Minrank: On the complexity of orbit closures

Christian Ikenmeyer

UNIVERSITY OF LIVERPOOL

IPAM
Tensor Methods and Emerging Applications to the Physical and Data Sciences
Workshop IV: Efficient Tensor Representations for Learning and Computational Complexity
2021-May-17

based on "On the Orbit Closure Containment Problem and Slice Rank of Tensors", SODA21
and "Variety Membership Testing, Algebraic Natural Proofs, and Geometric Complexity Theory"
with Bläser, Lysikov, Pandey, Schreyer

Agenda

Agenda

- A matrix A has rank $\leq r$ iff all $(r+1) \times (r+1)$ minors of A vanish.
- Given a sequence of matrices $A_1, A_2, A_3, \dots$ in $\mathbb{C}^{n \times n}$ with $\lim_{i\to\infty} A_i = A$ and $\forall i : \mathrm{rk}(A_i) \leq r$. Then by the continuity of the minors we have $\mathrm{rk}(A) \leq r$.

Definition (tensor rank)

A tensor $t \in U \otimes V \otimes W$ has **rank 1** (also called **decomposable**) iff there exist vectors $u \in U, v \in V, w \in W$ with $t = u \otimes v \otimes w$.
The **tensor rank** $R(t)$ is the smallest r such that t can be written as a sum of r rank 1 tensors.

- $(e_1 + \frac{1}{i} e_2)^{\otimes 3} = e_1^{\otimes 3} + \frac{1}{i}(e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1) + \frac{1}{i^2}(\dots) + \frac{1}{i^3}(\dots)$
- Hence $\lim_{i\to\infty} i\left((e_1 + \frac{1}{i} e_2)^{\otimes 3} - e_1^{\otimes 3}\right) = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1$
- $e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1$ is a rank 3 tensor that is a limit of rank 2 tensors.

Definition (border rank)

The **border rank** $\underline{R}(t)$ is the smallest r such that t can be approximated arbitrarily closely by tensors of rank $\leq r$.

Example: $\underline{R}(e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1) = 2$

Given t and r, it is NP-hard to decide whether or not $R(t) \leq r$ [Håstad 1990].

Open Question

Given t and r, how hard is it to decide $\underline{R}(t) \leq r$?

Agenda

Agenda

The Minrank problem

Given a tensor $t \in U \otimes V \otimes W$, the **minrank** of t is defined as the smallest rank r of any matrix that can be obtained as a nontrivial linear combination of the $\dim U$ many slice matrices of format $\dim V \times \dim W$.
In other words,

$$\mathrm{minrank}(t) = \min_{0 \neq \varphi \in U^*} \{\mathrm{rk}(\varphi t)\}.$$

Example: Let $U = V = W = \mathbb{C}^2$. Let t be the $2 \times 2 \times 2$ tensor with slices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We have $\mathrm{rk}(A_1 + iA_2) = 1$, so $\mathrm{minrank}(t) = 1$. (Remark: over the reals the minrank is 2.)

- We can define **border minrank** analogously to border rank:
  The border minrank of t, denoted $\underline{\mathrm{minrank}}(t)$, is the smallest r such that t can be approximated arbitrarily closely by tensors of minrank $\leq r$.
- We will see now that $\underline{\mathrm{minrank}}(t) = \mathrm{minrank}(t)$

Recall: $\mathrm{minrank}(t) = \min_{0\neq\varphi\in U^*}\{\mathrm{rk}(\varphi t)\}$.

Proposition: Minrank equals border minrank

For all t we have $\mathrm{minrank}(t) = \underline{\mathrm{minrank}}(t)$.

Proof: We mimic [Sawin-Tao, 2016] on slice rank: We write the minrank-variety as a projection along a projective variety.

Define the variety

$$\mathbb{P}\mathcal{X}_{U\otimes V\otimes W,r} = \{([t],[\varphi]) \in \mathbb{P}(U\otimes V\otimes W)\times\mathbb{P}U^* \mid \mathrm{rk}(\varphi t)\le r\} \subset \mathbb{P}(U\otimes V\otimes W)\times\mathbb{P}U^*.$$

Let $\pi\colon \mathbb{P}(U\otimes V\otimes W)\times\mathbb{P}U^* \to \mathbb{P}(U\otimes V\otimes W)$ be the projection onto the first component of the product.

$$\pi\mathbb{P}\mathcal{X}_{U\otimes V\otimes W,r} = \{[t]\in\mathbb{P}(U\otimes V\otimes W) \mid \exists\varphi\neq 0\colon \mathrm{rk}(\varphi t)\le r\}.$$

As an image of a projective variety, it is a closed subvariety of $\mathbb{P}(U\otimes V\otimes W)$ (see e. g. Shafarevich).
The affine cone over this subvariety is therefore also closed.
This affine cone is exactly the set of tensors of minrank $\le r$. $\square$

Agenda

Agenda

MinRank1

Given a tensor t with entries from $\mathbb{Q}$, the MinRank1 problem is to decide if $\operatorname{minrank}(t) \leq 1$.

Quad

Given a set of quadratic forms with coefficients from $\mathbb{Q}$, the Quad problem is to decide if they have a common solution.

Theorem

Quad is NP-hard, even if all coefficients are from $\{-1, 0, 1\}$.

Proof:

We reduce from graph 3-colorability. Let $G = (V, E)$ be a graph.

- Two variables for each vertex: x_v and y_v. One global additional variable z.
- Three equations for each vertex: $x_v y_v = 0, \;\; x_v^2 - x_v z = 0, \;\; y_v^2 - y_v z = 0$
- One equation for each edge (v, w): $x_v^2 + y_v^2 + x_w^2 + y_w^2 - x_v y_w - x_w y_w - z^2 = 0$

If $z = 0$, there is only the trivial solution. Assume $z \neq 0$, rescale $z = 1$.

When $z = 1$, the vertex equations give three possibilities: $(x_v, y_v) \in \{(0,0), (0,1), (1,0)\}$.

The edge equation on these inputs is satisfied iff $(x_v, y_v) \neq (x_w, y_w)$.

Hence, for $z = 1$ we get a 1:1 correspodence between solutions and 3-colorings. $\square$

MinRank1

Given a tensor t with entries from $\mathbb{Q}$, the MinRank1 problem is to decide if $\operatorname{minrank}(t) \leq 1$.

Quad (NP-hard)

Given a set of quadratic forms with coefficients from $\mathbb{Q}$, the Quad problem is to decide if they have a common solution.

Theorem

MinRank1 and Quad are polynomial-time equivalent. Hence MinRank1 is NP-hard.

Proof: One direction is clear, because MinRank1 is decided by the vanishing of 2×2 minors. Other direction:
Given k quadrics on $\mathbb{C}^n$. Each quadric $q(x) = \sum_{i,j} a_{ij} x_i x_j$ corresponds to the linear map $Q(x) = \sum_{i,j} a_{ij} x_{i,j}$ on $\operatorname{Sym}^2 \mathbb{C}^n$. We have $q(x) = 0$ iff $Q(x \otimes x) = 0$.
k linear forms give a linear map $L : \operatorname{Sym}^2 \mathbb{C}^n \to \mathbb{C}^k$ with $x \otimes x \in \ker(L)$ iff x is a common zero of the quadrics.
Compute $\ker(L) = \langle A_1, \ldots, A_m \rangle$ with linearly independent symmetric matrices A_i.
Set $t = \sum_{i=1}^m e_i \otimes A_i$.
If x is a nontrivial zero of the quadrics, then $x \otimes x \in \ker L$, hence $x \otimes x = \sum_{i=1}^m y_i A_i$, so $\operatorname{minrank}(t) \leq 1$.
If $\operatorname{minrank}(t) = 1$, then $\exists y : \sum_{i=1}^m y_i A_i$ is symmetric and rank 1, so $\sum_{i=1}^m y_i A_i = x \otimes x$, hence x is a common 0.
If $\operatorname{minrank}(t) = 0$ is impossible, because the A_i are linearly independent. $\square$

NP-hard:

$\textsc{MinRank1}$

Given a tensor t with entries from $\mathbb{Q}$, the $\textsc{MinRank1}$ problem is to decide if $\mathrm{minrank}(t) \leq 1$.

This makes determining lower bounds coNP-hard:

$\textsc{MinRank}{>}1$

Given a tensor t with entries from $\mathbb{Q}$, the $\textsc{MinRank}{>}1$ problem is to decide if $\mathrm{minrank}(t) > 1$.

If NP$\neq$coNP, then $\textsc{MinRank}{>}1$ is **not in NP**, i.e., there are no polytime-verifiable minrank lower bound proofs.

Agenda

Agenda

- $U = \mathbb{C}^k$, $V = W = \mathbb{C}^n$.
- Embed $\mathbb{C}^n \hookrightarrow \mathbb{C}^s$ with $s = n(k-1) + r$

$$t_{k,n,r} \quad := \quad e_1 \otimes \underbrace{(\sum_{j=1}^{r} e_{1j} \otimes e_{1j})}_{r \times r \text{ identity matrix block}} + \sum_{i=2}^{k} e_i \otimes \underbrace{(\sum_{j=1}^{n} e_{ij} \otimes e_{ij})}_{n \times n \text{ identity matrix block}} \quad \in \quad \mathbb{C}^k \otimes \mathbb{C}^s \otimes \mathbb{C}^s$$

Block diagonal, with each block on its own slice.

Group action of $G := \mathsf{GL}(U) \times \mathsf{GL}(V) \times \mathsf{GL}(W)$ on $U \otimes V \otimes W$: $(g_1, g_2, g_3)(u \otimes v \otimes w) := (g_1 u) \otimes (g_2 v) \otimes (g_3 w)$.

Proposition

For $t \in U \otimes V \otimes W$ we have $\mathrm{minrank}(t) \leq r$ iff $t \in \overline{(\mathsf{GL}_k \times \mathsf{GL}_s \times \mathsf{GL}_s) t_{k,n,r}}$.

(note the similarity of this formulation to the null cone problem)

Corollary

For tensors of order ≥ 3 and for the standard action of $\mathsf{GL} \times \mathsf{GL} \times \cdots \times \mathsf{GL}$ it is NP-hard to decide if a tensor lies in the orbit closure of another tensor.

Agenda

Agenda

- [Mulmuley-Sohoni 2001, 2008] introduce the method of **multiplicity obstructions** that can sometimes be used to prove that a point does not lie in an orbit closure.
- [Bürgisser-I 2011, 2013] use this method to prove modest lower bounds for the border rank of the matrix multiplication tensor.
- This method can also be used for lower bounds on minrank.

- $t_{k,n,r} := e_1 \otimes (\sum_{j=1}^{r} e_{1j} \otimes e_{1j}) + \sum_{i=2}^{k} e_i \otimes (\sum_{j=1}^{n} e_{ij} \otimes e_{ij}) \in \mathbb{C}^k \otimes \mathbb{C}^s \otimes \mathbb{C}^s$
- $G := \mathsf{GL}_k \times \mathsf{GL}_s \times \mathsf{GL}_s$
- The stabilizer $\mathrm{stab}_{k,n,r}$ of $t_{k,n,r}$ is isomorphic to $(\mathsf{GL}_r \times \mathbb{C}^\times) \times (\mathsf{GL}_n \times \mathbb{C}^\times)^{k-1} \rtimes \mathfrak{S}_{k-1}$.
- $Gt_{k,n,r} \simeq G/\mathrm{stab}_{k,n,r}$.

$G := \mathsf{GL}_k \times \mathsf{GL}_s \times \mathsf{GL}_s \qquad t_{k,n,r} := e_1 \otimes \left(\sum_{j=1}^r e_{1j} \otimes e_{1j}\right) + \sum_{i=2}^k e_i \otimes \left(\sum_{j=1}^n e_{ij} \otimes e_{ij}\right) \in \mathbb{C}^k \otimes \mathbb{C}^s \otimes \mathbb{C}^s$

Let $I(\overline{Gt_{k,n,r}})_\delta$ denote the homogeneous degree δ part of the vanishing ideal. It is a G-representation.
Let $\mathsf{mult}_{(\lambda,\mu,\nu)} I(\overline{Gt_{k,n,r}})_\delta$ denote the multiplicity of the irred. G-representation of type (λ,μ,ν) in $I(\overline{Gt_{k,n,r}})_\delta$.

We mimic Mulmuley-Sohoni's representation theoretic lower bounds method:

$$\mathsf{mult}_{(\lambda,\mu,\nu)} I(\overline{Gt_{k,n,r}})_\delta = \underbrace{k(\lambda,\mu,\nu)}_{\text{Kronecker coeff.}} - \underbrace{\mathsf{mult}_{(\lambda,\mu,\nu)} \mathbb{C}[\overline{Gt_{k,n,r}}]_\delta}_{\le \mathsf{mult}_{(\lambda,\mu,\nu)} \mathbb{C}[Gt_{k,n,r}]_\delta}$$

So to find representations in $I(\overline{Gt_{k,n,r}})_\delta$ it suffices to find (λ,μ,ν) with $\mathsf{mult}_{(\lambda,\mu,\nu)} \mathbb{C}[Gt_{k,n,r}]_\delta < k(\lambda,\mu,\nu)$.

$\mathsf{mult}_{(\lambda,\mu,\nu)} \mathbb{C}[Gt_{k,n,r}] = \mathsf{mult}_{(\lambda,\mu,\nu)} \mathbb{C}[G/\mathrm{stab}_{k,n,r}] = \mathsf{mult}_{(\lambda,\mu,\nu)} \mathbb{C}[G]^{\mathrm{stab}_{k,n,r}} =$

$$\sum_{\substack{\xi \trianglelefteq \lambda \quad \xi \vdash_{k-1} \\ \mu^1 \vdash_r |\lambda| - |\xi| \\ \mu^2,\ldots,\mu^k \vdash_m \\ |\mu^2| \le |\mu^3| \le \cdots \le |\mu^k| \le |\xi|}} \; \sum_{\substack{\delta^1,\ldots,\delta^{k-1} \vdash_{\ell(\xi)} \\ |\delta^i| = i\kappa_i}} c^{\mu}_{\mu^1,\ldots,\mu^k} c^{\nu}_{\mu^1,\ldots,\mu^k} c^{\xi}_{\delta^1,\ldots,\delta^{k-1}} \prod_{i=1}^{|\xi|} a_{\delta^i}(\kappa_i, i),$$

where the κ_i denotes the number of times i occurs in $(|\mu^2|,\ldots,|\mu^k|)$.

$\mathsf{mult}_{(\lambda,\mu,\nu)} \mathbb{C}[Gt_{2,3,1}]_6 = \boxed{0 < 1} = k(\lambda,\mu,\nu)$ for $(\lambda,\mu,\nu) \in$
$\Big\{ \Big((3,3),(2,2,2),(3,3)\Big), \Big(((3,3),(3,3),(2,2,2)\Big), \Big((3,3),(2,2,2),(4,1,1)\Big), \Big((3,3),(4,1,1),(2,2,2)\Big) \Big\}$

The only type we miss is $\Big((3,3),(3,2,1),(3,2,1)\Big)$, where both multiplicities are 2.

Summary

- Deciding tensor rank $\leq r$ is NP-hard.
- Open question: Can border rank $\leq r$ be decided in polynomial time?
- Minrank $=$ border minrank, and deciding minrank $\leq r$ is NP-hard. (also true for slice rank)
- Corollary: Detecting if a tensor is in the orbit closure of another tensor is NP-hard.
- If NP$\neq$coNP, then minrank lower bounds are not in NP.
- But still, representation theoretic multiplicites based on symmetries give equations for minrank. How far can this approach go in other settings such as geometric complexity theory?

Thank you for your attention! Questions/comments?