# Resurrection of the Payne-Whitham Pressure?

Benjamin Seibold (Temple University)

September $29^{\text{th}}$, 2015

## Collaborators and Students

Shumo Cui (Temple University)
Shimao Fan (Temple University & UIUC)
Louis Graup (Temple University)
Michael Herty (RWTH Aachen University)
Kathryn Lund (Temple University)
Rodolfo Ruben Rosales (MIT)

## Research Support

NSF CNS–1446690 . . . Control of vehicular traffic flow via low density autonomous vehicles

NSF DMS–1007899 . . . Phantom traffic jams, continuum modeling, and connections with detonation wave theory

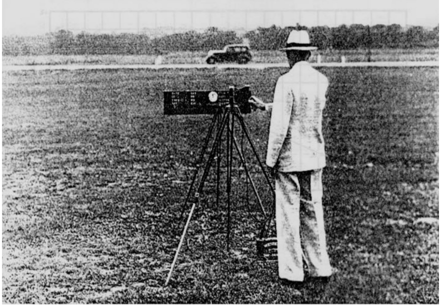
# Overview



TEMPLE UNIVERSITY

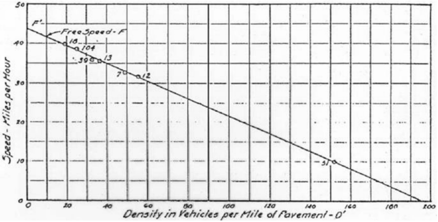
# Overview

## Bruce Greenshields collecting data (1933)



[This was only 25 years after the first Ford Model T (1908)]

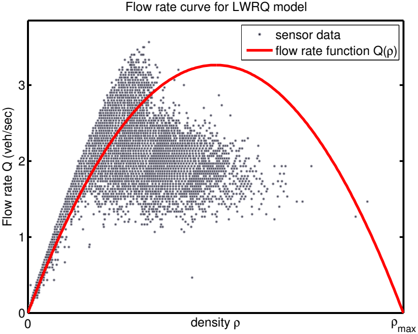
## Postulated density–velocity relationship




## Traffic flow theory

- Density $\rho$: #vehicles per unit length of road (at a fixed time)
- Flow rate $q$: #vehicles per unit time (passing a fixed position)
- Bulk velocity: $u = q/\rho$

## Contemporary measurements ($q$ vs. $\rho$)




[Fundamental Diagram of Traffic Flow]

## Continuum description

$\rho$ and $q$ aggregated over multiple lanes; position on road: $x$; time: $t$.

Number of vehicles between $a$ and $b$: $m(t) = \int_a^b \rho(x,t) \mathrm{d}x$

Traffic flow rate (= flux): $q = \rho u$

Change of number of vehicles equals inflow $q(a)$ minus outflow $q(b)$:

$$\int_a^b \rho_t \,\mathrm{d}x = \frac{d}{dt} m(t) = q(a) - q(b) = -\int_a^b q_x \,\mathrm{d}x$$

Equation holds for any choice of $a$ and $b$, thus

continuity equation $\rho_t + (\rho u)_x = 0$ (conservation of vehicles)

## First-order traffic models

Assume $u = U(\rho)$, and thus $q = \rho U(\rho) = Q(\rho)$ given by flow rate function.

Scalar hyperbolic conservation law.

## Second-order traffic models

Add a second equation, modeling vehicle acceleration, e.g.:

$u_t + u u_x = -\frac{p'(\rho)}{\rho} \rho_x + \frac{1}{\tau}(U(\rho) - u)$

$2 \times 2$ system of balance laws

## Lighthill-Whitham-Richards (LWR) model [Lighthill&Whitham: Proc. Roy. Soc. A 1955]

$$\rho_t + (\rho U(\rho))_x = 0$$

First-order model

## Payne-Whitham (PW) model [Whitham 1974], [Payne: Transp. Res. Rec. 1979]

$$\begin{cases} \rho_t + (\rho u)_x & = 0 \\ u_t + u u_x + \frac{1}{\rho} p(\rho)_x & = \frac{1}{\tau}(U(\rho) - u) \end{cases}$$

Parameters: pressure $p(\rho)$; desired velocity function $U(\rho)$; relaxation time $\tau$

Second-order model; vehicle acceleration: $u_t + u u_x = -\frac{p'(\rho)}{\rho}\rho_x + \frac{1}{\tau}(U(\rho) - u)$

## Inhomog. Aw-Rascle-Zhang (ARZ) model [Aw&Rascle: SIAM JAM 2000], [Zhang: TR–B 2002]

Second-order

$$\begin{cases} \rho_t + (\rho u)_x & = 0 \\ (u + h(\rho))_t + u(u + h(\rho))_x & = \frac{1}{\tau}(U(\rho) - u) \end{cases}$$

hesitation function $h(\rho)$; vehicle acceleration: $u_t + u u_x = \rho h'(\rho) u_x + \frac{1}{\tau}(U(\rho) - u)$

## Generalized ARZ models (GARZ) [Lebacque&Mammar&Haj-Salem, Proc. 17th ISTTT 2007]: “GSOM”

Second-order

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ w_t + u w_x \ = \frac{1}{\tau}(U(\rho) - u) \end{cases} \qquad \text{where} \quad u = V(\rho, w)$$

## Model shortcomings of Payne-Whitham

- Negative velocities can arise: $\rho(x,0) = 1 + \tanh(x/\varepsilon)$ and $u(x,0) = x^2$ for $x \in [-1,1]$. Then PW yields $u_t(0,0) = -\frac{1}{\varepsilon} p'(1) + \frac{1}{\tau} U(1) < 0$, if $\varepsilon$ sufficiently small. In contrast, (G)ARZ yields $u_t(0,0) = \frac{1}{\tau} U(1) > 0$.
- Some information travels faster than vehicles: $\lambda = u \pm c$, where $c = \sqrt{p'(\rho)}$. For ARZ: $\lambda_1 = u - \rho h'(\rho)$ and $\lambda_2 = u$.
- PW & ARZ have shocks that vehicles run into ($\rho_L < \rho_R$ and $s < u_R < u_L$). But: PW also admits shocks that overtake vehicles from behind ($\rho_L > \rho_R$ and $s > u_R > u_L$). (G)ARZ has contact discontinuities ($s = u_L = u_R$) instead.

## Current perspectives

**Math:** Models with a Payne-Whitham (i.e., density-based) pressure are flawed. The (G)ARZ form of the pressure is the correct one.

**Metanet:** [Papageorgiou et al.] PW's problems are fixed on a discrete level.

## Premise of this presentation

The Payne-Whitham pressure should be considered — in a PDE sense!

### Lighthill-Whitham-Richards (LWR) model (1950s)

Velocity is uniquely determined by density: $u = U(\rho)$.

Lighthill, Whitham, *On kinematic waves. II. A theory of traffic flow on long crowded roads*, Proc. Roy. Soc. A, 1955 ; Richards, *Shock waves on the highway*, Operations Research, 1956

### Payne-Whitham (PW) model (1970s)

$\rho$ and $u$ are independent variables.

Whitham, *Linear and nonlinear waves*, John Wiley and Sons, New York, 1974
Payne, *FREEFLO: A macroscopic simulation model of freeway traffic*, Transp. Res. Rec., 1979

### Requiem for second-order models (1990s)

PW: drivers look back, shocks can overtake vehicles, $U < 0$ can happen.

Daganzo, *Requiem for second-order fluid approximations of traffic flow*, Transp. Res. B, 1995

### Resurrection of second-order models (2000s)

Not 2$^{\text{nd}}$ order models are flawed, just PW; fixed via a different pressure.

Aw, Rascle, *Resurrection of second order models of traffic flow?*, SIAM J. Appl. Math., 2000
Zhang, *A non-equilibrium traffic model devoid of gas-like behavior*, Transp. Res. B, 2002

Now: Resurrection of the Payne-Whitham pressure?

TEMPLE UNIVERSITY

# Overview



TEMPLE UNIVERSITY

# $\longrightarrow$ Talk by Michael Herty

## LWR model

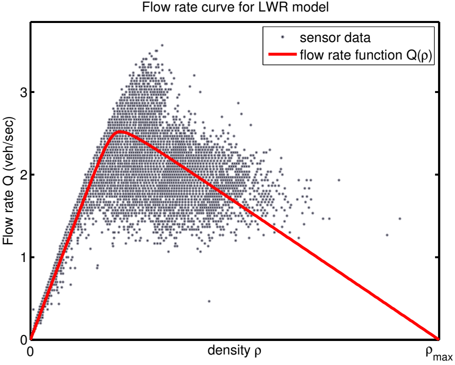
First-order model: $\rho_t + Q(\rho)_x = 0$

(does not reflect spread in FD)

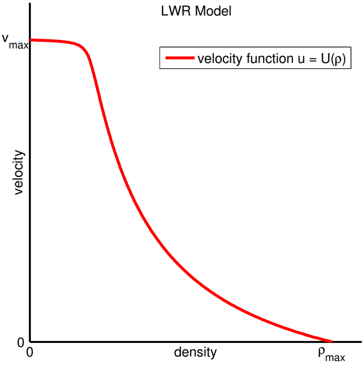
[Fan, S: *Data-fitted first-order traffic models and their second-order generalizations. Comparison by trajectory and sensor data*, Transportat. Res. Rec. 2391:32–43, 2013]

[Fan, Herty, S: *Comparative model accuracy of a data-fitted generalized Aw-Rascle-Zhang model*, Netw. Heterog. Media 9:239–268, 2014]

## Data-fitted flux $Q(\rho)$ — via LSQ-fit




## Induced velocity curve $U(\rho)$



## Aw-Rascle-Zhang (ARZ) model

$$\rho_t + (\rho u)_x = 0$$
$$(u + h(\rho))_t + u(u + h(\rho))_x = 0$$

where $h'(\rho) > 0$ and, WLOG, $h(0) = 0$.

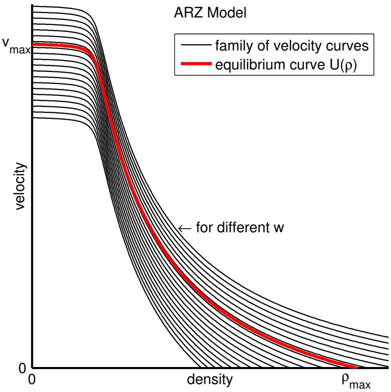
## Equivalent formulation

$$\rho_t + (\rho u)_x = 0$$
$$w_t + u w_x = 0$$
where $u = w - h(\rho)$

Interpretation 1: Each vehicle (moving with velocity $u$) carries a characteristic value, $w$, which is its empty-road velocity. The actual velocity $u$ is then: $w$ reduced by the *hesitation function* $h(\rho)$.

Interpretation 2: ARZ is a generalization of LWR: different drivers have different $u_w(\rho)$.

## ARZ model – velocity curves

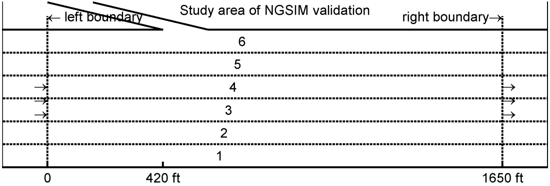



one-parameter family of curves:
$u = u_w(\rho) = u(w, \rho) = w - h(\rho)$

here: $h(\rho) = v_{\max} - U(\rho)$

equilibrium (i.e., LWR) curve:
$U(\rho) = u(v_{\max}, \rho)$

## NGSIM (I-80, Emeryville, CA; 2005)




- three 15 minute intervals
- precise trajectories of all vehicles (in 0.1s intervals)
- historic FD provided separately

## Approach

- Construct macroscopic fields $\rho$ and $u$ from vehicle positions (via kernel density estimation)
- Use data to prescribe i.c. at $t = 0$ and b.c. at left and right side of domain
- Run PDE model to obtain $\rho^{\text{model}}(x, t)$ and $u^{\text{model}}(x, t)$ and error

$$E(x,t) = \frac{\left|\rho^{\text{data}}(x,t) - \rho^{\text{model}}(x,t)\right|}{\rho_{\max}} + \frac{\left|u^{\text{data}}(x,t) - u^{\text{model}}(x,t)\right|}{u_{\max}}$$

- Evaluate model error in a macroscopic ($L^1$) sense: $$E = \frac{1}{TL}\int_0^T\int_0^L E(x,t)\,\mathrm{d}x\mathrm{d}t$$

## Space-and-time-averaged model errors for NGSIM data

| Data set | LWR | ARZ |
|---|---|---|
| 4:00–4:15 | 0.127 (+73%) | 0.073 |
| 5:00–5:15 | 0.115 (+36%) | 0.085 |
| 5:15–5:30 | 0.124 (+6%) | 0.117 |

Even better results with GARZ.

## Outcome

Second-order models reproduce real traffic dynamics better than LWR.

# Overview

# $\longrightarrow$ Talk by Rodolfo Ruben Rosales

## LWR model

$$\rho_t + (\rho U(\rho))_x = 0$$

has char. velocity $\mu = (\rho U(\rho))'$.

## Why do second-order macrosc. models need a pressure at all?

If not: pressureless gas eqns.

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + u u_x \quad = \frac{1}{\tau}(U(\rho) - u) \end{cases}$$

$\lambda_\pm = u$ with Jordan-block; vehicles pile up on top of each other (Dirac delta shocks). Prevented by pressure, which makes system hyperbolic.

[S, Flynn, Kasimov, Rosales: *Constructing set-valued fundamental diagrams from jamiton solutions in second order traffic models*, Netw. Heterog. Media 8(3):745–772, 2013]

[Flynn, Kasimov, Nave, Rosales, S: *Self-sustained nonlinear waves in traffic flow*, Phys. Rev. E 79(5):056113, 2009]

## PW pressure

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ \frac{1}{\rho}\frac{dp}{d\rho} & u \end{pmatrix} \cdot \begin{pmatrix} \rho \\ u \end{pmatrix}_x = \begin{pmatrix} 0 \\ \frac{1}{\tau}(U - u) \end{pmatrix}$$

yields $\lambda_1 = u - c$ and $\lambda_2 = u + c$, where $c = \sqrt{p'(\rho)}$.

## ARZ pressure

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ 0 & u - \rho\frac{dh}{d\rho} \end{pmatrix} \cdot \begin{pmatrix} \rho \\ u \end{pmatrix}_x = \begin{pmatrix} 0 \\ \frac{1}{\tau}(U - u) \end{pmatrix}$$

yields $\lambda_1 = u - \rho h'(\rho)$ and $\lambda_2 = u$.

## Mathematical structure (here for PW): System of balance laws

$$\underbrace{\begin{pmatrix}\rho\\ u\end{pmatrix}_t + \begin{pmatrix} u & \rho \\ \frac{1}{\rho}\frac{dp}{d\rho} & u\end{pmatrix}\cdot\begin{pmatrix}\rho\\ u\end{pmatrix}_x}_{\text{hyperbolic part}} = \underbrace{\begin{pmatrix}0\\ \frac{1}{\tau}(U(\rho)-u)\end{pmatrix}}_{\text{relaxation term}} \qquad \begin{array}{l}\lambda_1 = u - c\\ \lambda_2 = u + c\end{array}$$

## Relaxation to equilibrium

Formally, we can consider the limit $\tau \to 0$. Then: $u = U(\rho)$, i.e., the system reduces to LWR ("reduced equation"). Char. vel.: $\mu = (\rho U(\rho))'$.

## Sub-characteristic condition (SCC) [Whitham: Comm. Pure Appl. Math 1959]

Solutions of the $2 \times 2$ system converge to solutions of LWR, only if the SCC, $\lambda_1 \leq \mu \leq \lambda_2$, is satisfied. Otherwise: jamitons.

## Example (PW): intrinsic phase transition to "phantom traffic jams"

(SCC) $\iff U(\rho) - c(\rho) \leq U(\rho) + \rho U'(\rho) \leq U(\rho) + c(\rho) \iff \frac{c(\rho)}{\rho} \geq -U'(\rho)$.

For $p(\rho) = \frac{\beta}{2}\rho^2$ and $U(\rho) = u_m(1-\rho/\rho_m)$: uniform flow stable iff $\rho \leq \beta\rho_m^2/u_m^2$.

## PW model

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + u u_x + \frac{1}{\rho} p(\rho)_x = \frac{1}{\tau}(U(\rho) - u) \end{cases}$$

## Traveling wave ansatz

$\rho = \rho(\eta)$, $u = u(\eta)$, with self-similar variable $\eta = \frac{x - st}{\tau}$.

Then $\rho_t = -\frac{s}{\tau}\rho'$ , $\quad \rho_x = \frac{1}{\tau}\rho'$ , $\quad u_t = -\frac{s}{\tau}u'$ , $\quad u_x = \frac{1}{\tau}u'$

and $\quad p_x = \frac{1}{\tau}c^2\rho'$ , $\quad c^2 = \frac{dp}{d\rho}$

## Continuity equation

$$\rho_t + (u\rho)_x = 0$$

$$-\frac{s}{\tau}\rho' + \frac{1}{\tau}(u\rho)' = 0$$

$$(\rho(u-s))' = 0$$

$$\rho = \frac{m}{u-s}$$

$$\rho' = -\frac{\rho}{u-s}u'$$

## Momentum equation

$$u_t + u u_x + \frac{p_x}{\rho} = \frac{1}{\tau}(U - u)$$

$$-\frac{s}{\tau}u' + \frac{1}{\tau}u u' + \frac{dp}{d\rho}\frac{\rho'}{\rho} = \frac{1}{\tau}(U - u)$$

$$(u-s)u' - c^2\frac{1}{u-s}u' = U - u$$

$$u' = \frac{(u-s)(U-u)}{(u-s)^2 - c^2}$$

## Ordinary differential equation for $u(\eta)$

$$u' = \frac{(u-s)(U(\rho)-u)}{(u-s)^2 - c(\rho)^2} \quad \text{where} \quad \rho = \frac{m}{u-s}$$

where

$$s = \text{travel speed of jamiton}$$

$$m = \text{mass flux of vehicles through jamiton}$$

## Key point for jamitons

$m$ and $s$ can not be chosen independently:

Denominator has root at $u = s + c$. Solution can only pass smoothly through this singularity (the sonic point), if $u = s + c$ implies $U = u$.

Using $u = s + \frac{m}{\rho}$, we obtain for this sonic density $\rho_S$ that:

$$\begin{cases} \text{Denominator} & s + \frac{m}{\rho_S} = s + c(\rho_S) \implies m = \rho_S c(\rho_S) \\ \text{Numerator} & s + \frac{m}{\rho_S} = U(\rho_S) \implies s = U(\rho_S) - c(\rho_S) \end{cases}$$

Algebraic condition (Chapman-Jouguet condition [Chapman, Jouguet (1890)]) that relates $m$ and $s$ (and $\rho_S$).

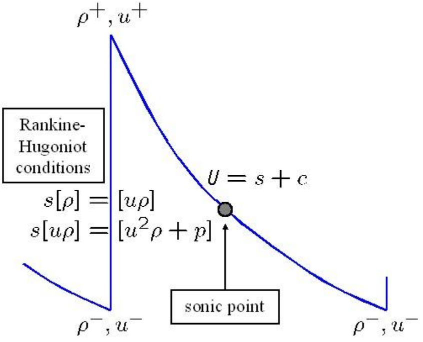
## Jamiton ODE

$$u' = \frac{(u-s)(U(\frac{m}{u-s})-u)}{(u-s)^2 - c(\frac{m}{u-s})^2}$$

## Construction

1. Choose $m$. Obtain $s$ by matching root of denominator with root of numerator.
2. Choose $u^-$. Obtain $u^+$ by Rankine-Hugoniot conditions.
3. ODE can be integrated through sonic point (from $u^+$ to $u^-$).
4. Yields length (from shock to shock) of jamiton, $\lambda$, and number of cars, $N = \int_0^\lambda \rho(x)dx$.

## Periodic case

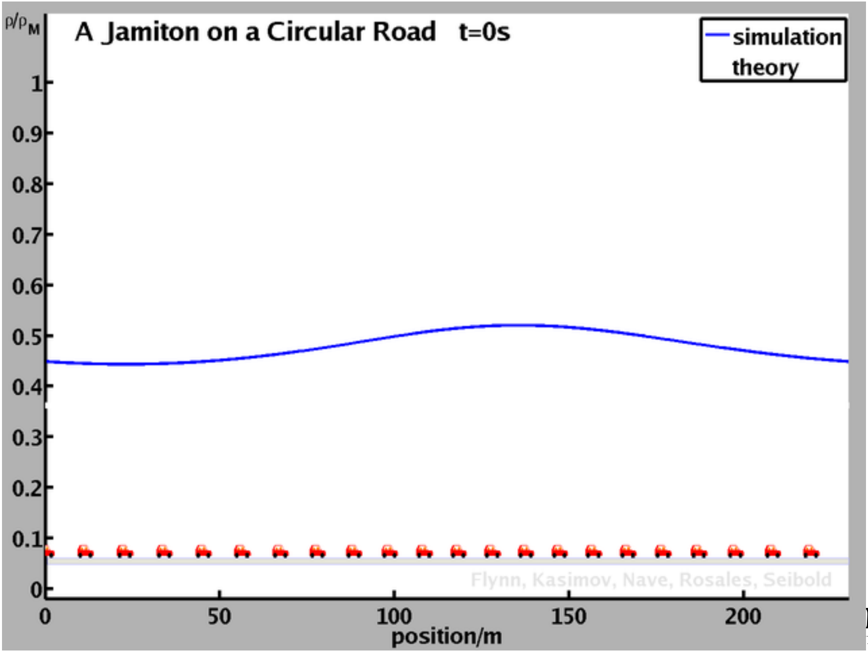



jamiton length $= O(\tau)$

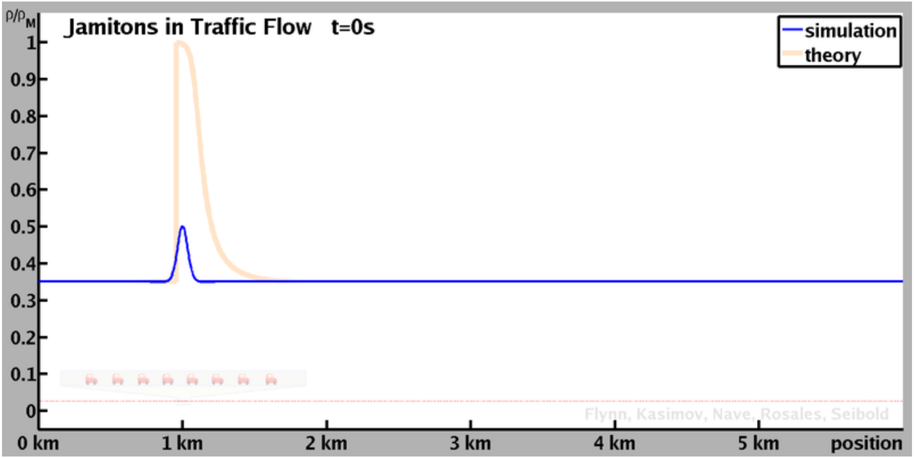
## Inverse construction

If $\lambda$ and $N$ are given, find jamiton by iteration.

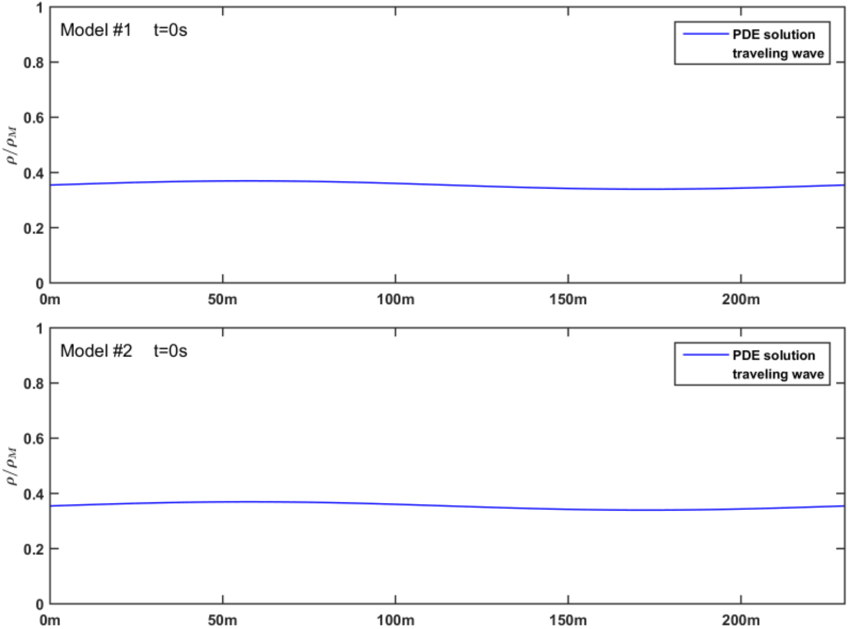
## Experiment: Jamitons on circular road [Sugiyama et al.: New J. of Physics 2008]


A Jamiton on a Circular Road t=0s
simulation
theory
ρ/ρ_M
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Flynn, Kasimov, Nave, Rosales, Seibold
0
50
100
150
200
position/m

Infinite road; lead jamiton gives birth to a chain of “jamitinos”.



## Comparison of ARZ & PW solutions. Can you tell which one is which?



## Recall: continuity equation

$$\rho_t + (u\rho)_x = 0$$

## Traveling wave ansatz

$\rho = \rho(\eta)$, $u = u(\eta)$, where $\eta = \frac{x-st}{\tau}$, yields

$$\rho_t + (u\rho)_x = 0$$

$$-\frac{s}{\tau}\rho' + \frac{1}{\tau}(u\rho)' = 0$$

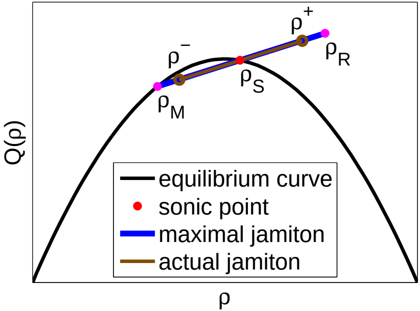
$$(\rho(u-s))' = 0$$

$$\rho(u-s) = m$$

$$q = m + s\rho$$

Hence: Any traveling wave is a line segment in the fundamental diagram.

## A jamiton in the FD




- Plot equilibrium curve $Q(\rho) = \rho U(\rho)$
- Chose a $\rho_S$ that violates the SCC
- Mark sonic point $(\rho_S, Q(\rho_S))$ (red)
- Set $m = \rho_S c(\rho_S)$ and $s = U(\rho_S) - c(\rho_S)$
- Draw maximal jamiton line (blue)
- Other jamitons with the same $m$ and $s$ are sub-segments (brown)

## Construction of jamiton FD

For each $\rho_S$ that violates the SCC: construct maximal jamiton.

## Temporal aggregation of jamitons

At fixed position $\bar{x}$, calculate all possible temporal average ($\Delta t = \alpha\,\tau$) densities

$$\bar{\rho} = \frac{1}{\Delta t}\int_0^{\Delta t} \rho(\bar{x}, t)\mathrm{d}t\ .$$

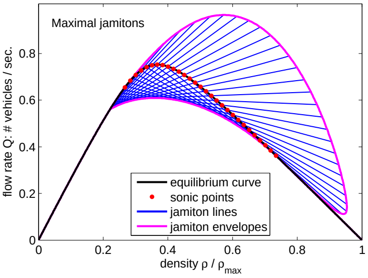
Average flow rate: $\bar{q} = m + s\bar{\rho}$.

This reduces the spread in the FD.

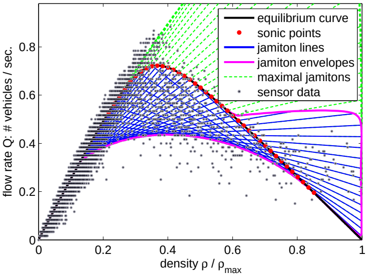
## Conclusions

- Set-valued FDs can be explained via traveling waves in second-order traffic models.
- No fundamental difference between ARZ and PW in terms of jamitons.

## Jamiton fundamental diagram




## Emulating sensor aggregation



# Overview



TEMPLE UNIVERSITY

## Macroscopic fields from NGSIM data

- NGSIM: all vehicle trajectories on 500m highway over 45 minutes.
- Construct macroscopic fields via kernel density estimation:
  Using Gaussian kernels $G(x) = Z^{-1}e^{-(x/h)^2}$, define
  $\rho(x) = \sum_j G(x - x_j)$ and $q(x) = \sum_j \dot{x}_j G(x - x_j)$ and
  $u(x) = q(x)/\rho(x)$. [Fan, S: Transportat. Res. Rec. 2013]

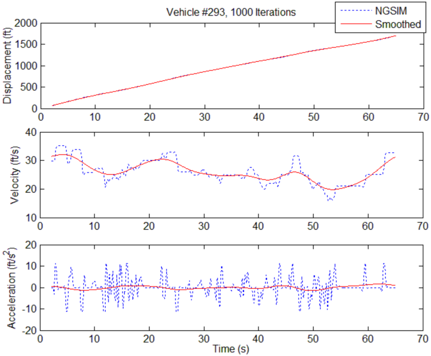
## Idea

- PW vehicle acceleration field
  $a = u_t + uu_x = -\frac{p'(\rho)}{\rho}\rho_x + \frac{1}{\tau}(U(\rho) - u) = A_{\text{PW}}(\rho, u, \rho_x)$
  is independent of $u_x$.
- ARZ vehicle acceleration field
  $a = u_t + uu_x = \rho h'(\rho)u_x + \frac{1}{\tau}(U(\rho) - u) = A_{\text{ARZ}}(\rho, u, u_x)$
  is independent of $\rho_x$ (same structure for GARZ).
- No model reproduces data exactly, but:
  If ARZ is a better model than PW, then true acceleration field should depend substantially more strongly on $u_x$ than on $\rho_x$.

## Idea

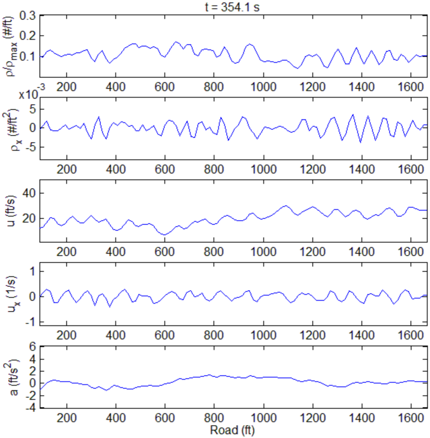
- PW acceleration field $u_t + uu_x$ is independent of $u_x$.
- (G)ARZ acceleration field $u_t + uu_x$ is independent of $\rho_x$.
- Does true acceleration field depend more strongly on $u_x$ than on $\rho_x$?

## Smoothing of vehicle trajectories




**Note:** macroscopic acceleration is very different from vehicle acceleration.

## Macroscopic fields



## Idea

- PW acceleration field $u_t + uu_x$ is independent of $u_x$.
- (G)ARZ acceleration field $u_t + uu_x$ is independent of $\rho_x$.
- Does true acceleration field depend more strongly on $u_x$ than on $\rho_x$?

## Approach

- Consider 100 positions (5m) along road, and 4500 time steps (0.6s). Yields 450,000 data points.
- At each data point (in $x$–$t$ domain), evaluate fields $\rho$, $u$, $\rho_x$, $u_x$, $a$.
- Divide 4-dimensional domain ($\rho$, $u$, $\rho_x$, $u_x$) into boxes ($20 \times 20 \times 20 \times 20$). For each box that contains data points, assign the average $a$-value.
- For each $(\rho, u, \rho_x)$, look at all boxes in $u_x$-direction. Calculate variation $\max a - \min a$ over this strip. Then average these $a$-variations over all strips with at least 2 $a$-values.
- Same idea: for each $(\rho, u, u_x)$, look at boxes in $\rho_x$-direction; calculate variation $\max a - \min a$ over strip; then average.
- Result: $\rho_x$-variation: $0.4697\,\mathrm{ft/s^2}$; $u_x$-variation: $0.4909\,\mathrm{ft/s^2}$.
  Data shows no strong preference for ARZ vs. PW.

# Overview

## Follow-the-leader (FTL) model

[Gazis, Herman, Rothery, Operat. Res. 1960]

$$\ddot{x}_j = b \frac{\dot{x}_{j+1} - \dot{x}_j}{x_{j+1} - x_j}$$

Divers equilibrate velocities.

## Optimal velocity model (OVM)

[Bando et al., PRE 1995]

$$\ddot{x}_j = a \cdot \left( U\left(\frac{\Delta X}{x_{j+1} - x_j}\right) - \dot{x}_j \right)$$

$\Delta X$ is reference distance, s.t. $\frac{\Delta X}{x_{j+1} - x_j}$ is a normalized density.

## Macro limit of combined FTL-OVM model [Aw,Klar,Materne,Rascle: SIAM J. Appl. Math. 2002]

$$\ddot{x}_j = b \frac{\dot{x}_{j+1} - \dot{x}_j}{x_{j+1} - x_j} + a \cdot \left( U\left(\frac{\Delta X}{x_{j+1} - x_j}\right) - \dot{x}_j \right)$$

converges to ARZ model (with $h(\rho) \propto b \log(\rho)$) as $\Delta X \to 0$ and number of vehicles $N \propto 1/\Delta X \to \infty$.

**Proof:** Show equivalence of forward Euler discretization of the microscopic model and a Godunov discretization of ARZ in Lagrangian variables.

## Problem

Argument fails for pure OVM (case $b = 0$): ARZ becomes pressureless gas eqns. (always unstable), while OVM is stable if $a$ is sufficiently large.

## Stability of uniform flow in optimal velocity model

OVM: $\ddot{x}_j = a \cdot (U(\frac{\Delta X}{x_{j+1} - x_j}) - \dot{x}_j)$

Base flow: $\bar{x}_j(t) = d\Delta X\, j + V(d)t$, where spacing $d$ given

Perturbed flow $x_j = \bar{x}_j + y_j$ yields: $\ddot{y}_j = a \cdot (\frac{-\rho^2 U'(\rho)}{\Delta X}(y_{j+1} - y_j) - \dot{y}_j)$

Basic waves $y_j = e^{zt + i\xi d \Delta X j}$ yield stability, iff $\frac{-\rho^2 U'(\rho)}{\Delta X} < \frac{1}{2}a$.

## Proper scaling

- If $a$ fixed, then the OVM becomes always unstable as $\Delta X \to 0$ (consistent with pressureless gas limit of ARZ)
- When scaling $a = \frac{A}{\Delta X}$, then we have stability condition, $-\rho^2 U'(\rho) < \frac{1}{2}A$, that persists in the macroscopic limit ($\Delta X \to 0$).
- Now the pressureless gas equation

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + u u_x = a \cdot (U(\rho) - u) \end{cases}$$

is not a proper macroscopic limit.

## Scaled OVM

$\ddot{x}_j = \frac{A}{\Delta X} \cdot \left(U\left(\frac{\Delta X}{x_{j+1}-x_j}\right) - \dot{x}_j\right)$ is stable if $-\rho^2 U'(\rho) < \frac{1}{2}A$.

## Macroscopic limit

- Pressureless gas limit is obtained when interpreting $\frac{\Delta X}{x_{j+1}-x_j} \approx \rho(x_j, t)$.
- Now interpret $\frac{\Delta X}{x_{j+1}-x_j} \approx \rho(x_{j+\frac{1}{2}}, t)$, where $x_{j+\frac{1}{2}} = \frac{1}{2}(x_j + x_{j+1})$ is midpoint between vehicles.
- Taylor expansion leads to:

$$\frac{\Delta X}{x_{j+1}-x_j} \approx \rho(x_j) + \rho_x(x_j)\frac{x_{j+1}-x_j}{2} \overset{\frac{\Delta X}{x_{j+1}-x_j}\approx\rho(x_j)}{\approx} \rho(x_j) + \rho_x(x_j)\frac{1}{2\rho(x_j)}\Delta X$$

and thus: $U\left(\frac{\Delta X}{x_{j+1}-x_j}\right) \approx U(\rho(x_j)) + U'(\rho(x_j))\rho_x(x_j)\frac{1}{2\rho(x_j)}\Delta X$

- Leads to Payne-Whitham model:

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + uu_x + a\Delta X \frac{-U'(\rho)}{2\rho}\rho_x = a \cdot (U(\rho) - u) \end{cases}$$

## Scaled OVM

$\ddot{x}_j = \frac{A}{\Delta X} \cdot \left(U\left(\frac{\Delta X}{x_{j+1}-x_j}\right) - \dot{x}_j\right)$ is stable if $-\rho^2 U'(\rho) < \frac{1}{2}A$.

## Macroscopic limit: Payne-Whitham model

$$\begin{cases} \rho_t + (\rho u)_x &= 0 \\ u_t + uu_x + A\frac{-U'(\rho)}{2\rho}\rho_x &= \frac{A}{\Delta X} \cdot (U(\rho) - u) \end{cases}$$

has char. velocities $\lambda_{1,2} = U(\rho) \pm \sqrt{-\frac{1}{2}AU'(\rho)}$. Hence, it is stable (SCC satisfied) if $\rho^2(-U'(\rho)) < \frac{1}{2}A$. Exactly matches OVM stability.

Note: shock behavior strongly affected by relaxation term...

## FTL-OVM model

Same approach yields pressure-hesitation model as macroscopic limit

$$\begin{cases} \rho_t + (\rho u)_x &= 0 \\ u_t + uu_x - \rho h'(\rho) u_x - A\frac{U'(\rho)}{2\rho}\rho_x &= \frac{A}{\Delta X}(U(\rho) - u) \end{cases}$$

## First-order limit (viscous LWR)

If SCC is satisfied (OVM is stable), an asymptotic expansion ($\Delta X \ll 1$) of the PW model leads to

$$\rho_t + (\rho U(\rho))_x = \Delta X \left(D(\rho)(\rho)_x\right)_x$$

with $D(\rho) = -U'(\rho)\left(\frac{1}{2} + \frac{1}{A}\rho^2 U'(\rho)\right)$.

## First-order limit (viscous LWR)

If SCC is satisfied (OVM is stable), an asymptotic expansion ($\Delta X \ll 1$) of the PW model leads to

$$\rho_t + (\rho U(\rho))_x = \Delta X \left(D(\rho)(\rho)_x\right)_x$$

with $D(\rho) = -U'(\rho)(\frac{1}{2} + \frac{1}{A}\rho^2 U'(\rho))$.

**Connection:**

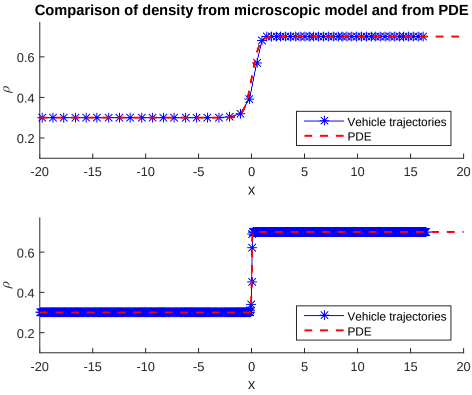
OVM stability $\Longleftrightarrow$ PW SCC

$\Longleftrightarrow D(\rho) > 0$.

**Moreover:**

Natural transfer of bounded acceleration from micro $\longrightarrow$ PW $\longrightarrow$ viscous LWR.

## A smeared shock in the OVM model




## One key message

Real traffic is microscopic. Ideally, accurate macroscopic models should not focus on the limit $N \to \infty$, but represent the solution with true #vehicles $N$. PW-type pressures play an important role in this.

# Overview



TEMPLE UNIVERSITY

## Key messages

- Second-order models have clear advantages over LWR in terms of reproducing data and modeling (multi-valued FDs).
- Phantom jam phase transition and jamiton behavior are very reasonable with PW (as with other second-order models). Bad shocks do not persist.
- Trajectory data does not favor the (G)ARZ structure over a PW structure.
- The PW pressure arises naturally when studying macroscopic limits of microscopic descriptions of traffic flow.

## Suggestion

Consider pressure-hesitation models with density- and velocity-driven pressures:

$$\begin{cases} \rho_t + (\rho u)_x & = 0 \\ u_t + u u_x - \alpha(\rho, u) u_x + \beta(\rho, u)\rho_x & = \frac{1}{\tau}(U(\rho) - u) \end{cases}$$

Example: GARZ-PW model

$$\begin{cases} \rho_t + (\rho u)_x & = 0 \\ w_t + u w_x + \frac{p(\rho)_x}{\rho} & = \frac{1}{\tau}(U(\rho) - u) \end{cases} \qquad \text{where } u = V(\rho, w)$$

## Modeling discussion

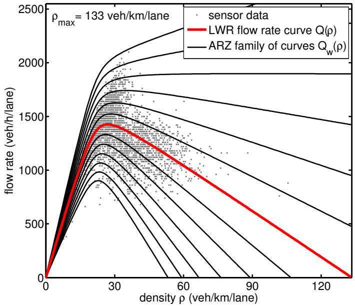
- Information traveling faster than vehicles (in PW): Are drivers really unaffected by a big truck approaching in the rear-view mirror?
- Aw&Rascle argue that, when traffic ahead is denser but faster, that drivers should speed up, rather than slow down (argument neglects relaxation term). Does that make sense, when traffic ahead is only a bit faster, but highly dense?
- Bad shocks in PW appear not to arise dynamically. Still, they can be produced via i.c. However, the microscopic reality of traffic should disallow too large $\rho_x$ in i.c.
- Negative velocities. Density-driven pressure should (in some way) vanish as $u \to 0$. Note: macroscopic description in the creeping regime ($u < 2\text{m/s}$) is challenging anyways.
- New phenomenon: more complex shock laws; even if $p(\rho)$ and $h(\rho)$ convex, pressure-hesitation models may have composite waves.

  $\longrightarrow$ cf. non-convex flux functions (capacity drop)

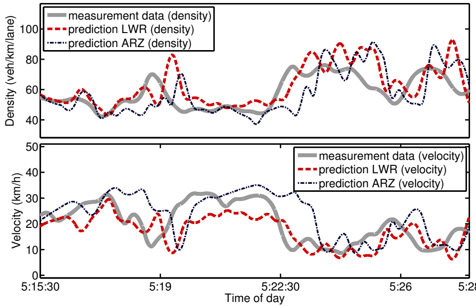
  $\longrightarrow$ cf. traffic models with non-convexity [Li, Liu: Comm. Math. Sci. 2005]

  $\longrightarrow$ non-convexity near jamming density [Fan, S: `arxiv.org/abs/1308.0393`]

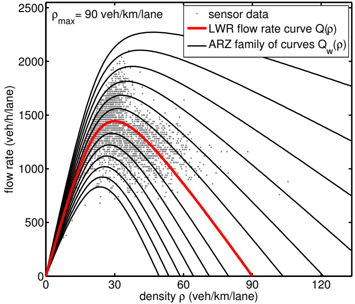
Flow rate curves for the NGSIM FD data



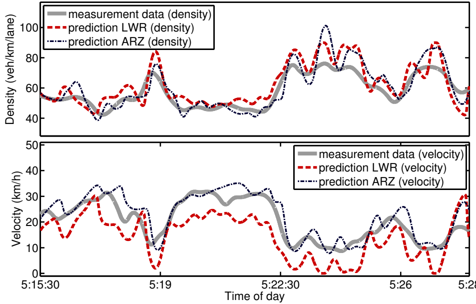

Model prediction at center for NGSIM data with stagnation density 133 veh/km/lane




Flow rate curves for the NGSIM FD data

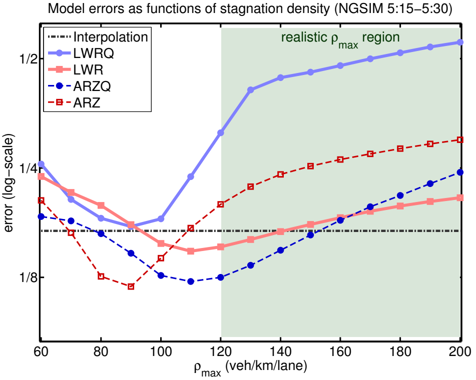



Model prediction at center for NGSIM data with stagnation density 90 veh/km/lane



## Non-convexity near jamming

Physical $\rho_{\max} \approx 133$ veh/km/lane can only be reached by $Q(\rho)$ if inflection point near $u = 0$ is inserted.




## Final words

- When modeling real traffic flow, the Payne-Whitham pressure should be considered.
- In light of autonomous vehicles, PW-pressure may become even more relevant. Should autonomous vehicles take into account traffic behind them? If so, how would that affect the macroscopic behavior?