical problems
- Statistical analysis: understanding and identifying characteristics/motivation of cancelation in LOB, identifying hidden liquidity (example: non-displayable orders)
- Control problems and MFGs for market making, optimal execution, and order placement

Several references on HFTs

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