Nonlocal Conservation Laws in the Modeling of Vehicular and Pedestrian Traffic

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Mathematical Foundations of Traffic
IPAM, UCLA

NonLocal Conservation Laws

Conservation Laws
- Analytic Techniques

NonLocal Conservation Laws
- Examples
- Numerical Methods

Crowd Dynamics
- Crowd ÷ Agent Interaction
- Crowd ÷ Agent Interaction – Alternative Construction
- General Case

Conservation Laws

Conservation Laws - Analytic Theory

$$\partial_t u + \operatorname{div}_x f(t,x,u) = g(t,x,u)$$

x	$\in$	$\mathbb{R}^N$	space
t	$\in$	$\mathbb{R}^+$	time
u	$\in$	$\mathbb{R}^n$	unknown
f	is	smooth	flow ($n \times N$)
g	is	smooth	o.d.e.

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Scalar MultiD	Systems in 1D
$n = 1$ and $N \geq 1$	$n \geq 1$ and $N = 1$

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Existence

(Kružkov: Mat.Sb., 1970)

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Conservation Laws - Analytic Theories

$$\partial_t u + \mathrm{div}_x f(t,x,u) = g(t,x,u)$$

$x \in \mathbb{R}^N$ space
$t \in \mathbb{R}^+$ time
$u \in \mathbb{R}^n$ unknown
f is smooth flow ($n \times N$)
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Dependence from data

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Dependence from f, g

(Colombo, Mercier, Rosini: CMS, 2009)

Systems in 1D

$n \geq 1$ and $N = 1$

$$\partial_t u + \partial_x f(u) = 0$$

Existence

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Dependence from data

(Bressan & c.: 1995, 2000)

Dependence from f

(Bianchini, Colombo: PAMS, 2002)

Techniques – Wave Front Tracking $(N = 1 \quad n \geq 1)$

- Riemann Problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(0,x) = \begin{cases} u^l & x < 0 \\ u^r & x > 0 \end{cases} \end{cases}$$

(Lax: CPAM, 1957)

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- Riemann Solver

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- Riemann Solver
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- Approximate Riemann Solver
- Piecewise constant initial datum

  (Dafermos: JMAA, 1972)

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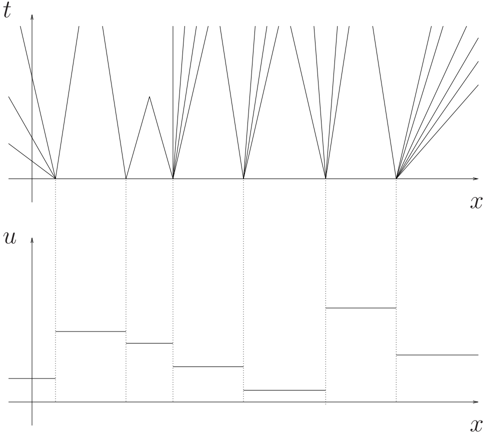

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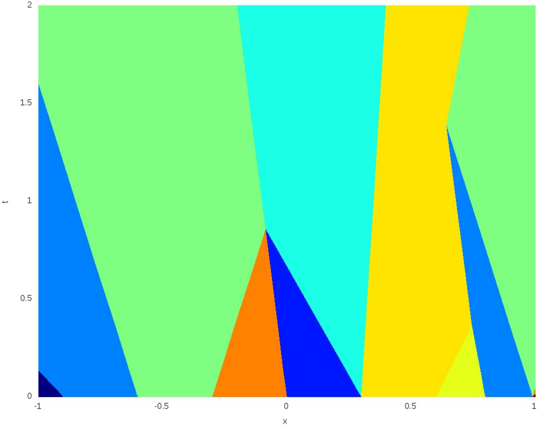

$\varepsilon = 1/4$

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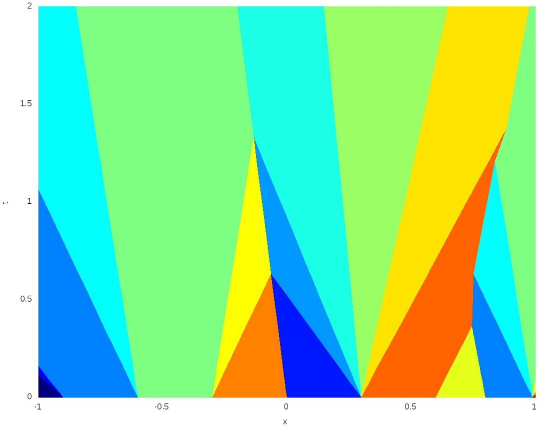

$\varepsilon = 1/8$

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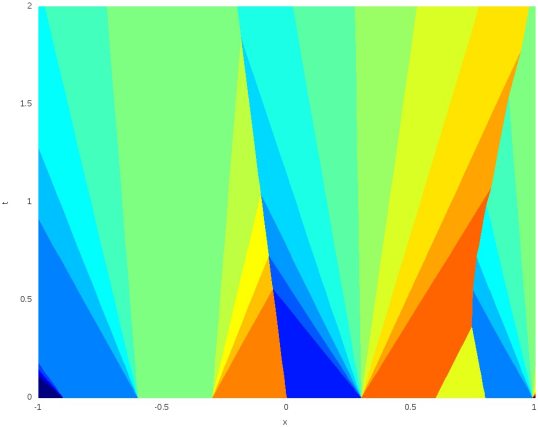

$\varepsilon = 1/16$

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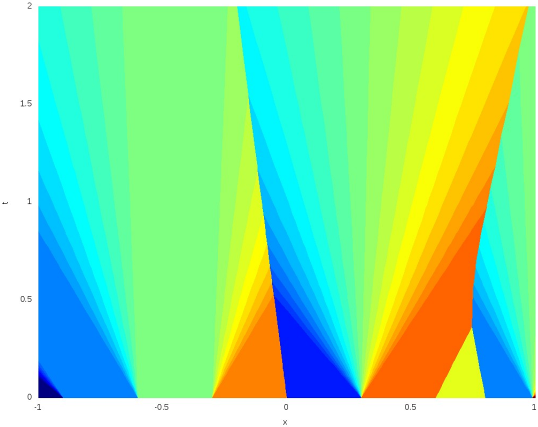

$\varepsilon = 1/32$

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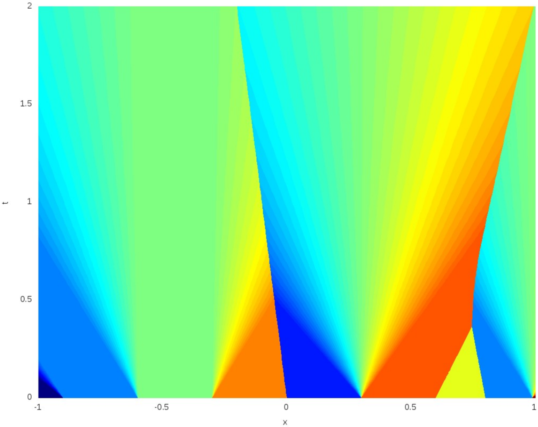

$\varepsilon = 1/64$

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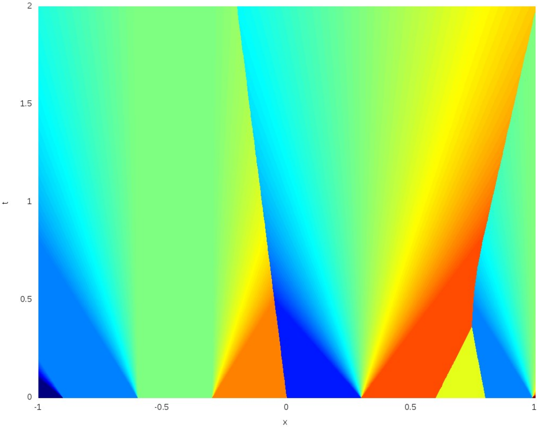

$\varepsilon = 1/128$

Techniques – Wave Front Tracking ($N = 1 \quad n \geq 1$)

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$\Rightarrow$ Existence of solutions

(Glimm: CPAM, 1965)

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$\Rightarrow$ Existence of solutions

$\Rightarrow$ Uniqueness + $\mathbf{L^1}$ Lipschitz dependence from u_o

(Bressan: ARMA, 1995)

(Bressan, Colombo: ARMA, 1995)

(Bressan, Crasta & Piccoli: Mem.AMS, 2000)

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$\Rightarrow$ $\mathbf{L^1}$ Lipschitz continuous dependence from f and g

(Bianchini & Colombo: PAMS, 2002)

Techniques – Kružkov Theory $(N \geq 1 \quad n = 1)$

1. Solve $\begin{cases} \partial_t u + \mathrm{div}_x f(t,x,u) = g(t,x,u) + \varepsilon \Delta u \\ u(0,x) = u_o(x) \end{cases}$

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(Colombo, Mercier & Rosini: CMS, 2009)

NonLocal Conservation Laws

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Measure Density

NonLocal Conservation Laws

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Measure	Density
Wasserstein distance	$\mathbf{L^1}$ distance

NonLocal Conservation Laws

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+ micro–macro	+ local–nonlocal

NonLocal Conservation Laws

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Measure	Density
Wasserstein distance	$\mathbf{L^1}$ distance
+ micro–macro	+ local–nonlocal
(Piccoli, Rossi: ARMA, 2014)	
(Piccoli, Tosin: ARMA, 2011)	

NonLocal Conservation Laws – g NonLocal

$$\begin{cases} \partial_t u + \operatorname{div} f(t,x,u) = g(t,x,u) & (t,x) \in \mathbb{R}^+ \times \Omega \\ u(t,y) = u_b(t,y) & (t,y) \in \mathbb{R}^+ \times \partial\Omega \\ u(0,x) = u_o(x) & x \in \Omega \end{cases}$$

$$t \in \mathbb{R}^+, \qquad x \in \Omega,\ \Omega \subseteq \mathbb{R}^N, \qquad u \in \mathbb{R}^n.$$

g NonLocal in t: $g(t,x,u) = \displaystyle\int_{t-T}^{t} G\left(\tau, x, u(\tau,x)\right) \mathrm{d}\tau$

$$\boxed{\boxed{\text{Memory effects}}}$$

(Christoforou: JHDE, 2007) $N = 1,\ n \geq 1$

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g NonLocal in x: $g(t,x,u) = \int_{\mathbb{R}^N} G\left(x, \xi, u(t,\xi)\right) \mathrm{d}\xi$

$$\boxed{\boxed{\text{Radiating Gas}}}$$

(Lattanzio, Marcati: JDE, 2003) $N = 1,\ n = 1$

(Colombo, Guerra: CPDE, 2007) $N = 1,\ n \geq 1$

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Vehicular Traffic

(Colombo, Herty, Mercier: COCV, 2011)	$N \geq 1,\ n = 1$
(Li, Li: NHM, 2011)	$N = 1,\ n = 1$
(Blandin, Goatin: Numer.Math., 2015)	$N = 1,\ n = 1$

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Crowd Dynamics

(Colombo, Mercier: Acta Math.Sc., 2011)	$N \geq 1$, $n \geq 1$
(Colombo, Garavello, Mercier: M3AS, 2012)	$N \geq 1$, $n = 1$
(Bellomo, Piccoli, Tosin: M3AS, 2012)	review
(Crippa, Mercier: NoDEA, 2012)	$N \geq 1$, $n = 1$
(Amadori, Goatin, Rosini: JMAA, 2014)	$N = 1$, $n = 1$

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Granular Matter

(Amadori, Shen: JHDE, 2012) $N = 1,\ n = 1$

(Guerra, Shen: JDE, 2014) $N = 1,\ n = 1$

NonLocal Conservation Laws – f NonLocal

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$$\boxed{\boxed{\text{Bolts – Conveyor Belts}}}$$

(Göttlich, Hoher, Schindler, Schleper: App.Mat.Mod., 2014) $N = 2,\ n = 1$

NonLocal Conservation Laws – f and g NonLocal

$$\begin{cases} \partial_t u + \operatorname{div} f(t,x,u) = g(t,x,u) & (t,x) \in \mathbb{R}^+ \times \Omega \\ u(t,y) = u_b(t,y) & (t,y) \in \mathbb{R}^+ \times \partial\Omega \\ u(0,x) = u_o(x) & x \in \Omega \end{cases}$$

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f and g NonLocal in x: $f(t,x,u) = \displaystyle\int_{\mathbb{R}^N} F\left(t,x,\xi,u(t,\xi)\right) \mathrm{d}\xi$

Laser Cutting

(Colombo, Marcellini: JDE, To appear) $N \geq 1,\ n = 2$

NonLocal Conservation Laws – u_b NonLocal

$$\begin{cases} \partial_t u + \operatorname{div} f(t,x,u) = g(t,x,u) & (t,x) \in \mathbb{R}^+ \times \Omega \\ u(t,y) = u_b(t,y) & (t,y) \in \mathbb{R}^+ \times \partial\Omega \\ u(0,x) = u_o(x) & x \in \Omega \end{cases}$$

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u_b NonLocal in x: $u_b(t,y) = \displaystyle\int_\Omega F\left(t,x,y,u(t,x)\right) \mathrm{d}x$

Structured Populations

(Perthame: Book, 2007)	$N = 1,\ n = 1$
(Carrillo, Gwiazda, Ulikowska: M3AS, 2014)	$N = 1,\ n = 1$
(Colombo, Garavello: MBE, 2015)	$N = 1,\ n \geq 1$

NonLocal Conservation Laws – Numerical Methods

$$\partial_t u + \operatorname{div} f\left(t, x, u(t,x), \left(u(t) * \eta\right)(x)\right) = g(t,x,u)$$

- Lax Friedrichs
- multiD ⇒ Dimensional Splitting
- source ⇒ Operator Splitting

1D	(Betancourt, Bürger, Karlsen, Tory: Nonlin., 2003)
	(Amorim, Colombo & Teixeira: ESAIM M2AN, 2015)
multiD	(Aggarwal, Colombo & Goatin: SINUM, 2015)
hyp $\oplus$ para	(Rossi, Schleper: ESAIM M2AN, To appear)

NonLocal Conservation Laws – Numerical Methods

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Computational cost!

Crowd Dynamics

Crowd Dynamics

Macroscopic

(Bellomo, Dogbé: M3AS, 2008)

(Degond, Appert-Rolland, Moussaid, Pettré, Theraulaz: J.Stat.Phys., 2013)

(Twarogowska, Goatin, Duvigneau: App.Math.Model., 2014)

(Andreianov, Donadello, Rosini: M3AS, 2014)

(Hoogendoorn, van Wageningen-Kessels, Daamen, Duives, Sarvib: Trasp.Res.Proc., 2015)

Crowd Dynamics

$$\begin{cases} v = \text{speed modulus} \\ \vec{v} = \text{velocity direction} \end{cases}$$

$$\partial_t \rho + \mathrm{div}\left[\rho\, v(\rho)\, \vec{v}(x)\right] = 0$$

(Kružkov: Mat.Sb., 1970)

Video clip

Crowd Dynamics – NonLocal

$$\boxed{\boxed{\partial_t \rho + \mathrm{div}_x \left(\rho \, v(\rho * \eta) \, \vec{v}(t,x)\right) = 0}}$$

Crowd Dynamics – NonLocal

Theorem (Colombo, Herty & Mercier: ESAIM COCV, 2011)

If $v, \vec{v}, \eta$ are sufficiently regular, then

$$\boxed{\boxed{\partial_t \rho + \operatorname{div}_x \left(\rho\, v(\rho * \eta)\, \vec{v}(t,x)\right) = 0}}$$

generates a semigroup

$$S \colon \mathbb{R} \times (\mathbf{L^1} \cap \mathbf{L^\infty} \cap \mathbf{BV})(\mathbb{R}^N; \mathbb{R}) \to (\mathbf{L^1} \cap \mathbf{L^\infty} \cap \mathbf{BV})(\mathbb{R}^N; \mathbb{R})$$

such that

Crowd Dynamics – NonLocal

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such that

✓ $t \to S_t \rho_o$ *is the solution with datum* ρ_o

✓ S_t *is* $\mathbf{L^1}$ *Lipschitz in the initial datum*

✓ S_t *is Gâteaux differentiable and* $\left(DS_t(\rho_o)\right)(r_o)$ *solves*

$$\begin{cases} \partial_t r + \mathrm{div}_x \Big(r\, V(\rho) + \rho\, \left(DV(\rho)\right)(r) \Big) = 0 \\ r(0,x) = r_o(x) \end{cases} \quad \textit{where } \rho(t) = S_t \rho_o$$

Also for systems (Colombo, Mercier: Acta Math.Sc., 2011)

Crowd Dynamics – NonLocal

Each individual chooses an optimal path

Crowd Dynamics – NonLocal

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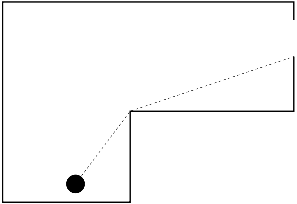


Hughes Model:

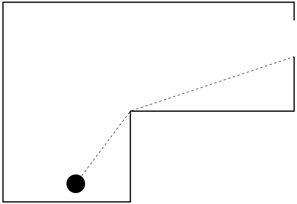
$$\vec{v} = -\operatorname{grad}_x \Phi$$

$$\begin{cases} \partial_t \rho + \operatorname{div}_x \left(\rho\, v(\rho)\, \operatorname{grad}_x \Phi\right) = 0 \\ \left\|\operatorname{grad}_x \Phi\right\|^2 = 1/v(\rho) \end{cases}$$

(Hughes: Transp.Res.B, 2002)

Crowd Dynamics – NonLocal

Each individual chooses an optimal path



Hughes Model Regularized

$$\vec{v} = -\operatorname{grad}_x \Phi$$

$$\begin{cases} \partial_t \rho + \operatorname{div}_x \left(\rho\, v(\rho)\, \operatorname{grad}_x \Phi\right) = 0 \\ \left\|\operatorname{grad}_x \Phi\right\|^2 = 1 \Big/ \left(v(\rho) + \varepsilon\right) + \varepsilon\, \Delta\Phi \end{cases}$$

In 1D: (Di Francesco, Markowich, Pietschmann, Wolfram: JDE, 2010)

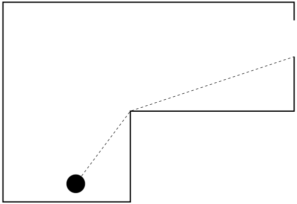
(Amadori & Di Francesco: Acta Math.Sc., 2011)

(El-Khatib, Goatin & Rosini: ZAMP, 2012)

(Goatin & Mimault: SIAM J.Sci.Comput., 2012)

Crowd Dynamics – NonLocal

Each individual chooses an optimal path



Hughes Model Regularized

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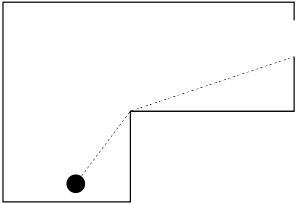
$$\begin{cases} \partial_t \rho + \operatorname{div}_x \left(\rho\, v(\rho)\, \operatorname{grad}_x \Phi\right) = 0 \\ \left\|\operatorname{grad}_x \Phi\right\|^2 = 1 \Big/ \left(v(\rho) + \varepsilon\right) + \varepsilon\, \Delta\Phi \end{cases}$$

In 2D: (Colombo, Garavello, Lécureux-Mercier: M3AS, 2012)

Crowd Dynamics – NonLocal

Each individual chooses an optimal path



Hughes Model Modified

$$\vec{v} = -\operatorname{grad}_x \Phi$$

$$\begin{cases} \partial_t \rho + \operatorname{div}_x \left(\rho\, v(\rho)\, \operatorname{grad}_x \Phi\right) = 0 \\ \left\|\operatorname{grad}_x \Phi\right\|^2 = 1 + \varepsilon\, \Delta\Phi \end{cases}$$

In 2D: (Colombo, Gokieli, Rosini: Work in progress, 2015?)

Crowd Dynamics – NonLocal

$$\partial_t \rho + \mathrm{div}_x \left(\rho\, v(\rho) \left(\vec{v}(x) \qquad + \qquad \text{deviation} \qquad \right) \right) = 0$$

Crowd Dynamics – NonLocal

$$\partial_t \rho + \mathrm{div}_x \left(\rho\, v(\rho) \left(\vec{v}(x) \qquad + \qquad \begin{array}{c} \text{avoid} \\ \text{high} \\ \text{density} \end{array} \qquad \right) \right) = 0$$

Crowd Dynamics – NonLocal

$$\partial_t \rho + \operatorname{div}_x \left(\rho\, v(\rho) \left(\vec{v}(x) - \frac{\kappa\ \operatorname{grad}_x (\rho * \eta)}{\sqrt{1 + \left\| \operatorname{grad}_x (\rho * \eta) \right\|^2}} \right) \right) = 0$$

Crowd Dynamics – NonLocal

$$\partial_t \rho + \mathrm{div}_x \left(\rho \, v(\rho) \left(\vec{v}(x) - \frac{\kappa \ \mathrm{grad}_x (\rho * \eta)}{\sqrt{1 + \left\| \mathrm{grad}_x (\rho * \eta) \right\|^2}} \right) \right) = 0$$

If: v is smooth, non decreasing, $v(0) = V$, $v(\rho) = 0$;
$\vec{v}$ is smooth;
η is smooth with compact support;

Crowd Dynamics – NonLocal

$$\partial_t \rho + \mathrm{div}_x \left(\rho\, v(\rho) \left(\vec{v}(x) - \frac{\kappa\ \mathrm{grad}_x\,(\rho * \eta)}{\sqrt{1 + \left\| \mathrm{grad}_x\,(\rho * \eta) \right\|^2}} \right) \right) = 0$$

If:
- v is smooth, non decreasing, $v(0) = V$, $v(\rho) = 0$;
- $\vec{v}$ is smooth;
- η is smooth with compact support;

Then:
- ✓ Existence, Uniqueness in $\mathbf{L^1}$, with $\rho \in [0, 1]$
- ✓ Lipschitz Continuity from Data and Equation
- ✓ Viability (discomfort)
- ✓ Lanes formation

Crowd Dynamics – NonLocal

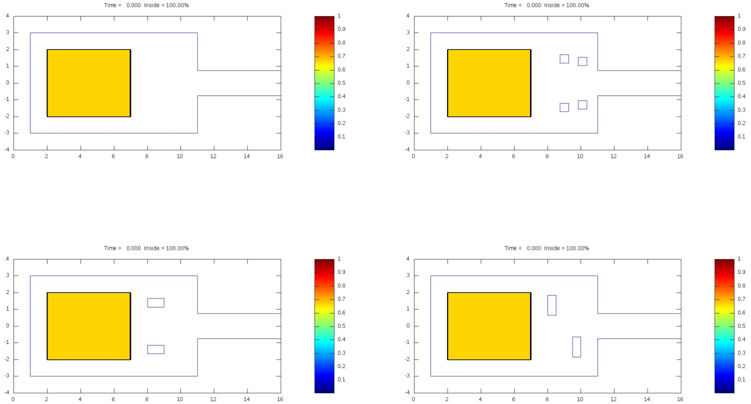
$$\partial_t \rho + \operatorname{div}_x \left(\rho\, v(\rho) \left(\vec{v}(x) - \frac{\kappa\ \operatorname{grad}_x (\rho * \eta)}{\sqrt{1 + \left\| \operatorname{grad}_x (\rho * \eta) \right\|^2}} \right) \right) = 0$$

If: v is smooth, non decreasing, $v(0) = V$, $v(\rho) = 0$;
$\vec{v}$ is smooth;
η is smooth with compact support;

Then:
- ✓ Existence, Uniqueness in $\mathbf{L^1}$, with $\rho \in [0, 1]$
- ✓ Lipschitz Continuity from Data and Equation
- ✓ Viability (discomfort)
- ✓ Lanes formation

(Colombo, Garavello, Lécureux-Mercier: M3AS, 2012)
Systems (Colombo & Mercier: Acta Math.Sc., 2011)
Bolts (Göttlich, Hoher, Schindler, Schleper: App.Mat.Mod., 2014)

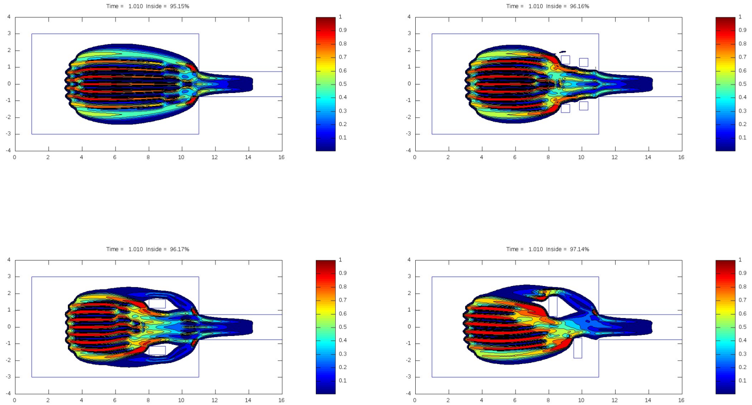
Crowd Dynamics – NonLocal



$t = 0.000$

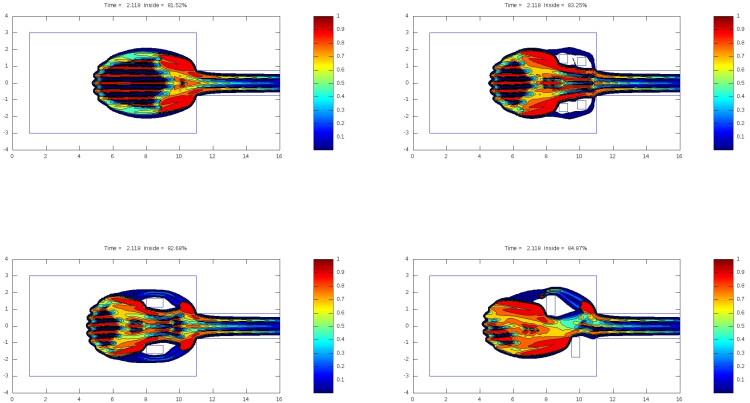
RMColombo

Crowd Dynamics – NonLocal



$t = 1.010$

Crowd Dynamics – NonLocal


Time = 2.118 Inside = 81.52%

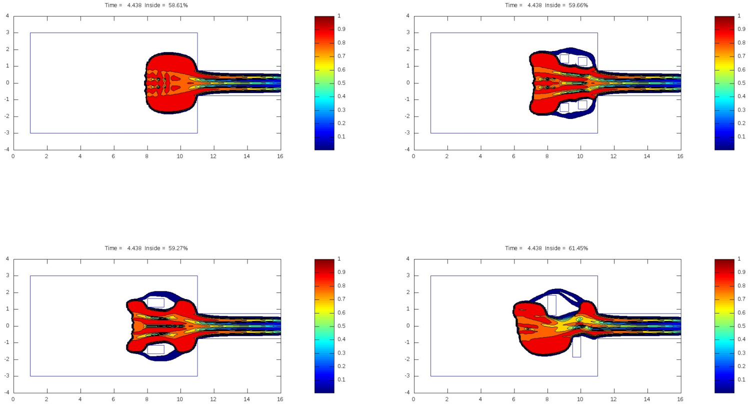
Time = 2.118 Inside = 83.25%

Time = 2.118 Inside = 82.68%

Time = 2.118 Inside = 84.87%

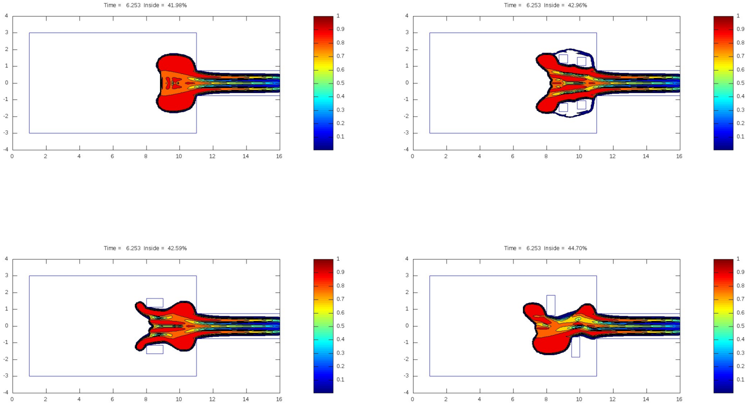
$$t = 2.118$$

Crowd Dynamics – NonLocal



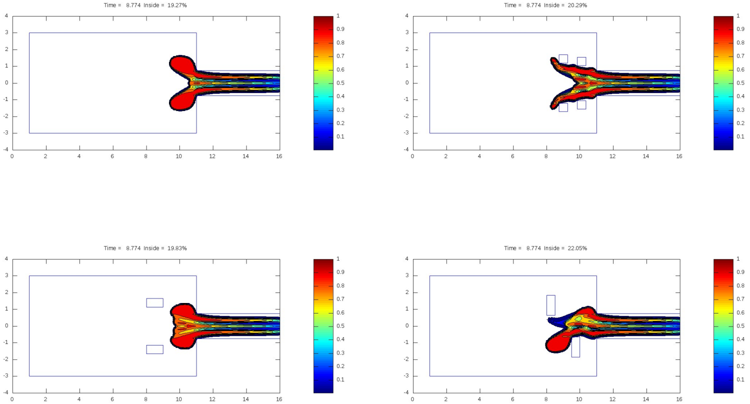
$t = 4.438$

Crowd Dynamics – NonLocal



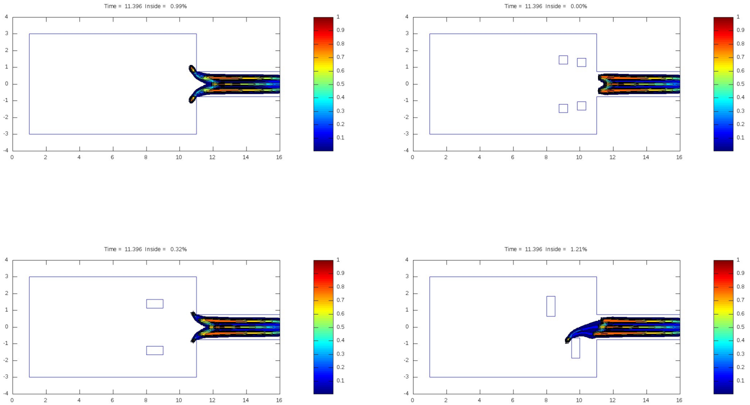
$t = 6.253$

Crowd Dynamics – NonLocal



$t = 8.774$

Crowd Dynamics – NonLocal – Braess Paradox



$t = 11.396$

Crowd Dynamics – NonLocal – Lanes' Formation

$$\partial_t \rho^1 + \operatorname{div}\left[\rho^1\, v(\rho^1)\left(\vec{v}^1(x) - \frac{\varepsilon_{11}\,\nabla(\rho^1 * \eta)}{\sqrt{1+\left\|\nabla(\rho^1*\eta)\right\|^2}} - \frac{\varepsilon_{12}\,\nabla(\rho^2 * \eta)}{\sqrt{1+\left\|\nabla(\rho^2*\eta)\right\|^2}}\right)\right] = 0$$

$$\partial_t \rho^2 + \operatorname{div}\left[\rho^2\, v(\rho^2)\left(\vec{v}^2(x) - \frac{\varepsilon_{21}\,\nabla(\rho^1 * \eta)}{\sqrt{1+\left\|\nabla(\rho^1*\eta)\right\|^2}} - \frac{\varepsilon_{22}\,\nabla(\rho^2 * \eta)}{\sqrt{1+\left\|\nabla(\rho^2*\eta)\right\|^2}}\right)\right] = 0$$

$$\vec{v}^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \delta \qquad \eta(x,y) = \left[1-(2x)^2\right]^3 \left[1-(2y)^2\right]^3 \chi_{[-0.5,0.5]^2}(x,y)$$

$$\vec{v}^2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \delta \qquad v(\rho) = 4\,(1-\rho) \qquad \begin{array}{rclrcl} \varepsilon_{11} & = & 0.3 & \varepsilon_{12} & = & 0.7 \\ \varepsilon_{21} & = & 0.7 & \varepsilon_{22} & = & 0.3 \end{array}$$

Crowd Dynamics – NonLocal – Lanes' Formation

Two populations moving in opposite directions

Crowd $\div$ Agent Interaction

Crowd $\div$ Agent Interaction – A Shepherd Dog

ρ = density of sheep p = position of the shepherd dog

$$\left\{\begin{array}{ll} \partial_t \rho + \operatorname{div}_x \big(\rho\, v(x, p, \rho)\big) = 0 & \text{sheep} \\ \dot p = \varphi(t) & \text{shepherd dog} \end{array}\right.$$

Crowd ÷ Agent Interaction – A Shepherd Dog

$\rho =$ density of sheep $\quad p =$ position of the shepherd dog

$$\begin{cases} \partial_t \rho + \mathrm{div}_x \left(\rho\, v(x, p, \rho)\right) = 0 & \text{sheep} \\ \dot{p} = \varphi(t) & \text{shepherd dog} \end{cases}$$

Kružkov Theorem ⇒ Existence of solutions

Crowd $\div$ Agent Interaction – A Shepherd Dog

ρ = density of sheep p = position of the shepherd dog

$$\begin{cases} \partial_t \rho + \operatorname{div}_x \left(\rho\, v(x,p,\rho)\right) = 0 & \text{sheep} \\ \dot p = \varphi(t) & \text{shepherd dog} \end{cases}$$

For example: $v(x,p,\rho) = V \cdot \left(1 - \frac{\rho}{\rho_{\max}}\right) \cdot \left(P(x,p) + R(x,p)\right)$

Crowd ÷ Agent Interaction – A Shepherd Dog

$\rho =$ density of sheep $\quad p =$ position of the shepherd dog

$$\begin{cases} \partial_t \rho + \operatorname{div}_x \left(\rho \, v(x,p,\rho)\right) = 0 & \text{sheep} \\ \dot p = \varphi(t) & \text{shepherd dog} \end{cases}$$

For example: $v(x,p,\rho) = V \cdot \left(1 - \frac{\rho}{\rho_{\max}}\right) \cdot \left(P(x,p) + R(x,p)\right)$

$$\begin{array}{ll} P(x,p) = x / \sqrt{1 + \|x\|^2} & \text{Propagation} \\ R(x,p) = -(p-x) \exp\left(-\|p-x\|\right) & \text{Repulsion} \end{array}$$

Crowd ÷ Agent Interaction – Confinement problem

$\rho =$ density of sheep $\quad p =$ position of the shepherd dog

$$\begin{cases} \partial_t \rho + \operatorname{div}_x \left(\rho\, v(x,p,\rho)\right) = 0 & \text{sheep} \\ \dot p = \varphi(t) & \text{shepherd dog} \end{cases}$$

For example: $v(x,p,\rho) = V \cdot \left(1 - \frac{\rho}{\rho_{\max}}\right) \cdot \left(P(x,p) + R(x,p)\right)$

$$P(x,p) = x \Big/ \sqrt{1 + \|x\|^2} \qquad \text{Propagation}$$

$$R(x,p) = -(p-x)\exp\left(-\|p-x\|\right) \qquad \text{Repulsion}$$

Given:	$B(0,r)$	the allowed area
	$\rho_o \left(\operatorname{spt}\rho_o \subseteq B(0,1)\right)$	initial sheep distribution
	T	deadline
Find:	φ	dog's trajectory
So that:	$\int_{\|x\|>r} \rho(T,x)\,\mathrm{d}x = 0$	for all $t \in [0,T]$

Crowd $\div$ Agent Interaction – A Good Shepherd Dog

$\rho =$ density of sheep $\quad p =$ position of the shepherd dog

$$\begin{cases} \partial_t \rho + \mathrm{div}_x \left(\rho\, v(x, p, \rho)\right) = 0 & \text{sheeps} \\ \dot p = \text{depends on } \rho\,! & \text{shepherd dog} \end{cases}$$

Crowd ÷ Agent Interaction – A Good Shepherd Dog

$\rho =$ density of sheep $\quad p =$ position of the shepherd dog

$$\begin{cases} \partial_t \rho + \mathrm{div}_x \left(\rho \, v(x,p,\rho)\right) = 0 & \text{sheeps} \\ \dot p = \varphi\left(p, \rho(t,p)\right) & \text{shepherd dog} \end{cases}$$

Crowd ÷ Agent Interaction – A Good Shepherd Dog

$\rho =$ density of sheep $\quad$ $p =$ position of the shepherd dog

$$\begin{cases} \partial_t \rho + \operatorname{div}_x \big(\rho\, v(x, p, \rho)\big) = 0 & \text{sheeps} \\ \dot p = \varphi\Big(p, \big(\rho(t) *_x \eta\big)(p)\Big) & \text{shepherd dog} \end{cases}$$

Crowd $\div$ Agent Interaction – A Good Shepherd Dog

$\rho =$ density of sheep $\quad p =$ position of the shepherd dog

$$\begin{cases} \partial_t \rho + \mathrm{div}_x \left(\rho\, v(x, p, \rho)\right) = 0 & \text{sheeps} \\ \dot p = \varphi\left(p, \left(\rho(t) *_x \eta\right)(p)\right) & \text{shepherd dog} \end{cases}$$

For example: $v \;=\; V \cdot \left(1 - \frac{\rho}{\rho_{\mathrm{max}}}\right) \cdot \left(P(x,p) + R(x,p)\right)$

Crowd $\div$ Agent Interaction – A Good Shepherd Dog

$\rho =$ density of sheep $\quad$ $p =$ position of the shepherd dog

$$\begin{cases} \partial_t \rho + \operatorname{div}_x \left(\rho\, v(x, p, \rho)\right) = 0 & \text{sheeps} \\ \dot p = \varphi\left(p, \left(\rho(t) *_x \eta\right)(p)\right) & \text{shepherd dog} \end{cases}$$

For example: $v = V \cdot \left(1 - \frac{\rho}{\rho_{\max}}\right) \cdot \left(P(x,p) + R(x,p)\right)$

$\varphi =$ orthogonal to $(\nabla\rho)(t, p)$

Crowd ÷ Agent Interaction – A Good Shepherd Dog

$\rho =$ density of sheep $\quad p =$ position of the shepherd dog

$$\begin{cases} \partial_t \rho + \operatorname{div}_x \left(\rho\, v(x, p, \rho)\right) = 0 & \text{sheeps} \\ \dot p = \varphi\left(p, \left(\rho(t) *_x \eta\right)(p)\right) & \text{shepherd dog} \end{cases}$$

For example:

$$\begin{aligned} v &= V \cdot \left(1 - \frac{\rho}{\rho_{\max}}\right) \cdot \left(P(x,p) + R(x,p)\right) \\ \varphi &= \text{orthogonal to } \left(\nabla \left(\rho(t) *_x \eta\right)\right)(p) \end{aligned}$$

Crowd ÷ Agent Interaction – A Good Shepherd Dog

$\rho =$ density of sheep $\quad p =$ position of the shepherd dog

$$\begin{cases} \partial_t \rho + \operatorname{div}_x \left(\rho\, v(x, p, \rho)\right) = 0 & \text{sheeps} \\ \dot p = \varphi\left(p, \left(\rho(t) *_x \nabla\eta\right)(p)\right) & \text{shepherd dog} \end{cases}$$

For example:
$$\begin{aligned} v &= V \cdot \left(1 - \frac{\rho}{\rho_{\max}}\right) \cdot \left(P(x, p) + R(x, p)\right) \\ \varphi &= \hat k \wedge \left(\left(\rho(t) *_x \nabla\eta\right)(p)\right) \end{aligned}$$

Crowd ÷ Agent Interaction – A Good Shepherd Dog

$\rho =$ density of sheep $p =$ position of the shepherd dog

$$\begin{cases} \partial_t \rho + \operatorname{div}_x \left(\rho\, v(x,p,\rho)\right) = 0 & \text{sheeps} \\ \dot p = \varphi \left(p, \left(\rho(t) *_x \nabla\eta\right)(p)\right) & \text{shepherd dog} \end{cases}$$

For example:
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Kružkov Theorem does not apply.

Crowd ÷ Agent Interaction – A Good Shepherd Dog

Given: $$\left\{\begin{array}{ll} \partial_t \rho + \operatorname{div}_x \left(\rho\, v(x, \rho, p)\right) = 0 & \text{HCL} \\ \dot p = \varphi\left(t, p, \left(\rho(t) * \eta\right)(p)\right) & \text{ODE} \end{array}\right.$$

Crowd $\div$ Agent Interaction – A Good Shepherd Dog

Given:

$$\begin{cases} \partial_t \rho + \operatorname{div}_x \left(\rho\, v(x, \rho, p)\right) = 0 & \text{HCL} \\ \dot p = \varphi\left(t, p, \left(\rho(t) * \eta\right)(p)\right) & \text{ODE} \end{cases}$$

IF: $v \in \mathbf{C^2}([0,R] \times \mathbb{R}^N \times \mathbb{R}^N; \mathbb{R}^N)$ is such that …

$\eta \in \mathbf{C^1_c}(\mathbb{R}^N; \mathbb{R})$

φ Caratheodory, Locally Lip. and sublinear

Crowd $\div$ Agent Interaction – A Good Shepherd Dog

Given:
$$\begin{cases} \partial_t \rho + \operatorname{div}_x \left(\rho\, v(x,\rho,p)\right) = 0 & \text{HCL} \\ \dot p = \varphi\left(t, p, \left(\rho(t) * \eta\right)(p)\right) & \text{ODE} \end{cases}$$

IF: $v \in \mathbf{C^2}([0,R] \times \mathbb{R}^N \times \mathbb{R}^N; \mathbb{R}^N)$ is such that …
$\eta \in \mathbf{C^1_c}(\mathbb{R}^N; \mathbb{R})$
φ Caratheodory, Locally Lip. and sublinear

Then: There exists a solution (u, w), with

- ✓ $\rho = \rho(t,x)$ weak entropy solution to HCL
- ✓ $p = p(t)$ Caratheodory solution to ODE
- ✓ stability estimates

$$\begin{aligned} &\left\|(\rho_1 - \rho_2)(t)\right\|_{\mathbf{L^1}} + \left\|(p_1 - p_2)(t)\right\| \\ \leq\ & C(t) \cdot \Big(\left\|\partial_\rho (v_1 - v_2)\right\|_{\mathbf{L^\infty}} + \left\|\operatorname{div}(v_1 - v_2)\right\|_{\mathbf{L^1}} \\ &\quad + \|\varphi_1 - \varphi_2\|_{\mathbf{L^\infty}} + \|\eta_1 - \eta_2\|_{\mathbf{L^1}} \\ &\quad + \|\bar\rho_1 - \bar\rho_2\|_{\mathbf{L^1}} + \|\bar p_1 - \bar p_2\| \Big) \end{aligned}$$

Crowd $\div$ Agent Interaction – A Good Shepherd Dog

Assumptions on v:

(v.1) $v(0,x,p) = v(R,x,p) = 0\ \forall (x,p)$

(v.2) $\rho \mapsto v(\rho,x,p) \in \mathbf{L}^\infty$, $\forall (x,p)$

$(\rho,x) \mapsto \partial_\rho v(\rho,x,p) \in \mathbf{L}^\infty$, $\forall p$

$(\rho,x) \mapsto \partial_\rho \nabla_x v(\rho,x,p) \in \mathbf{L}^\infty$, $\forall p$;

(v.3) $(\rho,p) \mapsto \int_{\mathbb{R}^N} \left\| \nabla_x \operatorname{div}_x v(\rho,x,p) \right\| \mathrm{d}x \in \mathbf{L}^\infty_{\mathbf{loc}}$

(v.4) $(\rho,p) \mapsto \int_{\mathbb{R}^N} \left\| \operatorname{div}_x v(\rho,x,p) \right\| \mathrm{d}x \in \mathbf{L}^\infty_{\mathbf{loc}}$

(v.5) $(\rho,p) \mapsto \int_{\mathbb{R}^N} \left\| \nabla_p \operatorname{div}_x v(\rho,x,p) \right\| \mathrm{d}x \in \mathbf{L}^\infty_{\mathbf{loc}}$

(v.6) $x \mapsto \nabla_p \partial_\rho v \in \mathbf{L}^\infty$, $\forall (\rho,p)$

(v.7) $\int_{\mathbb{R}^N} \left\| v(\cdot,x,\cdot) \right\|_{\mathbf{L}^\infty} \mathrm{d}x < \infty$, $\forall K \subset \mathbb{R}^N$, K compact

(Colombo, Mercier: JNLS, 2012)

Crowd $\div$ Agent Interaction – A Good Shepherd Dog

Video clip

(Colombo, Mercier: JNLS, 2012)

Crowd ÷ Agent Interaction – A Good Shepherd Dog

Video clip

(Colombo, Mercier: JNLS, 2012)

Crowd $\div$ Agent Interaction – Alternative Construction

Shepherd Dog

Shepherd Dog without p.d.e.s

Shepherd Dog without p.d.e.s

A Sheep:

Shepherd Dog without p.d.e.s

A Sheep: $x = x(t)$

Shepherd Dog without p.d.e.s

A Sheep: $x = x(t)$

A Wandering Sheep:

Shepherd Dog without p.d.e.s

A Sheep:	$x = x(t)$
A Wandering Sheep:	$\dot{x} \in B(0, c)$

Shepherd Dog without p.d.e.s

A Sheep: $x = x(t)$

A Wandering Sheep: $\dot{x} \in B(0, c)$

A Shepherd Dog:

Shepherd Dog without p.d.e.s

A Sheep: $x = x(t)$
A Wandering Sheep: $\dot{x} \in B(0, c)$
A Shepherd Dog: $\xi = \xi(t)$

Shepherd Dog without p.d.e.s

A Sheep:	$x = x(t)$
A Wandering Sheep:	$\dot x \in B(0, c)$
A Shepherd Dog:	$\xi = \xi(t)$

Sheep & Dog:

Shepherd Dog without p.d.e.s

A Sheep: $x = x(t)$
A Wandering Sheep: $\dot x \in B(0, c)$
A Shepherd Dog: $\xi = \xi(t)$

Sheep & Dog: $\dot x = v\left(\xi(t), x\right) + B(0, c)$

Shepherd Dog without p.d.e.s

A Sheep: $x = x(t)$
A Wandering Sheep: $\dot{x} \in B(0, c)$
A Shepherd Dog: $\xi = \xi(t)$

Sheep & Dog: $\dot{x} = v\left(\xi(t), x\right) + B(0, c)$

Differential Inclusions

(Aubin, Cellina: Differential Inclusions, 1984)

Shepherd Dog without p.d.e.s

A Sheep: $x = x(t)$
A Wandering Sheep: $\dot{x} \in B(0, c)$
A Shepherd Dog: $\xi = \xi(t)$

Sheep & Dog: $\dot{x} = v\left(\xi(t), x\right) + B(0, c)$

Differential Inclusions

(Aubin, Cellina: Differential Inclusions, 1984)

CONFINEMENT

(Bressan, Zhang: SetVal.Var.An., 2012)
(Colombo, Pogodaev: SIAD, 2012)
(Colombo, Lorenz & Pogodaev: DCDS, 2015)

Shepherd Dogs – Differential Inclusions

Film

Successful

Shepherd Dogs – Differential Inclusions

Film

NOT Successful

Shepherd Dogs – Differential Inclusions

Film

Movement

General Case

General Case

$$\left\{\begin{array}{ll} \partial_t \rho_i + \mathrm{div}\left[q^i(\rho^i)\; v^i\left(t, x,\; \left(\mathcal{A}^i\left(\rho(t)\right)\right)(x), p\right)\right] = 0 & i = 1, \ldots, n \\ \dot{p} = F\left(t, p,\; \left(\mathcal{B}\left(\rho(t)\right)\right)(p)\right) & p \in \mathbb{R}^m \end{array}\right.$$

t	time
x	plane/space coordinate
$\rho^1, \ldots, \rho^n$	various populations in the crowd
p	agents' positions (and speeds)
$q^1, \ldots, q^n$	effects of density on speed
$v^1, \ldots, v^n$	velocities
F	agents' velocities

General Case

$$\begin{cases} \partial_t \rho_i + \mathrm{div}\left[q^i(\rho^i)\; v^i\left(t, x, \left(\mathcal{A}^i\left(\rho(t)\right)\right)(x), p\right)\right] = 0 & i = 1, \ldots, n \\ \dot{p} = F\left(t, p, \left(\mathcal{B}\left(\rho(t)\right)\right)(p)\right) & p \in \mathbb{R}^m \end{cases}$$

t	time
x	plane/space coordinate
$\rho^1, \ldots, \rho^n$	various populations in the crowd
p	agents' positions (and speeds)
$q^1, \ldots, q^n$	effects of density on speed
$v^1, \ldots, v^n$	velocities
F	agents' velocities
$\mathcal{A}^1, \ldots, \mathcal{A}^n$	NonLocal crowd÷crowd interaction (averages)
$\mathcal{B}^1, \ldots, \mathcal{B}^m$	NonLocal crowd÷agent interaction (averages)

General Case

$$\begin{cases} \partial_t \rho_i + \mathrm{div} \left[q^i(\rho^i)\; v^i\left(t, x, \left(\mathcal{A}^i\left(\rho(t)\right)\right)(x), p \right) \right] = 0 & i = 1, \ldots, n \\ \dot p = F\left(t, p, \left(\mathcal{B}\left(\rho(t)\right)\right)(p) \right) & p \in \mathbb{R}^m \end{cases}$$

General Case

$$\begin{cases} \partial_t \rho_i + \mathrm{div} \left[q^i(\rho^i)\, v^i \Big(t, x, \left(\mathcal{A}^i \left(\rho(t) \right) \right) (x), p \Big) \right] = 0 & i = 1, \ldots, n \\ \dot{p} = F \left(t, p, \left(\mathcal{B} \left(\rho(t) \right) \right) (p) \right) & p \in \mathbb{R}^m \end{cases}$$

- Follow the Leader
  [$m =$ (space dimension) × (n. of individuals)]

General Case

$$\begin{cases} \partial_t \rho_i + \mathrm{div} \left[q^i(\rho^i)\, v^i\left(t, x, \left(\mathcal{A}^i\left(\rho(t)\right)\right)(x), p \right)\right] = 0 & i = 1, \ldots, n \\ \dot{p} = F\left(t, p, \left(\mathcal{B}\left(\rho(t)\right)\right)(p)\right) & p \in \mathbb{R}^m \end{cases}$$

- Follow the Leader
- NonLocal model for a crowd $[n = 1]$

General Case

$$\begin{cases} \partial_t \rho_i + \mathrm{div} \left[q^i(\rho^i)\, v^i\left(t, x, \left(\mathcal{A}^i\left(\rho(t)\right)\right)(x), p\right)\right] = 0 & i = 1, \ldots, n \\ \dot p = F\left(t, p, \left(\mathcal{B}\left(\rho(t)\right)\right)(p)\right) & p \in \mathbb{R}^m \end{cases}$$

- Follow the Leader
- NonLocal model for a crowd
- Crowd ÷ Agent Interaction
  [$n = 1$ and $m =$ (space dimension) × individuals]

General Case

$$\begin{cases} \partial_t \rho_i + \mathrm{div} \left[q^i(\rho^i)\, v^i \left(t, x, \left(\mathcal{A}^i \left(\rho(t) \right) \right) (x), p \right) \right] = 0 & i = 1, \ldots, n \\ \dot{p} = F \left(t, p, \left(\mathcal{B} \left(\rho(t) \right) \right) (p) \right) & p \in \mathbb{R}^m \end{cases}$$

- Follow the Leader
- NonLocal model for a crowd
- Crowd ÷ Agent Interaction
- Crowd ÷ Crowd Interaction $[n > 1]$

General Case

$$\begin{cases} \partial_t \rho_i + \operatorname{div}\left[q^i(\rho^i)\, v^i\left(t, x, \left(\mathcal{A}^i\left(\rho(t)\right)\right)(x), p\right)\right] = 0 & i = 1, \ldots, n \\ \dot p = F\left(t, p, \left(\mathcal{B}\left(\rho(t)\right)\right)(p)\right) & p \in \mathbb{R}^m \end{cases}$$

- Follow the Leader
- NonLocal model for a crowd
- Crowd ÷ Agent Interaction
- Crowd ÷ Crowd Interaction
- Crowd ÷ Crowd ÷ Agent Interaction
  [$n > 1$ and $m =$ (space dimension) × (n. of individuals)]

General Case

$$\begin{cases} \partial_t \rho_i + \operatorname{div}\left[q^i(\rho^i)\; v^i\Big(t, x, \big(\mathcal{A}^i\left(\rho(t)\right)\big)(x), p\Big)\right] = 0 & i = 1, \ldots, n \\ \dot p = F\Big(t, p, \big(\mathcal{B}\left(\rho(t)\right)\big)(p)\Big) & p \in \mathbb{R}^m \end{cases}$$

- Follow the Leader
- NonLocal model for a crowd
- Crowd $\div$ Agent Interaction
- Crowd $\div$ Crowd Interaction
- Crowd $\div$ Crowd $\div$ Agent Interaction

Well posedness

(Borsche, Colombo, Garavello & Meurer: JNLS, 2015)

General Case ÷ Crowd ÷ Cars Interaction

$$\begin{cases} \partial_t \rho^i + \operatorname{div}\left[\rho^i(1-\rho^i) w^i(x,p)\left(V^i(x) - \mathcal{A}^i(\rho)\right)\right] = 0 & i = 1,2 \\ \dot p^k = g\left(\left(\mathcal{B}(\rho)\right)(p^k)\right)\ u(p^{k+1} - p^k) & k = 1,\ldots,m-1 \\ \dot p^m = u_L(t) \end{cases}$$

$w^i = w^i(x,p)$ i-th crowd speed,

$V^i = V^i(x)$ i-th crowd main direction

$\mathcal{A}^i = \mathcal{A}^i(\rho)$ changes of direction due to crowd interaction

$\mathcal{B} = \mathcal{B}(\rho)$ average total crowd density felt by drivers

$u = u(p^{k+1} - p^k)$ driver's speed

General Case – Crowd $\div$ Cars Interaction

Film

General Case – Hooligans

$$\left\{\begin{array}{l}\partial_t \rho_1 + \operatorname{div}\left[\rho^1(1-\rho^1)\left(w^1(x) + \mathcal{A}^1(\rho^1,\rho^3)\right)\right] = 0\\ \partial_t \rho_2 + \operatorname{div}\left[\rho^2(1-\rho^2)\left(w^2(x) + \mathcal{A}^2(\rho^1,\rho^2)\right)\right] = 0\end{array}\right.$$

w_1 upward; w^2 downward

$$\begin{array}{rcl}\mathcal{A}^1(\rho) & = & \frac{\varepsilon_{11}\ \eta*(\rho^1-\bar\rho)\ \nabla_x(\rho^1*\eta)}{\sqrt{1+\|\eta*(\rho^1-\bar\rho)\nabla_x(\rho^1*\eta)\|^2}} + \frac{\varepsilon_{12}\ \eta*(\rho^2-\rho^1)\nabla_x(\rho^2*\eta)}{\sqrt{1+\|\eta*(\rho^2-\rho^1)\nabla_x(\rho^2*\eta)\|^2}},\\ \mathcal{A}^2(\rho) & = & \frac{\varepsilon_{22}\ \eta*(\rho^2-\bar\rho)\ \nabla_x(\rho^2*\eta)}{\sqrt{1+\|\eta*(\rho^1-\bar\rho)\nabla_x(\rho^2*\eta)\|^2}} + \frac{\varepsilon_{21}\ \eta*(\rho^1-\rho^2)\nabla_x(\rho^1*\eta)}{\sqrt{1+\|\eta*(\rho^1-\rho^2)\nabla_x(\rho^1*\eta)\|^2}},\end{array}$$

General Case – Hooligans

Film 1

General Case – Hooligans

$$\left\{\begin{array}{ll}\partial_t \rho_i + \mathrm{div}\left[\rho^i(1-\rho^i)\left(-w^i(x,p)+\mathcal{A}^i(\rho)\right)\right] = 0 & i = 1,2\\ \dot{p}^k = I_k(p) + \mathcal{B}_k(\rho) & k = 1,\ldots,4\end{array}\right.$$

$$\begin{array}{rcl}\mathcal{A}^1(\rho) & = & \frac{\varepsilon_{11}\ \eta*(\rho^1-\bar\rho)\ \nabla_x(\rho^1*\eta)}{\sqrt{1+\|\eta*(\rho^1-\bar\rho)\nabla_x(\rho^1*\eta)\|^2}} + \frac{\varepsilon_{12}\ \eta*(\rho^2-\rho^1)\nabla_x(\rho^2*\eta)}{\sqrt{1+\|\eta*(\rho^2-\rho^1)\nabla_x(\rho^2*\eta)\|^2}},\\ \mathcal{A}^2(\rho) & = & \frac{\varepsilon_{22}\ \eta*(\rho^2-\bar\rho)\ \nabla_x(\rho^2*\eta)}{\sqrt{1+\|\eta*(\rho^1-\bar\rho)\nabla_x(\rho^2*\eta)\|^2}} + \frac{\varepsilon_{21}\ \eta*(\rho^1-\rho^2)\nabla_x(\rho^1*\eta)}{\sqrt{1+\|\eta*(\rho^1-\rho^2)\nabla_x(\rho^1*\eta)\|^2}},\\ \mathcal{B}_k(\rho)(p) & = & \varepsilon_1 \frac{\nabla_x((\bar\eta*\rho^1)(\bar\eta*\rho^2))(p^k)}{\sqrt{1+\|\nabla_x((\bar\eta*\rho^1)(\bar\eta*\rho^2))(p^k)\|^2}}\end{array}$$

General Case – Hooligans

Film 2

(Borsche, Colombo, Garavello & Meurer: JNLS, 2015)