A Boltzmann-type kinetic approach to the modeling of vehicular traffic

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Research outline

1. The spatially homogeneous model
   - Continuous velocity model
   - From continuous to discrete velocity model
   - Fundamental diagrams and the phase transition
   - Traffic safety
2. The spatially non-homogeneous model
   - Model on a single road
   - Model on road networks

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Basics of kinetic modeling

- Microscopic state of the vehicles: speed $v \in [0, 1]$
- Kinetic distribution function: $f = f(t, v)$ s.t.

$$f(t, v)\,dv = \text{fraction of vehicles with speed in } [v, v + dv] \text{ at time } t \geq 0$$

The Boltzmann-type kinetic equation

$$\partial_t f = Q(f, f) := \int_0^1 \int_0^1 \mathcal{P}(v_* \to v | v^*, \rho) f(t, v_*) f(t, v^*)\, dv_*\, dv^* - \rho f \quad (1)$$

- $\mathcal{P}(v_* \to v | v^*, \rho)$ probability distribution of speed transitions due to pairwise (binary) interactions among the vehicles:

$$\int_0^1 \mathcal{P}(v_* \to v | v^*, \rho)\, dv = 1 \qquad \forall\, v_*, v^*, \rho \in [0, 1] \quad (2)$$

- Mass conservation: $\rho(t) := \int_0^1 f(t, v)\, dv$ is constant, in fact from (1)-(2):

$$\frac{d}{dt} \int_0^1 f(t, v)\, dv = \int_0^1 Q(f, f)\, dv = 0$$

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Qualitative properties

- Let $\mathcal{P}(v_* \to \cdot|v^*,\, \rho) \in \mathscr{P}([0,\, 1])$ for all $v_*,\, v^*,\, \rho \in [0,\, 1]$. We assume

$$W_1(\mathcal{P}(v_* \to \cdot|v^*,\, \rho),\, \mathcal{P}(w_* \to \cdot|w^*,\, \varrho)) \leq \mathrm{Lip}(\mathcal{P})\left(|w_* - v_*| + |w^* - v^*| + |\varrho - \rho|\right),$$

where W_1 is the 1-Wasserstein metric for probability measures

Theorem (P. Freguglia, A. T., 2015 [4])

Fix $\rho \in [0,\, 1]$ and $f(0,\, v) =: f_0(v) \in \mathscr{M}_+^\rho([0,\, 1])$. There exists a unique $f \in C([0,\, +\infty);\, \mathscr{M}_+^\rho([0,\, 1]))$ which solves (1) in mild form:

$$f(t,\, v) = e^{-\rho t} f_0(v) + \int_0^t e^{\rho(s-t)} \int_0^1 \int_0^1 \mathcal{P}(v_* \to v|v^*,\, \rho) f(t,\, v_*) f(t,\, v^*)\, dv_*\, dv^*\, ds.$$

Given $f_{01},\, f_{02} \in \mathscr{M}_+^\rho([0,\, 1])$, the following continuous dependence estimate holds:

$$\sup_{t \in [0,\, T]} W_1(f_1(t),\, f_2(t)) \leq e^{2\max\{1,\, \mathrm{Lip}(\mathcal{P})\}T} W_1(f_{01},\, f_{02})$$

up to an arbitrarily large final time $T < +\infty$.

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Transition probabilities

- We consider the following probability distribution of speed transitions:

$$\mathcal{P}(v_* \to v | v^*, \rho) = \begin{cases} (1-P)\delta_{v_*}(v) + P\delta_{\min\{v_*+\Delta v,\, 1\}}(v) & \text{if } v_* \leq v^* \\ (1-P)\delta_{v^*}(v) + P\delta_{v_*}(v) & \text{if } v_* > v^* \end{cases} \tag{3}$$

where $0 < \Delta v < 1$ is given and

$$P = P(\rho) = 1 - \rho^\gamma \qquad (\gamma > 0)$$

is a probability of passing (cf. I. Prigogine, 1961)

- The time-asymptotic solution of (1), with transition probabilities (3), concentrates only on speeds which are multiples of Δv (G. Puppo, M. Semplice, A. T., G. Visconti, 2015 [6]) Picture

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Discrete velocity model

- Define a speed lattice

$$v_j = (j-1)\Delta v, \quad j = 1, \dots, n, \quad \Delta v = \frac{1}{n-1}$$

- Assume that $\mathcal{P}$ is a discrete probability distribution over v:

$$\mathcal{P}(v_* \to v | v^*, \rho) = \sum_{j=1}^{n} \mathcal{P}^j(v_*, v^*, \rho)\delta_{v_j}(v) \tag{4}$$

- Fix $\rho \in [0, 1]$ and take an initial condition of the form

$$f_0(v) = \sum_{j=1}^{n} f_j^0 \delta_{v_j}(v) \quad \text{with} \quad f_j^0 \geq 0, \ \sum_{j=1}^{n} f_j^0 = \rho \tag{5}$$

Theorem (P. Freguglia, A. T., 2015 [4])

The unique solution to (1) with transition probabilities (4) and initial condition (5) is $f(t, v) = \sum_{j=1}^{n} f_j(t)\delta_{v_j}(v)$, where the f_j's satisfy

$$\frac{df_j}{dt} = \sum_{h=1}^{n}\sum_{k=1}^{n} \mathcal{P}_{hk}^j(\rho) f_h f_k - \rho f_j, \qquad f_j(0) = f_j^0 \tag{6}$$

and $\mathcal{P}_{hk}^j(\rho) := \mathcal{P}^j(v_h, v_k, \rho)$.

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Discrete transition probabilities

- From (3) we deduce:

$$\mathcal{P}^j_{hk}(\rho) = \begin{cases} (1-P)\delta_{jh} + P\delta_{j,\min\{h+1,\,n\}} & \text{if } h \leq k \\ (1-P)\delta_{jk} + P\delta_{jh} & \text{if } h > k \end{cases} \tag{7}$$

which explicitly reads:

$$\mathcal{P}^j_{hk}(\rho) = \begin{cases} \begin{cases} 1-P & \text{if } j = h \\ P & \text{if } j = h+1 \\ 0 & \text{otherwise} \end{cases} & \text{if } h \leq k,\ h < n \\ \begin{cases} 1 & \text{if } j = h \\ 0 & \text{otherwise} \end{cases} & \text{if } h = k = n \\ \begin{cases} 1-P & \text{if } j = k \\ P & \text{if } j = h \\ 0 & \text{otherwise} \end{cases} & \text{if } h > k \end{cases}$$

- We recall that $P = P(\rho) = 1 - \rho^\gamma$ $(\gamma > 0)$

Asymptotic distributions

- We study the asymptotic speed distributions $f^\infty = \{f_j^\infty\}_{j=1}^n$ resulting from (6)-(7): $f_j^\infty = \lim_{t\to+\infty} f_j(t)$
- The f_j^∞'s form a one-parameter family, the parameter being the density ρ which is conserved

Theorem (L. Fermo, A. T., 2014 [2])

For every $n \geq 2$ and every $\rho \in [0, 1]$ there exists a unique stable and attractive equilibrium f^∞ of (6), which satisfies:

$$f_j^\infty \geq 0 \quad \forall\, j = 1, \dots, n, \qquad \sum_{j=1}^{n} f_j^\infty = \rho$$

- In more detail, setting $\rho_c := \left(\frac{1}{2}\right)^{\frac{1}{\gamma}}$,
  - for $\rho < \rho_c$ there exists only one stable and attractive equilibrium
  - for $\rho > \rho_c$ the previous equilibrium becomes unstable and a second stable and attractive one appears
- ρ_c is a critical value for equilibria inducing a supercritical bifurcation

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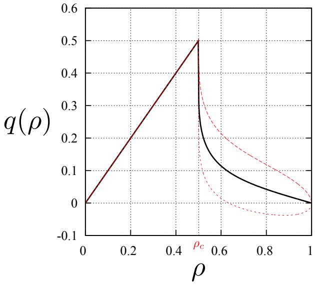
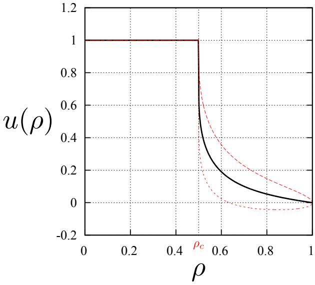
Fundamental and speed diagrams

- We compute the macroscopic flux q and the mean speed u at equilibrium:

$$q(\rho) := \sum_{j=1}^{n} v_j f_j^\infty(\rho), \qquad u(\rho) := \frac{q(\rho)}{\rho}$$

along with their standard deviations (dashed-red lines in the graphs below)





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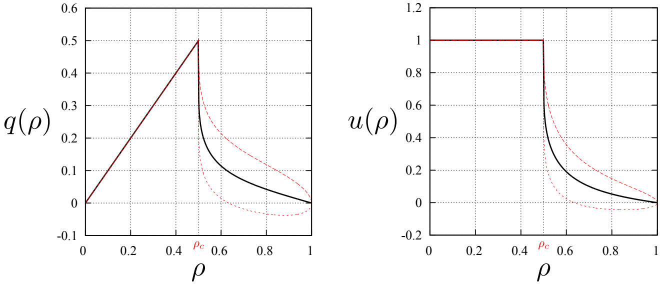

- Bifurcation $\rightsquigarrow$ phase transition, free ($\rho < \rho_c$) to congested flow ($\rho > \rho_c$)

Traffic as a mixture of different vehicles

- We consider two populations of vehicles, say cars (C) and trucks (T), with different microscopic characteristics
- Cars are shorter and faster while trucks are longer and slower
- Speed grids:

$$v_j = (j-1)\Delta v, \quad j = 1, \ldots, n^{\mathsf{p}}, \quad \mathsf{p} = \mathsf{C}, \mathsf{T}$$
$$\Delta v = \frac{1}{n^{\mathsf{C}} - 1}, \quad n^{\mathsf{T}} < n^{\mathsf{C}}$$

- Characteristic lengths: $\ell^{\mathsf{C}} = 1$, $\ell^{\mathsf{T}} > 1$
- Fraction of road occupancy:

$$s := \rho^{\mathsf{C}}\ell^{\mathsf{C}} + \rho^{\mathsf{T}}\ell^{\mathsf{T}},$$

the admissible pairs of densities $(\rho^{\mathsf{C}}, \rho^{\mathsf{T}}) \in [0, 1]^2$ being those s.t. $s \leq 1$

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$$\Delta v = \frac{1}{n^{\mathsf{C}} - 1}, \quad n^{\mathsf{T}} < n^{\mathsf{C}}$$

- Characteristic lengths: $\ell^{\mathsf{C}} = 1$, $\ell^{\mathsf{T}} > 1$
- Fraction of road occupancy:

$$s := \rho^{\mathsf{C}}\ell^{\mathsf{C}} + \rho^{\mathsf{T}}\ell^{\mathsf{T}},$$

the admissible pairs of densities $(\rho^{\mathsf{C}}, \rho^{\mathsf{T}}) \in [0, 1]^2$ being those s.t. $s \leq 1$

Traffic as a mixture of different vehicles

- We consider two populations of vehicles, say cars (C) and trucks (T), with different microscopic characteristics
- Cars are shorter and faster while trucks are longer and slower
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Multi-population model

- Model with self- and cross-interactions:

$$\frac{df_j^{\mathsf{p}}}{dt} = \underbrace{\sum_{h,k=1}^{n^{\mathsf{p}}} \mathcal{P}_{hk}^{\mathsf{p},j}(\rho) f_h^{\mathsf{p}} f_k^{\mathsf{p}}}_{\text{self-interactions}} + \underbrace{\sum_{h=1}^{n^{\mathsf{p}}} \sum_{k=1}^{n^{\mathsf{q}}} \mathcal{Q}_{hk}^{\mathsf{pq},j}(\rho) f_h^{\mathsf{p}} f_k^{\mathsf{q}}}_{\text{cross-interactions}} - (\rho^{\mathsf{C}} + \rho^{\mathsf{T}}) f_j^{\mathsf{p}} \qquad (\mathsf{q} := \neg\mathsf{p})$$

- In the transition probabilities $\mathcal{P}_{hk}^{\mathsf{p},j}$, $\mathcal{Q}_{hk}^{\mathsf{pq},j}$ the density ρ is replaced by the fraction of road occupancy s, i.e., the probability of passing is now

$$P = P(s) = 1 - s^{\gamma} \qquad (\gamma > 0)$$

Theorem (G. Puppo, M. Semplice, A. T., G. Visconti, 2015 [5])

If $n^{\mathsf{C}} = n^{\mathsf{T}}$ and $\ell^{\mathsf{C}} = \ell^{\mathsf{T}} = 1$ the total kinetic distribution function $f_j(t) := f_j^{\mathsf{C}}(t) + f_j^{\mathsf{T}}(t)$ solves the single-population model (6).

- $s_c := \left(\frac{1}{2}\right)^{\frac{1}{\gamma}}$ is again a critical value for equilibria. For $s = s_c$ the maximum flux is attained (G. Puppo, M. Semplice, A. T., G. Visconti, 2015 [5]) ▸ Picture

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Transition probabilities

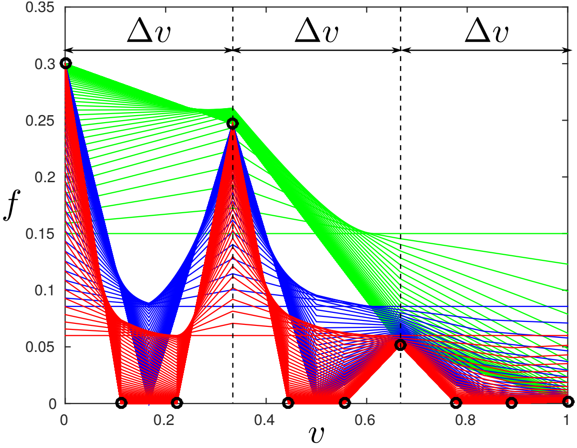

FIGURE: The asymptotic distribution function concentrates on multiples of Δv Back

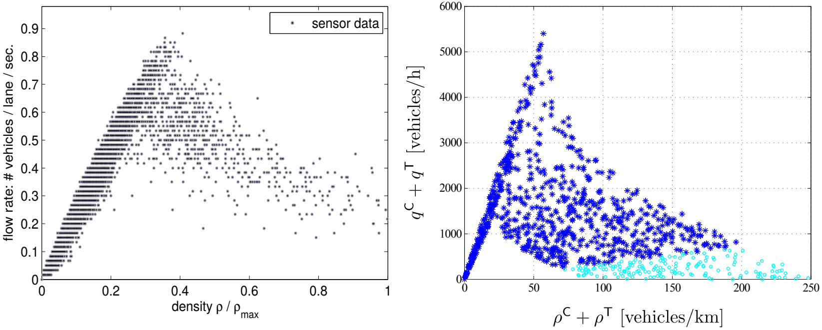

Figure: Left: fundamental diagram from experimental data (Minnesota Department of Transportation, 2003). Right: fundamental diagram from the multi-population model with $\gamma = 0.5$ Back