A Mathematical Introduction to Traffic Flow Theory

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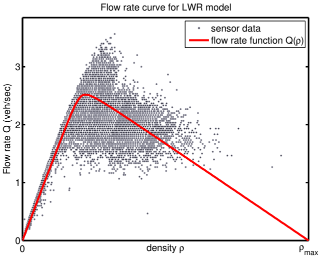

September 9–11, 2015

Tutorials Traffic Flow

Institute for Pure and Applied Mathematics, UCLA

Some References for Further Reading

- **general**: wikipedia (can be used for almost every technical term)
- **traffic flow theory**: wikibooks, "Fundamentals of Transportation/Traffic Flow"; Hall, "Traffic Stream Characteristics"; Immers, Logghe, "Traffic Flow Theory"
- **traffic models**: review papers by Bellomo, Dogbe, Helbing
- **traffic phase theory**: papers by Kerner, et al.
- **optimal velocity models and follow-the-leader models**: papers by Bando, Hesebem, Nakayama, Shibata, Sugiyama, Gazis, Herman, Rothery, Helbing, et al.
- **connections between micro and macro models**: papers by Aw, Klar, Materne, Rascle, Greenberg, et al.
- **first-order macroscopic models**: papers by Lighthill, Whitham, Richards, et al.
- **second-order macroscopic models**: papers by Whitham, Payne, Aw, Rascle, Zhang, Greenberg, Helbing, et al.
- **cellular models and cell transmission models**: papers by Nagel, Schreckenberg, Daganzo, et al.
- **numerical methods for hyperbolic conservation laws**: books by LeVeque
- **traffic networks**: papers by Holden, Risebro, Piccoli, Herty, Klar, Rascle, et al.

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Overview

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Overview

Two extreme situations: traffic flow modeling trivial/easy

Very low density

Negligible interaction between vehicles; everyone travels their desired speed

(Almost) complete stoppage

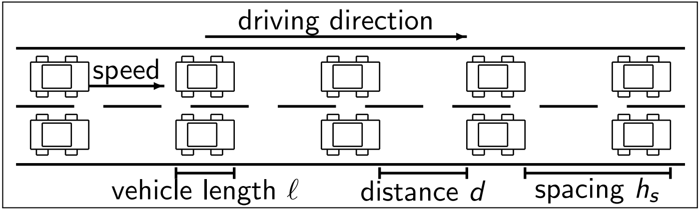
Flow dynamics irrelevant; simply queueing theory

Traffic flow modeling of importance, and challenging

Traffic dense but moving

- Micro-dynamics: by which laws do vehicles interact with each other?
- Laws of flow, e.g.: how does vehicle density relate with flow rate?
- Macro-view: temporal evolution of traffic density? traffic waves? etc.

Uniform traffic flow



Some fundamental quantities

- density ρ: number of vehicles per unit length (at fixed time)
- flow rate (throughput) f: number of vehicles passing fixed position per unit time
- speed (velocity) u: distance traveled per unit time
- time headway h_t: time between two vehicles passing fixed position
- space headway (spacing) h_s: road length per vehicle
- occupancy b: percent of time a fixed position is occupied by a vehicle

Relations

- $h_s = d + \ell$
- $\rho = \#\text{lanes}/h_s$
- $f = \#\text{lanes}/h_t$
- $f = \rho u$
- $\rho_{\max} = \#\text{lanes}/\ell$
- $\rho = b \cdot \rho_{\max}$
- etc.

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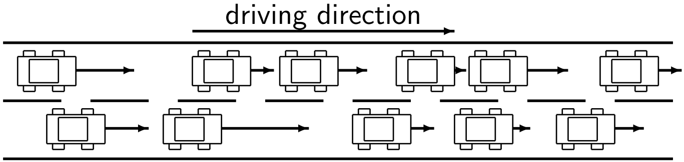
Non-uniform traffic flow



vehicles have different velocities $\Longrightarrow$ pattern changes

Fundamental quantities

- density ρ: number of vehicles per unit length (at fixed time) $\leftarrow$ as before
- flow rate f: number of vehicles passing fixed position per unit time $\leftarrow$ as before
- time mean speed: fixed position, average vehicle speeds over time
- space mean speed: fixed time, average speeds over a space interval
- bulk velocity: $u = f/\rho$ $\leftarrow$ usually meant in macroscopic perspective

Key questions

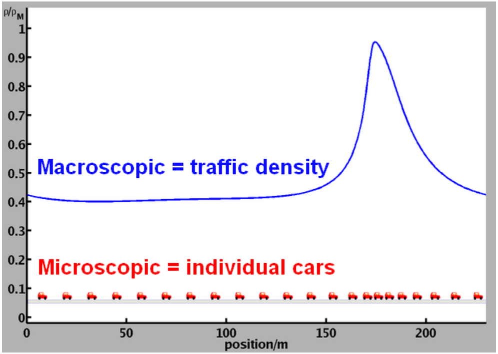
- Given vehicle positions and velocities, how do they evolve?
- Impossible to predict on level of trajectories (microscopic).
- But it may be possible on level of density and flow rate fields (macroscopic).
- Needed: connection between micro and macro description.

Microscopic description

- individual vehicle trajectories
- vehicle position: $x_j(t)$
- vehicle velocity: $\dot{x}_j(t) = \frac{\mathrm{d}x_j}{\mathrm{d}t}(t)$
- acceleration: $\ddot{x}_j(t)$

Macroscopic description

- field quantities, defined everywhere
- vehicle density: $\rho(x,t)$
- velocity field: $u(x,t)$
- flow-rate field: $f(x,t) = \rho(x,t)u(x,t)$



- Micro view distinguishes vehicles; macro view does not.
- Traffic flow has intrinsic irreproducibility. No hope to describe/predict precise vehicle trajectories via models [also, privacy issues]. But, prediction of large-scale field quantities may be possible.

Microscopic $\longrightarrow$ macroscopic: "kernel density estimation" (KDE)

Some approaches to go from $x_1, x_2, \dots, x_N$ to $\rho(x)$:

- piecewise constant: let $x_1 < x_2 < \dots, x_N$; for $x_j < x \leq x_{j+1}$, define $\rho(x) = 1/(x_{j+1} - x_j)$.

  $\ominus$ no unique density at vehicle positions; $\ominus$ does not work for aggregated lanes
- moving window: at position x, define $\rho(x)$ as the number of vehicles in $[x-h, x+h]$, divided by the window width $2h$.

  $\ominus$ integer nature; $\ominus$ discontinuous in time
- smooth moving kernels: each vehicle is a smeared Dirac delta; $\rho(x) = \sum_j G(x - x_j)$; use e.g. Gaussian kernels, $G(x) = Z^{-1} e^{-(x/h)^2}$.

Most KDE approaches require a length scale h that must be larger than the typical vehicle distance, and smaller than the actual length scale of interest (road length).

Macroscopic $\longrightarrow$ microscopic: "sampling"

Some approaches to go from $\rho(x)$ to $x_1, x_2, \dots, x_N$:

sampling via rejection, inverse transform, MCMC, Metropolis, Gibbs, etc.

Once we have a macroscopic (i.e. continuum) description

Vehicle density function $\rho(x,t)$; $x =$ position on road; $t =$ time.

Number of vehicles between point a and b:

$$m(t) = \int_a^b \rho(x,t)\mathrm{d}x$$

Traffic flow rate (flux) is product of density ρ and vehicle velocity u

$$f = \rho u$$

Change of number of vehicles equals inflow $f(a)$ minus outflow $f(b)$

$$\frac{d}{dt}m(t) = \int_a^b \rho_t \mathrm{d}x = f(a) - f(b) = -\int_a^b f_x \mathrm{d}x$$

Equation holds for any choice of a and b:

continuity equation $\qquad \rho_t + (\rho u)_x = 0 \qquad \Longleftrightarrow \qquad \rho_t + f_x = 0$

Note: The continuity equation merely states that no vehicles are lost or created. There is no modeling in it. In particular, it is not a closed model (1 equation for 2 unknowns).

A critical note on stochasticity

- It is frequently stated that "traffic is stochastic", or "traffic is random."
- Or even further: "traffic flow cannot be described via deterministic models."

Problems with these statements

- A stochastic description is only useful if one defines the random variables, and knows something about their (joint) probability distribution.
- Certain observables in a stochastic model may evolve deterministically. It matters what one is interested in: the evolution of real traffic flow may be reproduced quite well by a model, if one is interested in the density on length scales much larger than the vehicle spacing. In turn, attempting to model the precise trajectories of the vehicles is most likely hopeless.

A better formulation and research goals

- Traffic has an intrinsic irreproducibility (that cannot be overcome, say, via more a careful experiment), due to hidden variables (e.g. drivers' moods).
- Goals: 1) find models that describe macroscopic variables as well as possible ($\longrightarrow$ fundamental insight into mechanics); 2) combine these models with filtering to incorporate data ($\longrightarrow$ prediction).

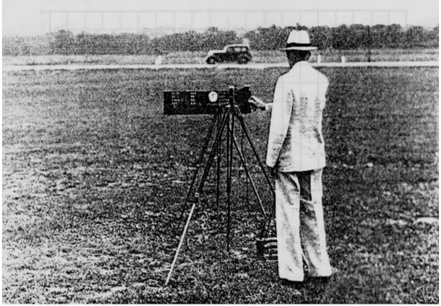
Examples of ways to measure traffic variables

- human in chair: count passing vehicles ($\longrightarrow$ flow rate)
- single loop sensor: at fixed position, count passing vehicles (flow rate) and measure occupancy ($\longrightarrow$ density and velocity indirectly, assuming average vehicle length)
- double loop sensor: as before, but also measure velocity directly
- aerial imaging: photo of section of highway; direct measure of density
- GPS: precise trajectories of a few vehicles on the road
- observers in the traffic stream: as GPS, but also information about nearby vehicles
- processed camera data: precise trajectories of all vehicles (NGSIM data set); perfect data, but not possible on large scale

Inferring macroscopic variables (density, flow rate, etc.)

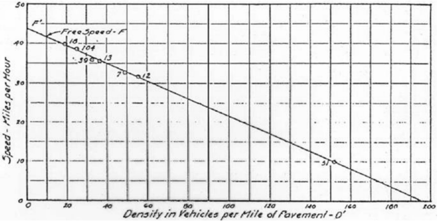
- Traditional means of data collection measure them (almost) directly.
- Modern data collection measures very different things. Inferring traffic flow variables (everywhere and at all times) is a key challenge.

Bruce Greenshields collecting data (1933)



[This was only 25 years after the first Ford Model T (1908)]

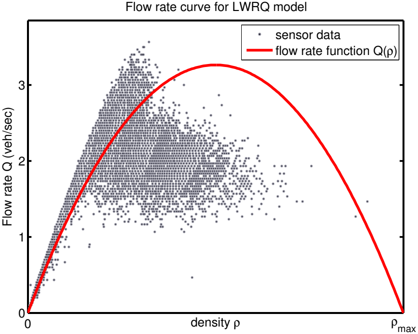
Postulated density–velocity relationship



Deduced relationship

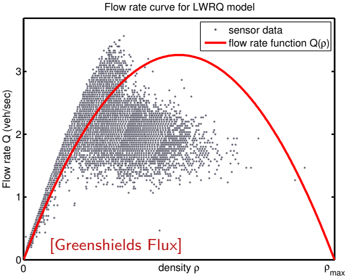
- $u = U(\rho) = u_{\max}(1 - \rho/\rho_{\max})$,
  $\rho_{\max} \approx 195$ veh/mi; $u_{\max} \approx 43$ mi/h
- Flow rate
  $f = Q(\rho) = u_{\max}(\rho - \rho^2/\rho_{\max})$

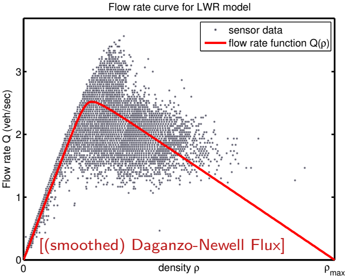
Contemporary measurements (Q vs. ρ)



[Fundamental Diagram of Traffic Flow]

Fundamental Diagram (FD) of traffic flow (sensor data)





Fascinating fact

FDs exhibit the same features, for highways everywhere in the world:

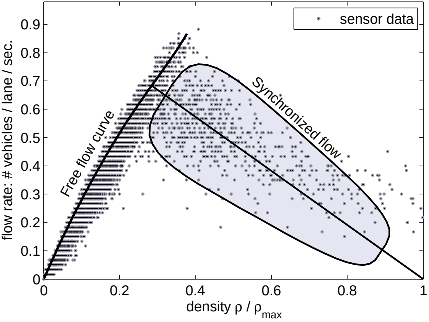
- $f = Q(\rho)$ for ρ small (free-flow)
- for ρ sufficiently large (congestion), FD becomes set-valued
- set-valued part: f decreases with ρ

Ideas and concepts

- ignore spread and define function $f = Q(\rho)$, e.g.: Greenshields (quadratic), Daganzo-Newell (piecewise linear) flux
- empirical classification and explanation $\longrightarrow$ traffic phase theory [Kerner 1996–2002]
- explanation of spread via "imperfections": sensor noise, driver inhomogeneities, etc.
- study of models that reproduce spread (esp. second-order models)

Traffic phase theory (here: 2 phases)

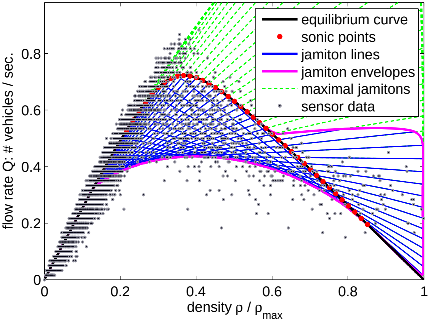
[Kerner 1996–2002]



Also: 3-phase theory (wide moving jams)

Phase transition induced by traffic waves in macroscopic model

[S, Flynn, Kasimov, Rosales, NHM 8(3):745–772, 2013]



A key challenge in traffic flow theory

Many open questions in explaining (phenomenologically), modeling, and understanding the spread observed in FDs.

Overview

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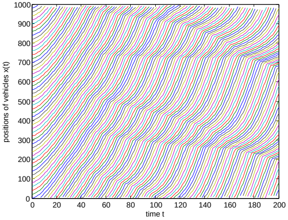
One can study traffic flow empirically, i.e., observe and classify what one sees and measures.

So why study traffic models?

- Can remove/add specific effects (lane switching, driver/vehicle inhomogeneities, road conditions, etc.) $\longrightarrow$ understand which effects play which role.
- Can study effect of model parameters (driver aggressiveness, etc.).
- Can be analyzed theoretically (to a certain extent).
- Can use computational resources to simulate.
- Yield quantitative predictions ($\longrightarrow$ traffic forecasting).
- We actually do not really know (exactly) how we drive. Models that produce correct emergent phenomena help us understand our driving behavior.
- Traffic models can be studied purely theoretically (mathematical properties). That is fun; but ultimately models must be validated with real data to investigate how well they reproduce real traffic flow.

Microscopic models

$$\ddot{x}_j = G(x_{j+1} - x_j, u_j, u_{j+1})$$



Idea

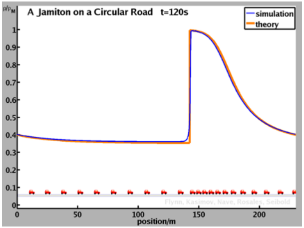
Describe behavior of individual vehicles (ODE system).

Micro $\longleftrightarrow$ Macro

- macro = limit of micro when #vehicles $\to \infty$
- micro = discretization of macro in Lagrangian variables

Macroscopic models

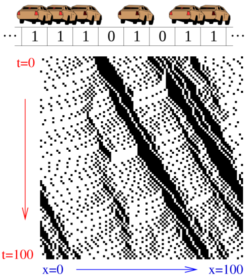
$$\begin{cases} \rho_t + (\rho u)_x &= 0 \\ (u+h)_t + u(u+h)_x &= \frac{1}{\tau}(U-u) \end{cases}$$



Methodology and role

- Describe aggregate/bulk quantities via PDE.
- Natural framework for multiscale phenomena, traveling waves, and shocks.
- Great framework to incorporate sparse data [Mobile Millennium Project].

Cellular models



Idea

Cell-to-cell propagation (space-time-discrete).

Cellular $\longleftrightarrow$ Macro

- macro = limit of cellular when #cells $\to \infty$
- cellular = discretization of macro in Eulerian variables

Microscopic Traffic Models — Typical Setup

N Vehicles on the road

- Position of j-th vehicle: x_j
- Velocity of j-th vehicle: v_j
- Acceleration of j-th vehicle: a_j

Physical principles

- Velocity is rate of change of position: $v_j = \dot{x}_j$
- Acceleration is rate of change of velocity: $a_j = \dot{v}_j = \ddot{x}_j$

"Follow the leader" model

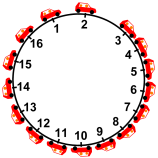
Accelerate/decelerate towards velocity of vehicle ahead of you:

$$a_j = \frac{v_{j+1} - v_j}{x_{j+1} - x_j}$$

"Optimal velocity" model

Accelerate/decelerate towards an optimal velocity that depends on your distance to the vehicle ahead:

$$a_j = V(x_{j+1} - x_j) - v_j$$



use computers to solve $\longrightarrow$

Combined Model

$$a_j = \alpha \frac{v_{j+1} - v_j}{x_{j+1} - x_j} + \beta \left(V(x_{j+1} - x_j) - v_j \right)$$

Microscopic Traffic Models — Philosophy

Philosophy of microscopic models

- Compute trajectories of each vehicle.
- Can be extended to multiple lanes (with lane switching), different vehicle types, etc.
- At the core of most micro-simulators (e.g., Aimsun, using the Gipps' model; or Vissim, using the Wiedemann model); usually with a discrete time-step.
- Many other car-following models, e.g., the intelligent driver model.
- May have many parameters, in particular free parameters that cannot be measured directly. Hence, calibration required.
- In most applications, initial positions of vehicles not known precisely. Ensembles of computations must be run.
- Good for simulation ("How would a lane-closure affect this highway section?"); not the best framework for estimation and prediction based on sparse/noisy data.

Macroscopic Traffic Models — Setup

Recall: continuity equation

Vehicle density $\rho(x,t)$. Number of vehicles in $[a,b]$: $m(t) = \int_a^b \rho(x,t)\mathrm{d}x$

Traffic flow rate (flux): $f = \rho u$

Change of number of vehicles equals inflow $f(a)$ minus outflow $f(b)$:

$$\frac{d}{dt}m(t) = \int_a^b \rho_t \mathrm{d}x = f(a) - f(b) = -\int_a^b f_x \mathrm{d}x$$

Equation holds for any choice of a and b: $\rho_t + (\rho u)_x = 0$

First-order models (Lighthill-Whitham-Richards)

Model: velocity uniquely given by density, $u = U(\rho)$. Yields flux function $f = Q(\rho) = \rho U(\rho)$. Scalar hyperbolic conservation law.

Second-order models (e.g., Payne-Whitham, Aw-Rascle-Zhang)

ρ and u are independent quantities; augment continuity equation by a second equation for velocity field (vehicle acceleration). System of hyperbolic conservation laws.

Macroscopic Traffic Models — Philosophy

Philosophy of macroscopic models

- Equations for macroscopic traffic variables.
- Usually lane-aggregated, but multi-lane models can also be formulated.
- Natural framework for multiscale phenomena, traveling waves, shocks.
- Established theory of control and coupling conditions for networks.
- Nice framework to fill gaps in incorporated measurement data.
- Mathematically related with other models, e.g., microscopic models, mesoscopic (kinetic) models, cell transmission models, stochastic models.
- Good for estimation and prediction, and for mathematical analysis of emergent features. Not the best framework if vehicle trajectories are of interest. Also, analysis and numerical methods for PDE are more complicated than for ODE.

Cellular Traffic Models — Cellular Automata

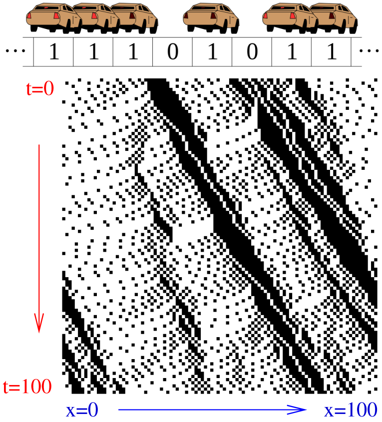
Nagel-Schreckenberg model

Cut up road into cells of width h.
Choose time step Δt.

Each cell is either empty or contains a single car. A car has a discrete speed.

In each time-step, do the following:

1. All cars have speed increased by 1.
2. For each car reduce its speed to the number of empty cells ahead.
3. Each car with speed ≥ 2 is slowed down by 1 with probability p.
4. Each car is moved forward its velocity number of cells.



Produces traffic jams that resemble quite well real observations.

Cellular Traffic Models — Cell Transmission Models

Cell Transmission Models (CTM)

- Cut up road into cells of width h. Choose time step Δt.
- Each cell contains an average density ρ_j. In each step, a certain flux of vehicles, $f_{j+\frac{1}{2}}$, travels from cell j to cell $j+1$. The fluxes between cells change the densities in the cells accordingly.
- The flux depends on the adjacent cells: $f_{j+\frac{1}{2}} = F(\rho_j, \rho_{j+1})$.
- The flux function F is based on a sending (demand) function of ρ_j, and a receiving (supply) function of ρ_{j+1}.
- The basic CTM is equivalent to a Godunov (finite volume) discretization of the LWR model.
- CTMs of second-order macroscopic models (e.g., ARZ) can be formulated.
- High-order accurate finite volume schemes can be formulated (both for LWR and for second-order models).

Traffic Models — Messages

Messages

- Cellular automaton models are easy to code, and resemble real jamming behavior. Not much further focus here.
- Cell-transmission models will be seen not as separate models, but as finite volume discretizations of macroscopic models.
- Microscopic models converge to macroscopic models as one scales vehicles smaller and smaller.
- First-order models (both micro and macro) exhibit shock (compression) waves, but no instabilities.
- Second-order models (both micro and macro) can have unstable uniform flow states (phantom traffic jams) that develop into traveling waves (traffic waves, jamitons).

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Overview

Continuity equation $\rho_t + (\rho u)_x = 0$

One equation, two unknown quantities ρ and u.

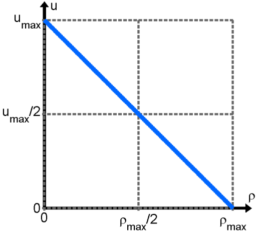
Simplest idea: model velocity u as a function of ρ.

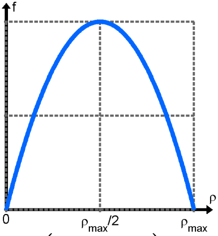
(i) alone on the road $\Rightarrow$ drive with speed limit: $u(0) = u_{\max}$

(ii) bumper to bumper $\Rightarrow$ complete clogging: $u(\rho_{\max}) = 0$

(iii) in between, use linear function: $u(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$

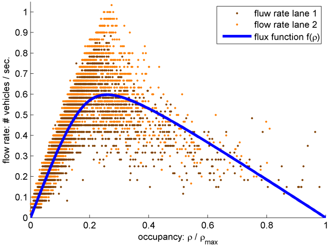
Lighthill-Whitham-Richards model (1950)





$$f(\rho) = \frac{\rho}{\rho_{\max}}\left(1 - \frac{\rho}{\rho_{\max}}\right) u_{\max}$$

A more realistic $f(\rho)$



Method of characteristics

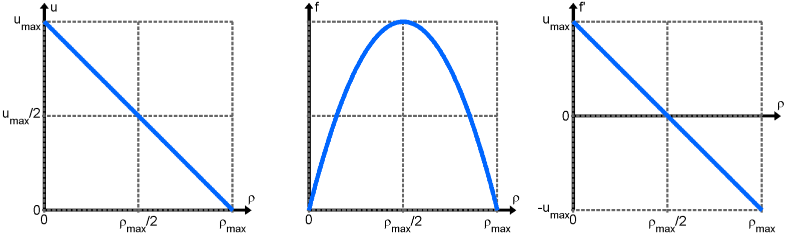
$$\rho_t + (f(\rho))_x = 0$$

Look at solution along a special curve $x(t)$. At this moving observer:

$$\frac{d}{dt}\rho(x(t),t) = \rho_x \dot{x} + \rho_t = \rho_x \dot{x} - (f(\rho))_x = \rho_x \dot{x} - f'(\rho)\rho_x = \left(\dot{x} - f'(\rho)\right)\rho_x$$

If we choose $\dot{x} = f'(\rho)$, then solution (ρ) is constant along the curve.

LWR flux function and information propagation

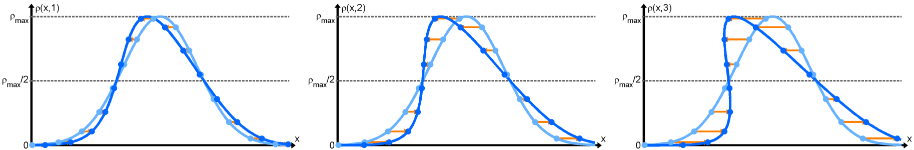


speed of vehicles | speed of information

Solution method

Let the initial traffic density $\rho(x,0) = \rho_0(x)$ be represented by points $(x, \rho_0(x))$. Each point evolves according to the characteristic equations

$$\begin{cases} \dot{x} = f'(\rho) \\ \dot{\rho} = 0 \end{cases}$$



Shocks

The method of characteristics eventually creates breaking waves.
In practice, a shock (= traveling discontinuity) occurs.
Interpretation: Upstream end of a traffic jam.

Note: A shock is a model idealization of a real thin zone of rapid braking.

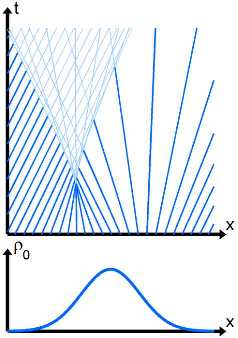
Characteristic form of LWR

LWR model $\qquad \rho_t + f(\rho)_x = 0 \qquad (1)$

in characteristic form: $\dot{x} = f'(\rho)$, $\dot{\rho} = 0$.

If initial conditions $\rho(x,0) = \rho_0(x)$ smooth (C^1), solution becomes non-smooth at time $t^* = -\frac{1}{\inf_x f''(\rho_0(x))\rho_0'(x)}$.

Reality exists for $t > t^*$, but PDE does not make sense anymore (cannot differentiate discont. function).



Weak solution concept

$\rho(x,t)$ is a weak solution if it satisfies

$$\int_0^\infty \int_{-\infty}^\infty \rho\phi_t + f(\rho)\phi_x \,\mathrm{d}x\mathrm{d}t = -\int_{-\infty}^\infty [\rho\phi]_{t=0}\,\mathrm{d}x \quad \forall \underbrace{\phi \in C_0^1}_{\text{test fct., } C^1 \text{ with compact support}} \qquad (2)$$

Theorem: If $\rho \in C^1$ ("classical solution"), then (1) $\iff$ (2).

Proof: integration by parts.

Weak formulation of LWR

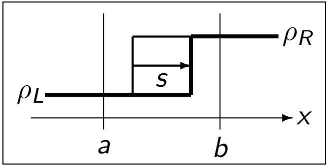
$$\int_0^\infty \int_{-\infty}^\infty \rho\phi_t + f(\rho)\phi_x \,\mathrm{d}x\mathrm{d}t = -\int_{-\infty}^\infty [\rho\phi]_{t=0}\,\mathrm{d}x \quad \forall\,\phi \in C_0^1$$

Every classical (C^1) solution is a weak solution.

In addition, there are discontinuous weak solutions (i.e., with shocks).

Riemann problem (RP)

$$\rho_0(x) = \begin{cases} \rho_L & x < 0 \\ \rho_R & x \geq 0 \end{cases}$$



Speed of shocks

The weak formulation implies that shocks move with a speed such that the number of vehicles is conserved:

RP: $\quad (\rho_L - \rho_R)\cdot s = \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \rho(x,t)\,\mathrm{d}x = f(\rho_L) - f(\rho_R)$

Yields: $\boxed{s = \frac{f(\rho_R)-f(\rho_L)}{\rho_R-\rho_L} = \frac{[f(\rho)]}{[\rho]}}$ Rankine-Hugoniot condition

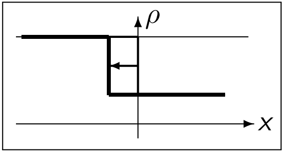
Weak formulation and Rankine-Hugoniot shock condition

$$\int_0^\infty \int_{-\infty}^\infty \rho\phi_t + f(\rho)\phi_x \,\mathrm{d}x\mathrm{d}t = -\int_{-\infty}^\infty [\rho\phi]_{t=0}\,\mathrm{d}x \;\forall \phi \in C_0^1 \quad ; \quad s = \frac{[f(\rho)]}{[\rho]}$$

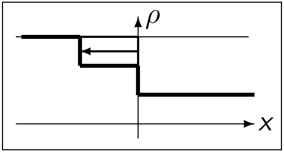
Problem

For RP with $\rho_L > \rho_R$, many weak solutions for same initial conditions.

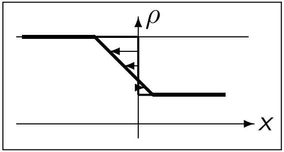
One shock



Two shocks



Rarefaction fan



Entropy condition

Single out a unique solution (the dynamically stable one $\longrightarrow$ vanishing viscosity limit) via an extra “entropy” condition:

Characteristics must go **into** shocks, i.e., $f'(\rho_L) > s > f'(\rho_R)$.

For LWR ($f''(\rho) < 0$): shocks must satisfy $\rho_L < \rho_R$.

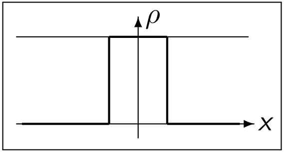
Exercises 1

Explain characteristic speed and shock speed graphically in FD plot.

Exercises 2 & 3

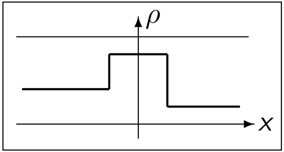
LWR model $\rho_t + f(\rho)_x = 0$ with Greenshields flux $f(\rho) = \rho(1-\rho)$.

Red light turning green



$$\rho_0(x) = \begin{cases} 0 & |x| < 1 \\ 1 & |x| \leq 1 \end{cases}$$

A bottleneck removed

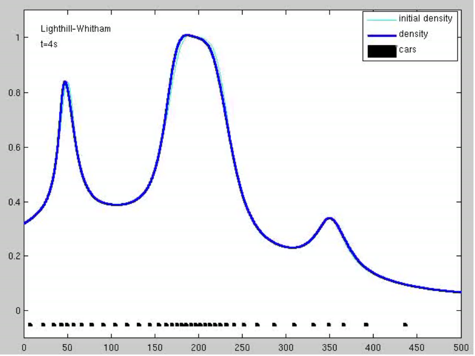


$$\rho_0(x) = \begin{cases} 0.4 & x < -1 \\ 0.8 & |x| \leq 1 \\ 0.2 & x > 1 \end{cases}$$

Exercise 4

What changes if we use Daganzo-Newell flux $f(\rho) = \frac{1}{2} - |\rho - \frac{1}{2}|$?

Time-evolution of density for LWR



Result

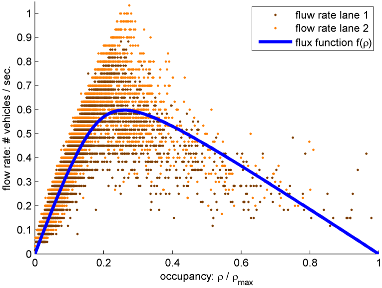
The LWR model quite nicely explains the shape of traffic jams (vehicles run into a shock).

Shortcomings of LWR

Perturbations never grow ("maximum principle").

Thus, phantom traffic jams cannot be explained with LWR.

And neither can multi-valued FD.



Overview

Modeling shortcoming of the LWR model

Vehicles adjust their velocity u instantaneously to the density ρ.
Conjecture: real traffic instabilities are caused by vehicles' inertia.

Payne-Whitham model (1979)

Vehicles at $x(t)$ moves with velocity $\dot{x}(t) = u(x(t), t)$. Thus its acceleration is

$$\frac{d}{dt}u(x(t), t) = u_t + u_x\dot{x} = u_t + u\,u_x$$

Model for acceleration: $u_t + u\,u_x = \frac{1}{\tau}\left(U(\rho) - u\right)$

Here $U(\rho)$ desired velocity (as in previous model), and τ relaxation time.

Full model:

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + u\,u_x \;\; = \frac{(p(\rho))_x}{\rho} + \frac{1}{\tau}\left(U(\rho) - u\right) \end{cases}$$

Here $p(\rho)$ traffic pressure (models preventive driving);
without this term, vehicles would collide.

Payne-Whitham model

$$\begin{cases} \rho_t + (\rho u)_x & = 0 \\ u_t + u\,u_x + \frac{(p(\rho))_x}{\rho} & = \frac{1}{\tau}\left(U(\rho) - u\right) \end{cases}$$

Examples for traffic pressure:

shallow water type: $p(\rho) = \frac{\beta}{2}\rho^2 \quad \Rightarrow \frac{p_x}{\rho} = \beta \rho_x$

singular at ρ_m: $p(\rho) = -\beta\left(\rho + \rho_m \log(1 - \frac{\rho}{\rho_m})\right) \quad \Rightarrow \frac{p_x}{\rho} = \frac{\beta\rho_x}{\rho_m - \rho}$

Model parameters:

- relaxation time τ ($\frac{1}{\tau}$ = aggressiveness of drivers).
- anticipation rate β.

System of hyperbolic conservation laws with relaxation term

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ \frac{1}{\rho}\frac{dp}{d\rho} & u \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = \begin{pmatrix} 0 \\ \frac{1}{\tau}(U(\rho) - u) \end{pmatrix}$$

Characteristic velocities: $v = u \pm c$, with speed of sound: $c^2 = \frac{dp}{d\rho}$.

Payne-Whitham (PW) Model [Whitham 1974], [Payne: Transp. Res. Rec. 1979]

$$\begin{cases} \rho_t + (\rho u)_x & = 0 \\ u_t + u u_x + \frac{1}{\rho} p(\rho)_x & = \frac{1}{\tau}(U(\rho) - u) \end{cases}$$

Parameters: pressure $p(\rho)$; desired velocity function $U(\rho)$; relaxation time τ

Second order model; vehicle acceleration: $u_t + u u_x = -\frac{p'(\rho)}{\rho}\rho_x + \frac{1}{\tau}(U(\rho) - u)$

Inhomogeneous Aw-Rascle-Zhang (ARZ) Model

[Aw&Rascle: SIAM J. Appl. Math. 2000], [Zhang: Transp. Res. B 2002]

$$\begin{cases} \rho_t + (\rho u)_x & = 0 \\ (u + h(\rho))_t + u(u + h(\rho))_x & = \frac{1}{\tau}(U(\rho) - u) \end{cases}$$

Parameters: hesitation function $h(\rho)$; velocity function $U(\rho)$; time scale τ

Second order model; vehicle acceleration: $u_t + u u_x = \rho h'(\rho) u_x + \frac{1}{\tau}(U(\rho) - u)$

Remark

The ARZ model addresses modeling shortcomings of PW (such as: negative speeds can arise, shocks may overtake vehicles from behind).

However, the arguments regarding instabilities and traffic waves apply to PW and ARZ in the same fashion. Below, we present things for PW (simpler expressions).

System of balance laws

$$\begin{pmatrix}\rho\\ u\end{pmatrix}_t + \begin{pmatrix}u & \rho\\ \frac{1}{\rho}\frac{dp}{d\rho} & u\end{pmatrix}\begin{pmatrix}\rho\\ u\end{pmatrix}_x = \begin{pmatrix}0\\ \frac{1}{\tau}(U(\rho)-u)\end{pmatrix}$$

Eigenvalues

$$\left\{\begin{array}{l}\lambda_1 = u - c\\ \lambda_2 = u + c\end{array}\right\} \quad c^2 = \frac{dp}{d\rho}$$

Linear stability analysis

(S) When are constant base state solutions $\rho(x,t) = \tilde{\rho}$, $u(x,t) = U(\tilde{\rho})$ stable (i.e. small perturbations do not amplify)?

Reduced equation

(R) When do solutions of the 2×2 system converge (as $\tau \to \infty$) to solutions of the reduced equation

$$\rho_t + (\rho U(\rho))_x = 0 \quad ?$$

Sub-characteristic condition (Whitham [1959])

(S) $\iff$ (R) $\iff \lambda_1 \le \mu \le \lambda_2$, where $\mu = (\rho U(\rho))'$

Stability for Payne-Whitham model

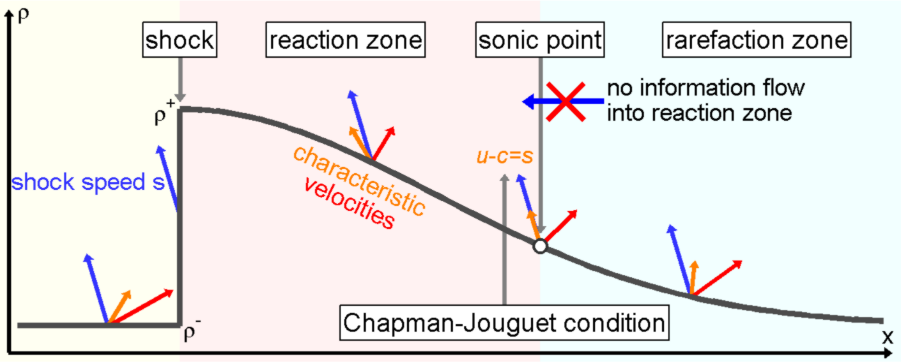
$$U(\rho) - c(\rho) \le U(\rho) + \rho U'(\rho) \le U(\rho) + c(\rho) \iff \frac{c(\rho)}{\rho} \ge -U'(\rho).$$

For $p(\rho) = \frac{\beta}{2}\rho^2$ and $U(\rho) = u_\mathrm{m}\left(1 - \frac{\rho}{\rho_\mathrm{m}}\right)$: Stability iff $\frac{\rho}{\rho_\mathrm{m}} \le \frac{\beta\rho_\mathrm{m}}{u_\mathrm{m}^2}$.

Phase transition: If enough vehicles on the road, perturbations grow.

Self-sustained detonation wave

Key Observation: ZND theory [Zel'dovich (1940), von Neumann (1942), Döring (1943)] applies to second order traffic models, such as PW and ARZ.



Reaction zone travels unchanged with speed of shock.
Rankine-Hugoniot conditions one condition short (unknown shock speed).
Sonic point is event horizon. It provides missing boundary condition.

PW model

$$\begin{cases} \rho_t + (\rho u)_x &= 0 \\ u_t + u u_x + \frac{1}{\rho} p(\rho)_x &= \frac{1}{\tau}(U(\rho) - u) \end{cases}$$

Traveling wave ansatz

$\rho = \rho(\eta)$, $u = u(\eta)$, with self-similar variable $\eta = \frac{x-st}{\tau}$.

Then $\rho_t = -\frac{s}{\tau}\rho'$, $\rho_x = \frac{1}{\tau}\rho'$, $u_t = -\frac{s}{\tau}u'$, $u_x = \frac{1}{\tau}u'$

and $p_x = \frac{1}{\tau}c^2\rho'$, $c^2 = \frac{dp}{d\rho}$

Continuity equation

$$\rho_t + (u\rho)_x = 0$$

$$-\frac{s}{\tau}\rho' + \frac{1}{\tau}(u\rho)' = 0$$

$$(\rho(u-s))' = 0$$

$$\rho = \frac{m}{u-s}$$

$$\rho' = -\frac{\rho}{u-s}u'$$

Momentum equation

$$u_t + u u_x + \frac{p_x}{\rho} = \frac{1}{\tau}(U - u)$$

$$-\frac{s}{\tau}u' + \frac{1}{\tau}uu' + \frac{dp}{d\rho}\frac{\rho'}{\rho} = \frac{1}{\tau}(U-u)$$

$$(u-s)u' - c^2\frac{1}{u-s}u' = U - u$$

$$u' = \frac{(u-s)(U-u)}{(u-s)^2 - c^2}$$

Ordinary differential equation for $u(\eta)$

$$u' = \frac{(u-s)(U(\rho)-u)}{(u-s)^2 - c(\rho)^2} \quad \text{where} \quad \rho = \frac{m}{u-s}$$

where

$$s = \text{travel speed of jamiton}$$
$$m = \text{mass flux of vehicles through jamiton}$$

Key point

In fact, m and s can not be chosen independently:

Denominator has root at $u = s + c$. Solution can only pass smoothly through this singularity (the sonic point), if $u = s + c$ implies $U = u$.

Using $u = s + \frac{m}{\rho}$, we obtain for this sonic density ρ_S that:

$$\begin{cases} \text{Denominator} & s + \frac{m}{\rho_S} = s + c(\rho_S) \implies m = \rho_S c(\rho_S) \\ \text{Numerator} & s + \frac{m}{\rho_S} = U(\rho_S) \implies s = U(\rho_S) - c(\rho_S) \end{cases}$$

Algebraic condition (Chapman-Jouguet condition [Chapman, Jouguet (1890)]) that relates m and s (and ρ_S).

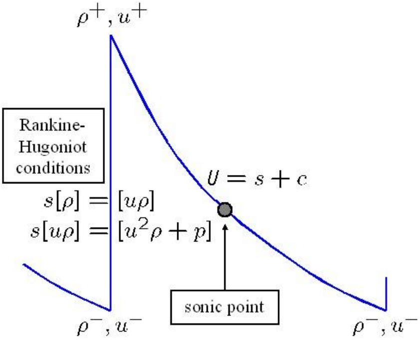
Jamiton ODE

$$u' = \frac{(u-s)(U(\frac{m}{u-s}) - u)}{(u-s)^2 - c(\frac{m}{u-s})^2}$$

Construction

1. Choose m. Obtain s by matching root of denominator with root of numerator.
2. Choose u^-. Obtain u^+ by Rankine-Hugoniot conditions.
3. ODE can be integrated through sonic point (from u^+ to u^-).
4. Yields length (from shock to shock) of jamiton, λ, and number of cars, $N = \int_0^\lambda \rho(x)dx$.

Periodic case

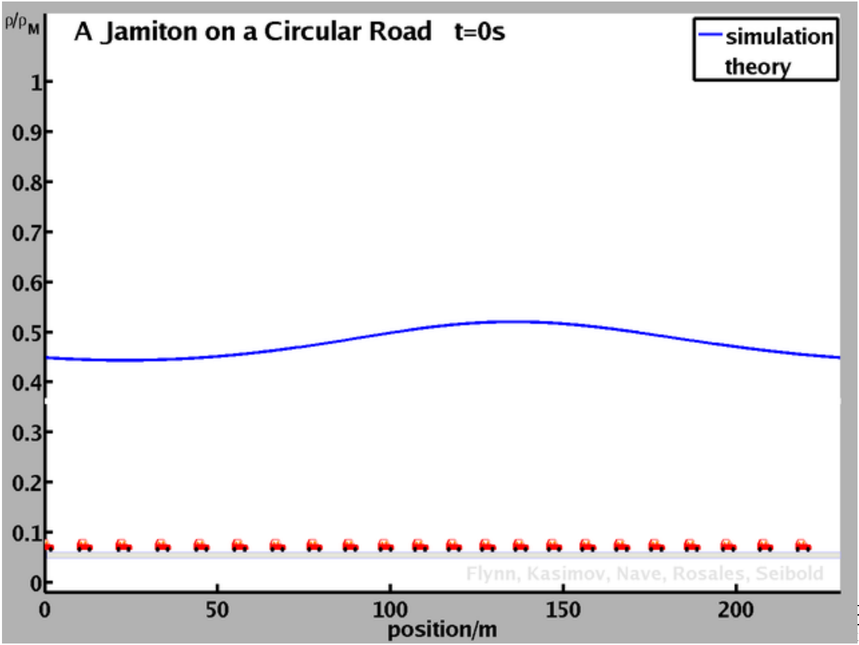


jamiton length $= O(\tau)$

Inverse construction

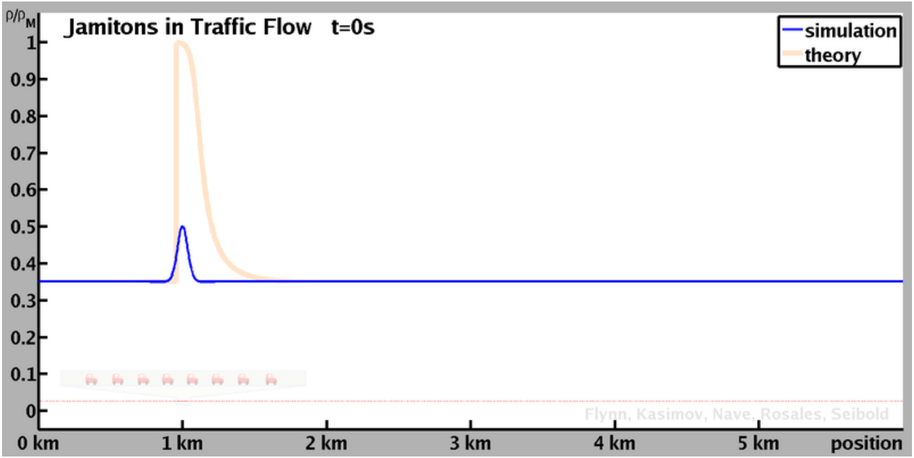
If λ and N are given, find jamiton by iteration.

Experiment: Jamitons on circular road [Sugiyama et al.: New J. of Physics 2008]



ρ/ρ_M
A Jamiton on a Circular Road t=0s
simulation
theory
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Flynn, Kasimov, Nave, Rosales, Seibold
0
50
100
150
200
position/m

Infinite road; lead jamiton gives birth to a chain of “jamitinos”.



Recall: continuity equation

$$\rho_t + (u\rho)_x = 0$$

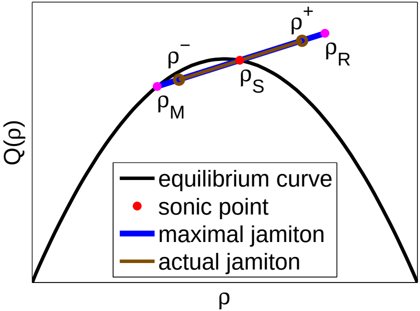
Traveling wave ansatz

$\rho = \rho(\eta)$, $u = u(\eta)$, where $\eta = \frac{x-st}{\tau}$, yields

$$\rho_t + (u\rho)_x = 0$$
$$-\frac{s}{\tau}\rho' + \frac{1}{\tau}(u\rho)' = 0$$
$$(\rho(u-s))' = 0$$
$$\rho(u-s) = m$$
$$q = m + s\rho$$

Hence: Any traveling wave is a line segment in the fundamental diagram.

A jamiton in the FD



- Plot equilibrium curve $Q(\rho) = \rho U(\rho)$
- Chose a ρ_S that violates the SCC
- Mark sonic point $(\rho_S, Q(\rho_S))$ (red)
- Set $m = \rho_S c(\rho_S)$ and $s = U(\rho_S) - c(\rho_S)$
- Draw maximal jamiton line (blue)
- Other jamitons with the same m and s are sub-segments (brown)

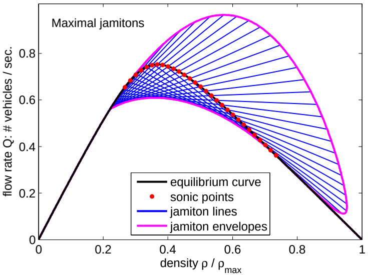
Construction of jamiton FD

For each sonic density ρ_S that violates the SCC: construct maximal jamiton.

Conclusion

Set-valued fundamental diagrams can be explained via traveling wave solutions in second-order traffic models.

Jamiton fundamental diagram



Temporal aggregation of jamitons

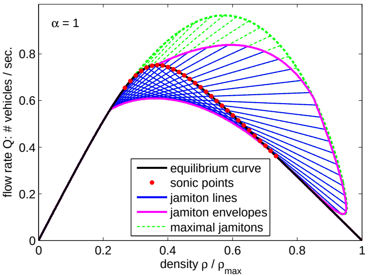
At fixed position $\bar{x}$, calculate all possible temporal average ($\Delta t = \alpha\,\tau$) densities

$$\bar{\rho} = \frac{1}{\Delta t}\int_0^{\Delta t} \rho(\bar{x},t)\mathrm{d}t\ .$$

Average flow rate: $\bar{q} = m + s\bar{\rho}$.

Resulting aggregated jamiton FD is a subset of the maximal jamiton FD.

Emulating sensor aggregation



Theorem

Jamitons always reduce the flow rate.

Remark

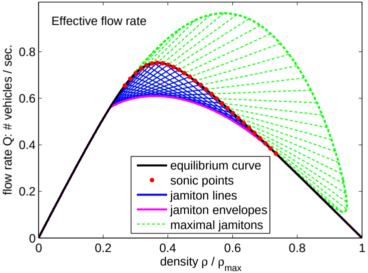
Statement is equivalent to: If $\Delta t \to \infty$, $(\bar{\rho}, \bar{q})$ lies below the equilibrium curve:

aggregate(=effective) flow rate of jamiton chain is lower than uniform base state with same average density.

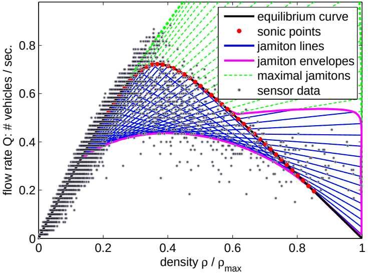
Good agreement with sensor data

We can reverse-engineer model parameters, such that the aggregated jamiton FD shows a good qualitative agreement with sensor data.

Infinite aggregation ($\Delta t \to \infty$)



Sensor data with jamiton FD



LWR model

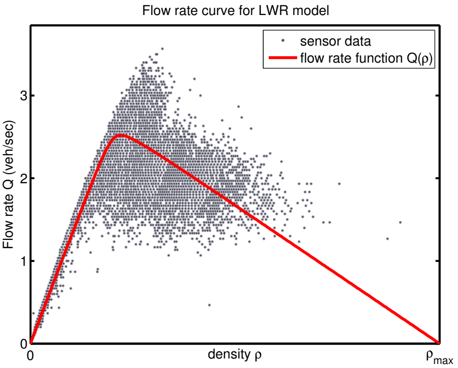
First-order model
(does not reflect spread in FD)

$$\left.\begin{aligned} \rho_t + (\rho U(\rho))_x &= 0 \\ \iff \quad \rho_t + Q(\rho)_x &= 0 \end{aligned}\right\} \longleftarrow \; Q(\rho) = \rho U(\rho)$$

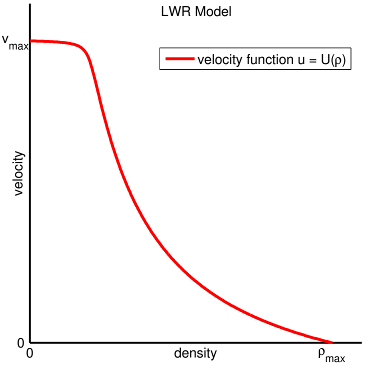
Construct function $Q(\rho)$

...via LSQ-fit to FD data.

Data-fitted flux $Q(\rho)$



Induced velocity curve $U(\rho)$



Aw-Rascle-Zhang (ARZ) model

$$\rho_t + (\rho u)_x = 0$$
$$(u + h(\rho))_t + u(u + h(\rho))_x = 0$$

where $h'(\rho) > 0$ and, WLOG, $h(0) = 0$.

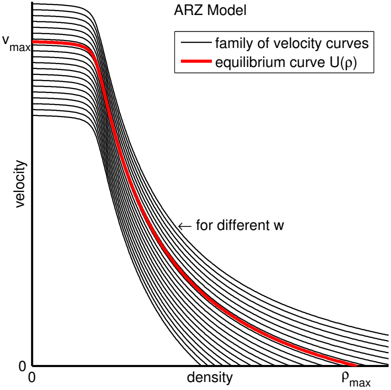
Equivalent formulation

$$\rho_t + (\rho u)_x = 0$$
$$w_t + u w_x = 0$$
$$\text{where } u = w - h(\rho)$$

Interpretation 1: Each vehicle (moving with velocity u) carries a characteristic value, w, which is its empty-road velocity. The actual velocity u is then: w reduced by the *hesitation function* $h(\rho)$.

Interpretation 2: ARZ is a generalization of LWR: different drivers have different $u_w(\rho)$.

ARZ model – velocity curves



one-parameter family of curves:

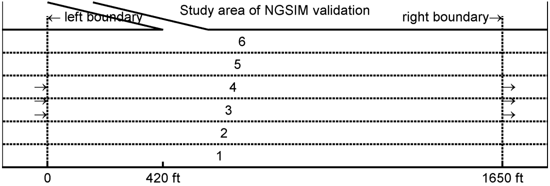
$$u = u_w(\rho) = u(w,\rho) = w - h(\rho)$$

here: $h(\rho) = v_{\max} - U(\rho)$

equilibrium (i.e., LWR) curve:

$$U(\rho) = u(v_{\max}, \rho)$$

NGSIM (I-80, Emeryville, CA; 2005)



- three 15 minute intervals
- precise trajectories of all vehicles (in 0.1s intervals)
- historic FD provided separately

Approach

- Construct macroscopic fields ρ and u from vehicle positions (via kernel density estimation)
- Use data to prescribe i.c. at $t = 0$ and b.c. at left and right side of domain
- Run PDE model to obtain $\rho^{\text{model}}(x, t)$ and $u^{\text{model}}(x, t)$ and error

$$E(x,t) = \frac{\left|\rho^{\text{data}}(x,t) - \rho^{\text{model}}(x,t)\right|}{\rho_{\max}} + \frac{\left|u^{\text{data}}(x,t) - u^{\text{model}}(x,t)\right|}{u_{\max}}$$

- Evaluate model error in a macroscopic (L^1) sense:

$$E = \frac{1}{TL}\int_0^T\int_0^L E(x,t)\,\mathrm{d}x\mathrm{d}t$$

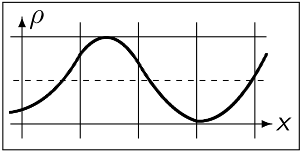
Space-and-time-averaged model errors for NGSIM data

Data set	LWR	ARZ
4:00–4:15	0.127 (+73%)	0.073
5:00–5:15	0.115 (+36%)	0.085
5:15–5:30	0.124 (+6%)	0.117

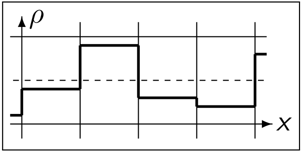
TEMPLE UNIVERSITY

Overview

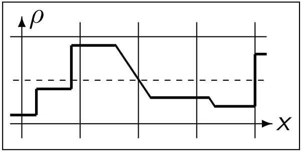
Initial condition



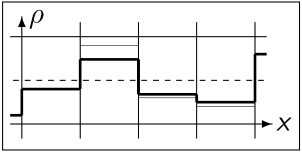
Cell-averaged initial cond.



Solution evolved exactly



Cell-averaged evolved solution



Godunov's method

REA = reconstruct–evolve–average

1. Divide road into cells of width h.
2. On each cell, store the average density ρ_j.
3. Assume solution is constant in each cell.
4. Evolve this piecewise constant solution exactly from t to $t + \Delta t$.
5. Average over each cell to obtain a pw-const. sol. again.
6. Go to step 4.

Godunov's method

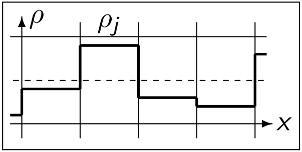
4. Evolve pw-const. sol. exactly from t to $t+\Delta t$.
5. Average over each cell.
6. Go to step 4.

Key Points

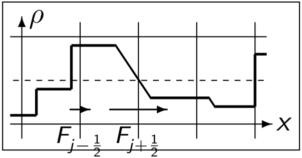
- If we choose $\Delta t < \frac{h}{2\max|f'|}$ ("CFL condition"), waves starting at neighboring cell interfaces never interact. Thus, can be solved as local Riemann problems.
- Because the exactly evolved solution is averaged again, all that matters for the change $\rho_j(t) \longrightarrow \rho_j(t+\Delta t)$ are the fluxes through the cell boundaries:

$$\rho_j(t+\Delta t) = \rho_j(t) + \frac{\Delta t}{h}(F_{j-\frac{1}{2}} - F_{j+\frac{1}{2}})$$

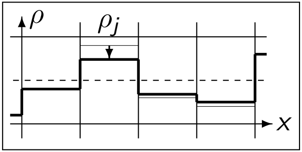
Solution at time t



Solution evolved exactly



Solution at time $t + \Delta t$



Godunov's method

$$\rho_j(t+\Delta t) = \rho_j(t) + \tfrac{\Delta t}{h}\left(F_{j-\frac{1}{2}} - F_{j+\frac{1}{2}}\right)$$

- right-going shock or rarefaction:
  $F_{j+\frac{1}{2}} = f(\rho_j)$
- left-going shock or rarefaction:
  $F_{j+\frac{1}{2}} = f(\rho_{j+1})$
- transsonic rarefaction:
  $F_{j+\frac{1}{2}} = f(\rho_c) = \max_\rho f(\rho)$

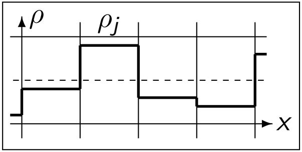
Equivalent formulation of fluxes $\longrightarrow$ CTM

$F_{j+\frac{1}{2}} = \min\{D(\rho_j), S(\rho_{j+1})\}$

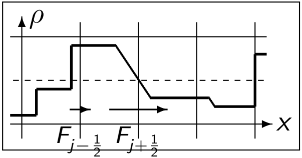
where:

- demand function: $D(\rho) = f(\min(\rho, \rho_c))$
- supply function: $S(\rho) = f(\max(\rho, \rho_c))$

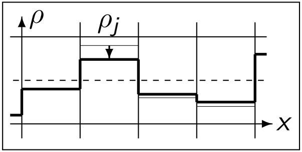
Solution at time t



Solution evolved exactly



Solution at time $t+\Delta t$



Cell transmission model (CTM)

$$\rho_j(t+\Delta t) = \rho_j(t) + \tfrac{\Delta t}{h}\left(F_{j-\frac{1}{2}} - F_{j+\frac{1}{2}}\right)$$

Flux between cells:

$$F_{j+\frac{1}{2}} = \min\{D(\rho_j), S(\rho_{j+1})\}$$

- demand function: $D(\rho) = f(\min(\rho, \rho_c))$
- supply function: $S(\rho) = f(\max(\rho, \rho_c))$

Principles

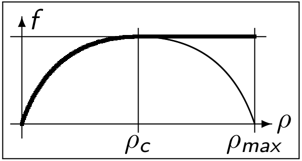
The flux of vehicles between cells j and $j+1$ cannot exceed:

a) the demand (on road capacity) that the sending cell j requires;

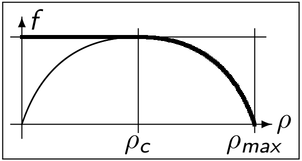
b) the supply (of road capacity) that the receiving cell $j+1$ provides.

The actual flux is the maximum flux that satisfies these constraints.

Demand function



Supply function



Convergence of the Godunov method (or CTM)

- As $\Delta t = Ch \to 0$, the sequence of approximate solutions converges (in L^1) to the unique weak entropy solution of the LWR model $\rho_t + f(\rho)_x = 0$.
- For any fixed Δt and h, the temporal evolution that Godunov=CTM provides is only an approximation to the LWR model.
- To leading order, Godunov=CTM solutions behave like solutions of the "modified equation" $\rho_t + f(\rho)_x = ch\rho_{xx}$.
- Hence, the error between Godunov=CTM and LWR scales like $O(h)$; first-order numerical scheme.
- Also, shocks get smeared out over several cells.

- Other, more accurate (high-order) numerical schemes for LWR exist (MUSCL, discontinuous Galerkin, etc.). These are more complicated.
- CTMs can also be cast for second-order traffic models.

Overview

The simplest network: 1 in-road; 1 out-road

Example: Greenshields flux

- road 1 has 4 lanes and speed limit 2: $f_1(\rho) = 2(1 - \rho/4)\rho$
- road 2 has 3 lanes and speed limit 4: $f_2(\rho) = 4(1 - \rho/3)\rho$

Cell transmission model (CTM)

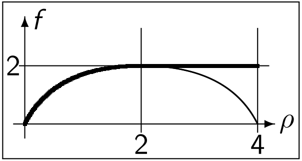
On each road segment, apply CTM as before.

Interface between last cell of road 1 and first cell of road 2: apply same principle as well.

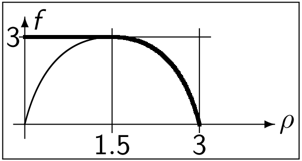
$$F_{\text{road } 1 \to \text{road } 2} = \min\{D_1(\rho_1), S_2(\rho_2)\}$$

- demand function: $D_1(\rho) = f_1(\min(\rho, \rho_c^1))$
- supply function: $S_2(\rho) = f_2(\max(\rho, \rho_c^2))$

Demand function



Supply function



Nice

As simple to code as the basic CTM.

Generalized Riemann problem

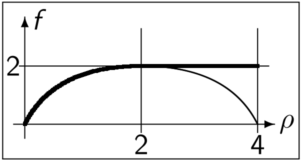
Given state ρ_1 on road 1, and state ρ_2 on road 2, determine new states $\hat{\rho}_1$ and $\hat{\rho}_2$ that solution will assume.

1. Calculate demand and supply: $D_1(\rho_1)$ and $S_2(\rho_2)$.
2. Maximum flux that does not exceed demand and supply: $F = \min\{D_1(\rho_1), S_2(\rho_2)\}$.
3. Choose new states so that no waves travel into the interface:

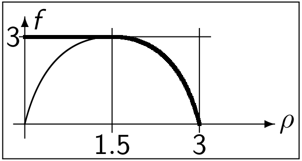
$$\hat{\rho}_1 = \begin{cases} \rho_1 & F = f_1(\rho_1) \\ f_1^{-1,\oplus}(F) & F \neq f_1(\rho_1) \end{cases}$$

$$\hat{\rho}_2 = \begin{cases} \rho_2 & F = f_2(\rho_2) \\ f_2^{-1,\ominus}(F) & F \neq f_2(\rho_2) \end{cases}$$

Demand function



Supply function



Exercise

$\rho_1 = 2.5$ and $\rho_2 = 2.5$

$D_1 = 2$ and $S_2 = 5/3$

$\hat{\rho}_1 \approx 2.8$ and $\hat{\rho}_2 = 2.5$

1-in-2-out node

In-road: $f_1(\rho_1)$; out-roads: $f_2(\rho_2)$ and $f_3(\rho_3)$.

Additional info: split ratios $\alpha_2 + \alpha_3 = 1$ (how many cars go where).

Generalized Riemann problem

1. Calculate demand and supplies: $D_1(\rho_1)$ and $S_2(\rho_2)$, $S_3(\rho_3)$.
2. Flux is maximum flux that does not exceed demand and supply.
   Constraints: $F \leq D_1(\rho_1)$; $\alpha_2 F \leq S_2(\rho_2)$; $\alpha_3 F \leq S_3(\rho_3)$.
   Yields: $F = \min\{D_1(\rho_1), \frac{S_2(\rho_2)}{\alpha_2}, \frac{S_3(\rho_3)}{\alpha_3}\}$.
3. If all we want is CTM, we are done ($F_1 = F$; $F_2 = \alpha_2 F$; $F_3 = \alpha_3 F$).
4. If we need new states: no waves travel into the interface.

$$\hat{\rho}_1 = \begin{cases} \rho_1 & F_1 = f_1(\rho_1) \\ f_1^{-1,\oplus}(F_1) & F_1 \neq f_1(\rho_1) \end{cases}$$

$$\hat{\rho}_2 = \begin{cases} \rho_2 & F_2 = f_2(\rho_2) \\ f_2^{-1,\ominus}(F_2) & F_2 \neq f_2(\rho_2) \end{cases} \qquad \hat{\rho}_3 = \begin{cases} \rho_3 & F_3 = f_3(\rho_3) \\ f_3^{-1,\ominus}(F_3) & F_3 \neq f_3(\rho_3) \end{cases}$$

Generalized Riemann problem for off-ramp

1. Calculate demand and supplies: $D_1(\rho_1)$ and $S_2(\rho_2)$, $S_3(\rho_3)$.
2. Flux is maximum flux that does not exceed demand and supply. Constraints: $F \leq D_1(\rho_1)$; $\alpha_2 F \leq S_2(\rho_2)$; $\alpha_3 F \leq S_3(\rho_3)$. Yields: $F = \min\{D_1(\rho_1), \frac{S_2(\rho_2)}{\alpha_2}, \frac{S_3(\rho_3)}{\alpha_3}\}$.

Wait a minute. . .

1. If the off-ramp is clogged ($\rho_3 = \rho_3^{\max}$), this produces $F = 0$, i.e., no traffic passes on along the highway.
2. No contradiction, because we required F_2 and F_3 to be distributed according to split ratios: FIFO (first-in-first-out) model.
3. One can also formulate non-FIFO models. Problem there: consequences of queuing of vehicles that want to exit are neglected.
4. Fix of these issues: [J. Somers: *Macroscopic Coupling Conditions with Partial Blocking for Highway Ramps*, MS Thesis, Temple University, 2015].

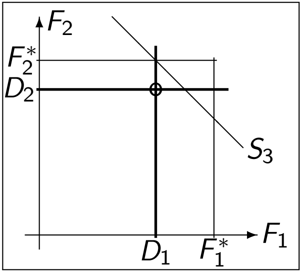
2-in-1-out node

In-roads: $f_1(\rho_1)$ and $f_2(\rho_2)$; out-road: $f_3(\rho_3)$.

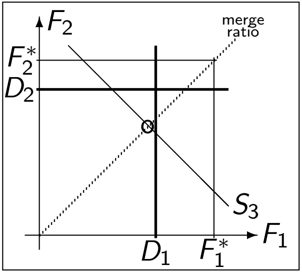
Generalized Riemann problem

1. Calculate demands and supply: $D_1(\rho_1)$, $D_2(\rho_2)$ and $S_3(\rho_3)$.
2. Now we have two fluxes to be determined: F_1 and F_2.
3. Constraints on flux vector $\vec{F} = (F_1, F_2)$: $0 \leq F_1 \leq D_1(\rho_1)$; $0 \leq F_2 \leq D_2(\rho_2)$; $F_1 + F_2 \leq S_3(\rho_3)$.
4. Maximize flux $F_1 + F_2$.
5. If that non-unique, use additional information: merge ratio $F_1/F_2 = \gamma$.
6. Use fluxes for CTM, or obtain new states, as before.

All traffic can get through



Not all traffic gets through



General Networks

- The presented 1-in-1-out, 1-in-2-out, and 2-in-1-out nodes already enable us to put together large road networks.
- Can use CTM to compute LWR solutions on such a network.
- Even for the simple on- and off-ramp case, there is still some modeling and validation work to be done with regards to the question which choices of fluxes are most realistic.
- The general n-in-m-out case can be treated similarly. However, the constrained flux maximization leads to a linear program that may need to be solved numerically (can be costly).
- Second-order models: coupling conditions for the ARZ model have been developed, albeit not for the most general case.

Overview

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Microscopic traffic models

- Vehicle trajectories: $x_1(t), x_2(t), \ldots, x_N(t)$.
- Popular framework: car-following models.
- First-order models: $\dot{x}_j = V(x_{j+1} - x_j)$.
- Second-order models: $\ddot{x}_j = G(x_{j+1} - x_j, \dot{x}_j, \dot{x}_{j+1})$.
- Scaling: introduce reference distance ΔX, scale space headway by $\frac{x_{j+1}-x_j}{\Delta X}$, and place $N \propto \frac{1}{\Delta X}$ on the roadway (spaced $\propto \Delta X$).
- First-order micro model converges (as $\Delta X \to 0$) to LWR with same velocity function.
- Certain second-order microscopic models converge to different types of second-order macroscopic models.

  Example: $\ddot{x}_j = \alpha\, \Delta X \frac{\dot{x}_{j+1}-\dot{x}_j}{x_{j+1}-x_j} + \beta\left(V\left(\frac{x_{j+1}-x_j}{\Delta X}\right) - \dot{x}_j\right)$

  converges to ARZ model with logarithmic pressure.
- Due to this connection, second-order microscopic models can exhibit unstable uniform flow ("phantom traffic jams") and traffic waves.

Car following model with parameters determined so that traffic waves of ring-road experiment are reproduced in quantitative agreement

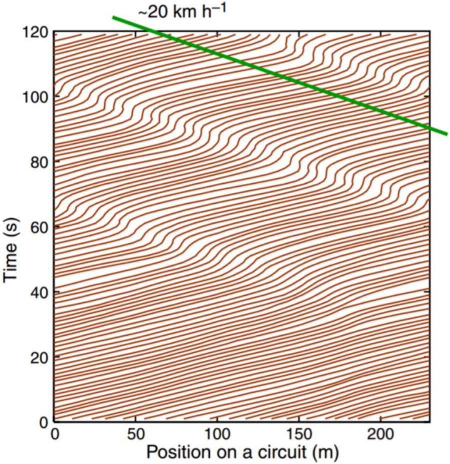

Figure 1: Trajectories of the experiment.

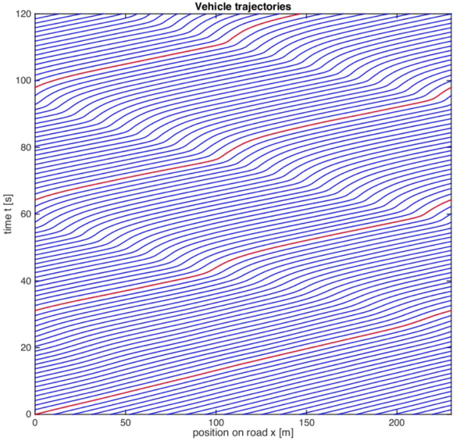

Figure 2: Trajectories of the model.