ANALYSIS OF RANDOM MEASUREMENTS

COMMENTARY AND BIBLIOGRAPHY FOR THE IPAM SHORT COURSE

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Lecture 1. The sparse reconstruction problem and random matrices

Some of this material can be found in my unedited lecture notes [25].

The sparse reconstruction problem. The ever-growing literature on the sparse reconstruction problem is documented at the *Compressed Sensing* webpage at Rice:

http://www.dsp.ece.rice.edu/cs/

The restricted isometry condition. The proof that the Restricted Isometry Condition is sufficient for sparse recovery using convex programming is due to Candes and Tao [2]. A simpler proof, also due to Candes and Tao, is in [4].

Applications of restricted isometries to vector quantization and democratic coding are found in [10], and applications to computing the condition number of random matrices are found in [18].

Asymptotic and non-asymptotic theory of random matrices. The survey [23] contains references to asymptotic (size $\to \infty$) and non-asymptotic (size fixed) estimates on the singular values of random matrices.

Lecture 2. Upper and lower bounds for subgaussian matrices

Most of this material can be found in my unedited lecture notes [25]. See also the survey [5] of non-asymptotic results on random matrices. The theory is most developed for gaussian matrices, for which very precise upper and lower bounds are known [6].

Sums of independent random variables. The book [13] has a wealth of deviation inequalities. Bernstein's inequality for subexponential random variables mentioned in the lecture can be found e.g. in [24].

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Reconstruction from subgaussian matrices. A proof of the fact that subgaussian matrices are restricted isometries can be found in [12]. A more general result is in [11].

Dudley's inequality. The standard proof of Dudley's inequality is found in the expository paper [19] and in the first pages of the book [21]. The monograph [9] contains a more general version of Dudley's inequality (Section 11.1).

Sharp bounds for subgaussian matrices. The theorem that the singular numbers of subgaussian matrices with aspect ratio y lie within $[1-C\sqrt{y}, 1+C\sqrt{y}]$ can be derived from more general results: ([8] Theorem 1.4) with constant probability, and ([11] Theorem D) with exponentially large probability.

Lecture 3. Random Fourier measurements

Reconstruction from Fourier measurements. The *non-uniform* result is due to [1]. It states that, for a given n-sparse signal f, the random set of $N \sim n \log d$ frequencies is good with high probability: f can be reconstructed correctly from these frequencies using a convex program. The dependence of N on n is optimal.

The *uniform* result states that, with high probability, a random set of $N \sim n \log^4 d$ frequencies is good for *every* f (in the above sense). This estimate is proved in [16, 17]; a weaker estimate $N \sim n \log^6 d$ is in [3].

The uniform result is deduced (via Candes-Tao's Restricted Isometry Condition) from the theorem that random N rows of the $d \times d$ Discrete Fourier Transform matrix form a restricted isometry (for sparsity level n). Proving this for N below $n \log\log d$ should be hard because of the rlations with the Λ_1 conjecture [20] (selecting a proportion of characters that spans a Euclidean subspace of L^1). This connection is explained in [7].

Khinchine's inequality for scalars and operators. An exposition of the classical Khinchine's inequality can be found in Section 4.1 of [9]. The non-commutative version (for linear operators) is covered in the book [14]; see also the paper [15].

Rudelson's Selection Theorem. This was proved in [15] using the non-commutative Khinchine inequality. The result is restated in the language of frames in [22].

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