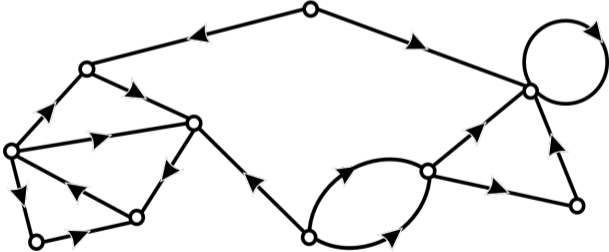


principal components along quiver representations

Anna Seigal

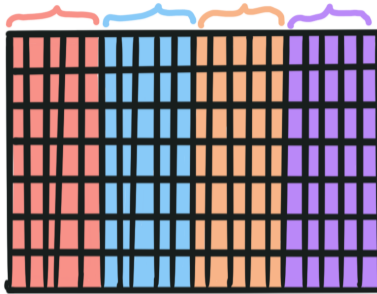
University of Oxford

joint work with Vidit Nanda and Heather Harrington



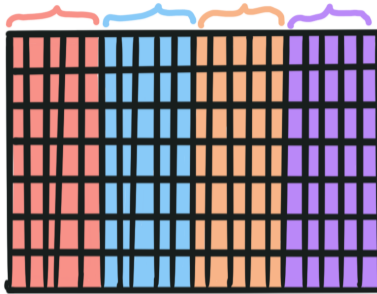
May 17, 2021

tensor of data



¹GTEx Consortium. The Genotype-Tissue Expression (GTEx) pilot analysis: Multitissue gene regulation in humans. 2015

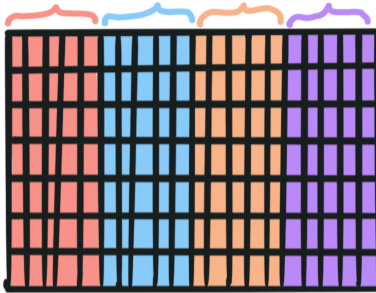
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- the rows are samples $\{y_1, \dots, y_N\}$

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tensor of data

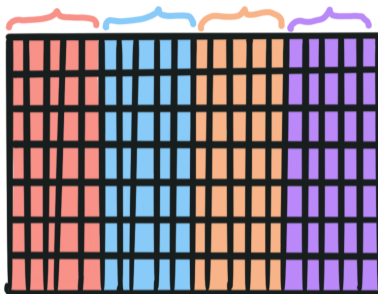


- the rows are samples $\{y_1, \dots, y_N\}$
- each column is a measurement

e.g. here there are 20 measurements

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tensor of data



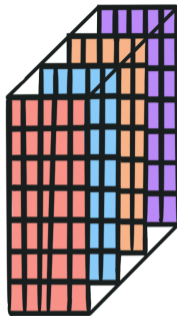
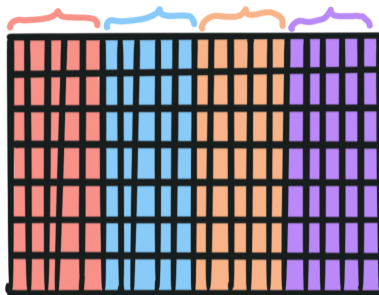
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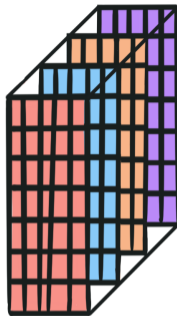
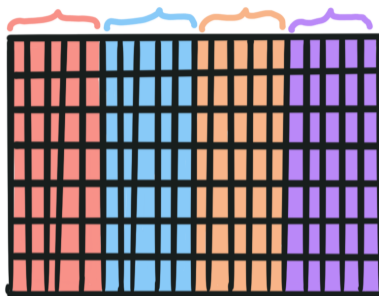
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e.g. here, it is of size $N \times 5 \times 4$

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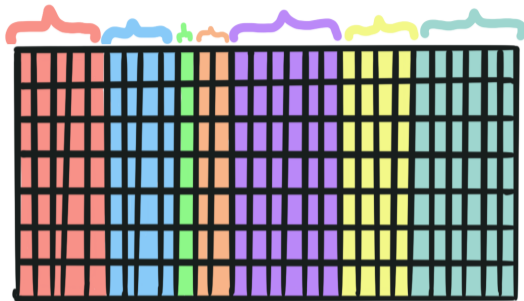
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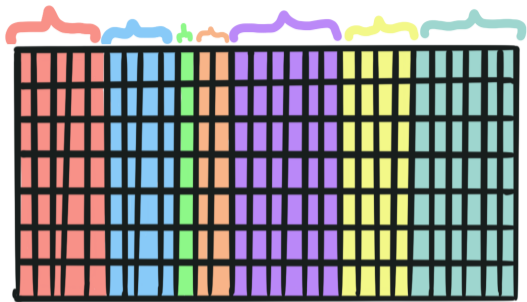
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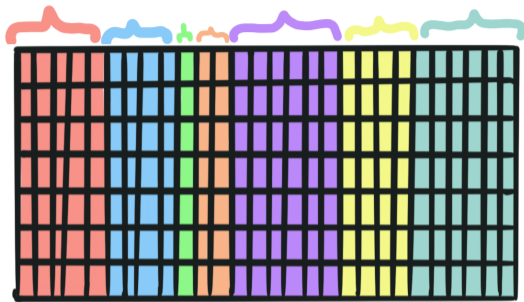
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blocks of measurements

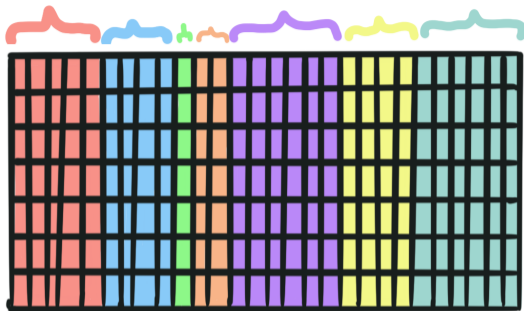


blocks of measurements



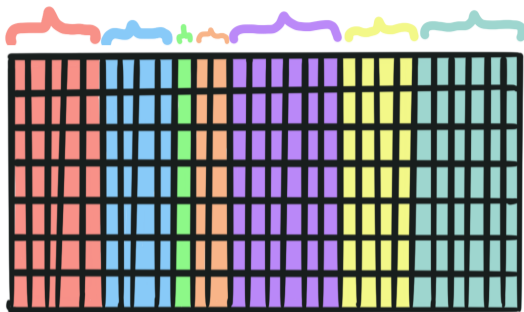
- blocks indexed by $v \in V$
e.g. here $|V| = 7$
- sample $y_i = (y_{i,v} : v \in V)$
- blocks not aligned,

blocks of measurements



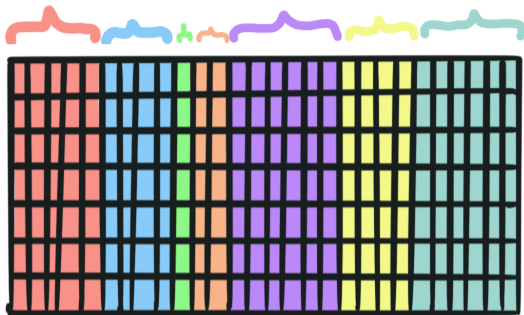
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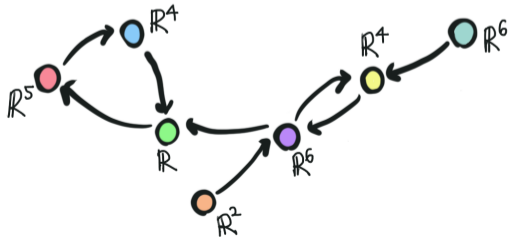


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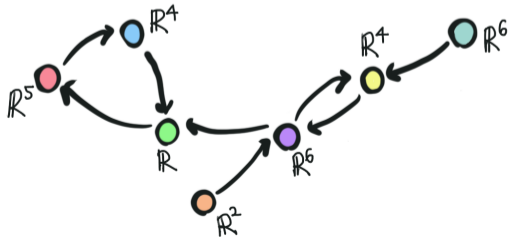
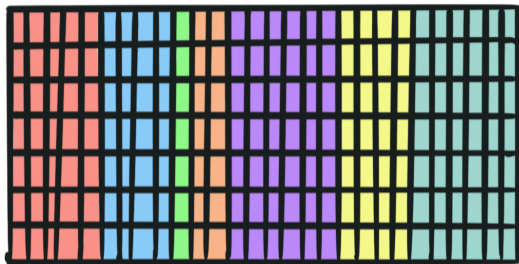
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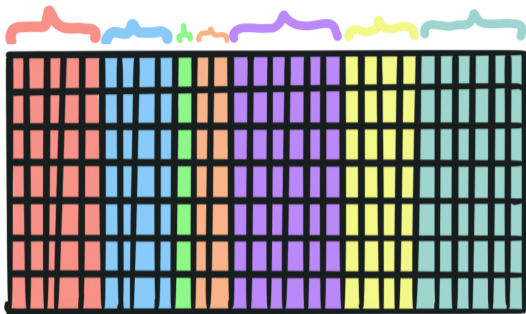


blocks of measurements

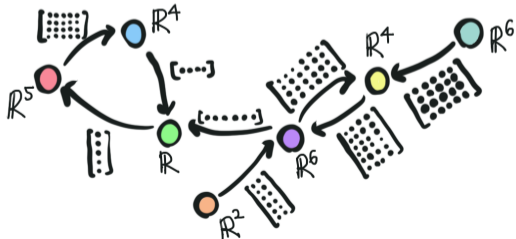


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- assumption: blocks related
under linear maps

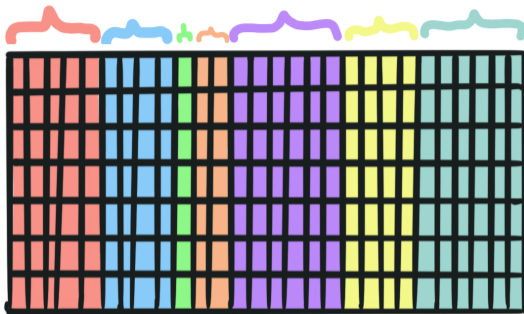
blocks of measurements



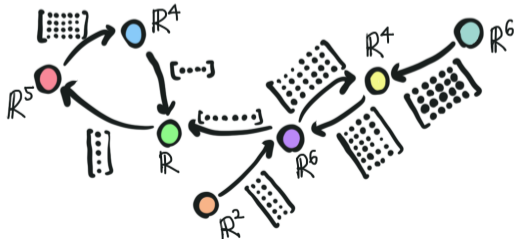
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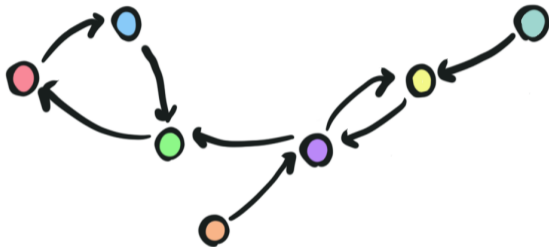
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... gives a **quiver representation**

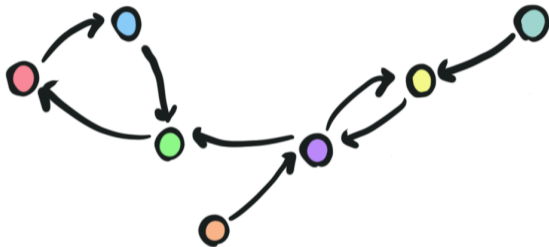
quiver representation

quiver $Q = (V, E)$ is a directed graph



quiver representation

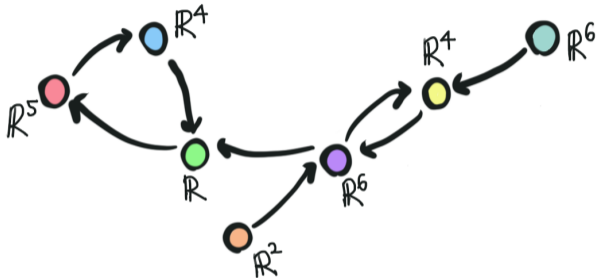
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a representation of Q assigns vector spaces to vertices

quiver representation

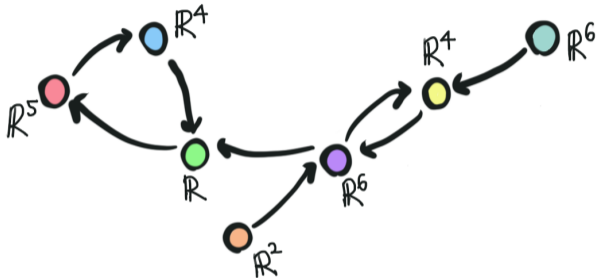
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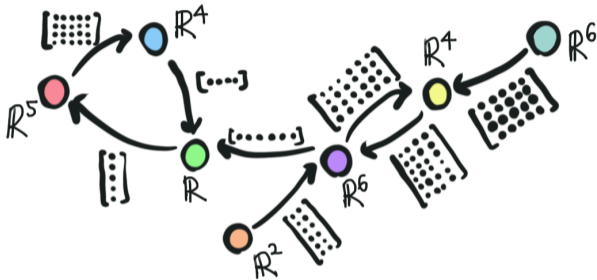
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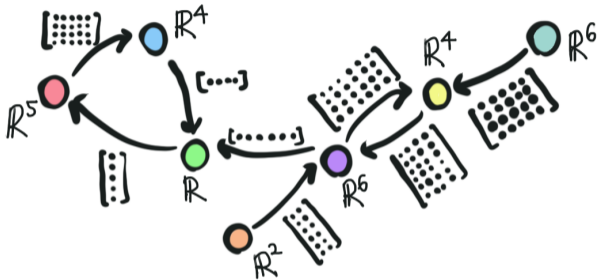
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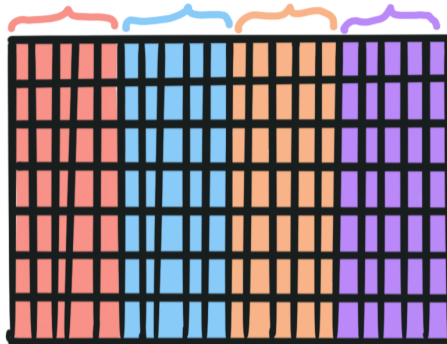


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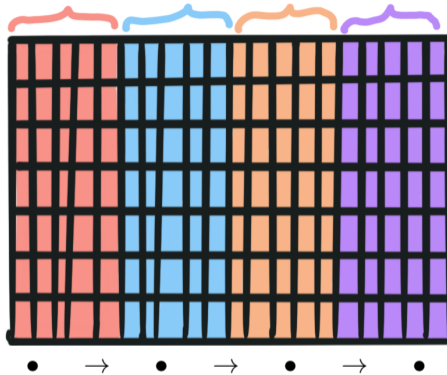
representation \mathbf{A}_\bullet assigns

- to each vertex v , a vector space \mathbf{A}_v
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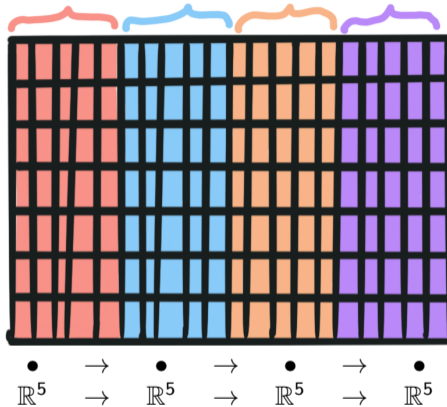
example: tensor of data



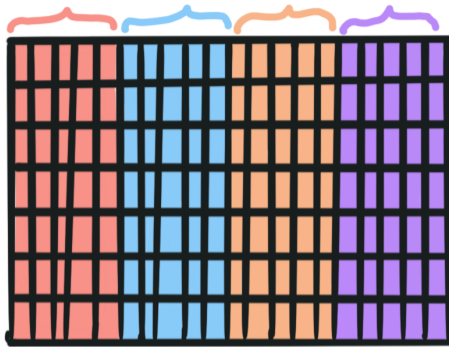
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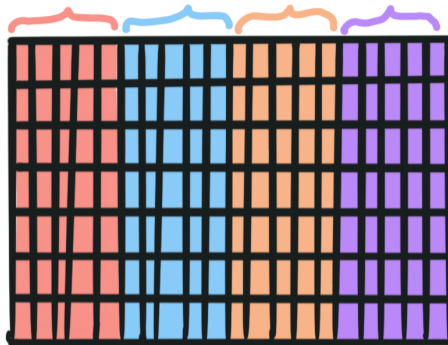
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$$\begin{array}{ccccccc} \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet \\ \mathbb{R}^5 & \rightarrow & \mathbb{R}^5 & \rightarrow & \mathbb{R}^5 & \rightarrow & \mathbb{R}^5 \\ & \lambda_1/5 & & \lambda_2/5 & & \lambda_3/5 & \end{array}$$

assume maps are scalar multiplication

example: tensor of data

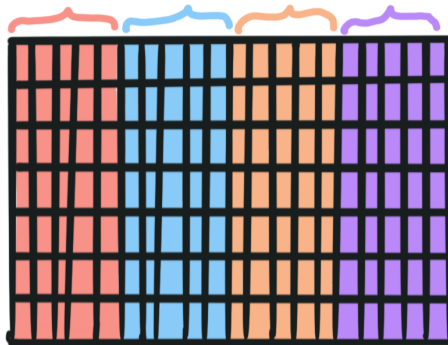


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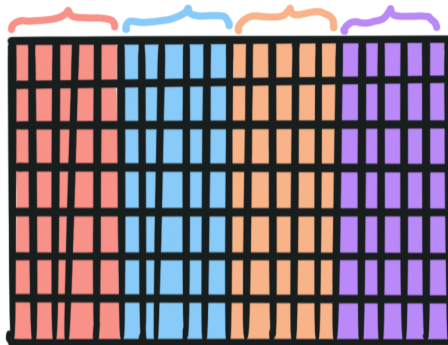
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best rank one approximation of each block, *subject to compatibility under maps*, has the form

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$$\begin{array}{cccc}
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$$\text{i.e. } T \approx a \otimes b \otimes c, \quad c = [1 \quad \lambda_1 \quad \lambda_1 \lambda_2 \quad \lambda_1 \lambda_2 \lambda_3].$$

the space of sections

finding a low-rank approximation *subject to compatibility under linear maps* is an optimisation problem over “the space of sections”.



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A *section* of \mathbf{A}_\bullet is a vector $\gamma_v \in \mathbf{A}_v$ for each $v \in V$ that is compatible with the linear map \mathbf{A}_e on the edge,

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goal: describe space of sections

rooted examples

quiver $Q = (V, E)$, vector space \mathbf{A}_v at each vertex v , linear map \mathbf{A}_e for each edge e ,
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rooted examples

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Example (path)



$\Gamma(Q; \mathbf{A}_\bullet) \simeq \mathbf{A}_r$, where r is the first vertex (root) of the path.

rooted examples

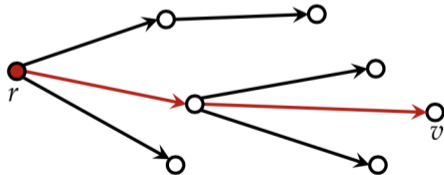
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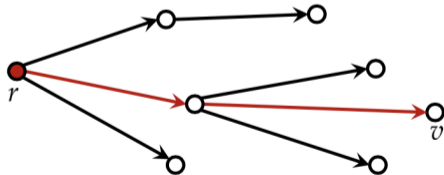
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... such a quiver is called an **arborescence**

small examples

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Example (loop)

single vertex v and loop e

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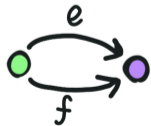


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Example (Kronecker quiver)

two vertices u, v and edges e, f .

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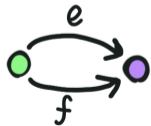


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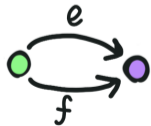


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small examples

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small examples

quiver $Q = (V, E)$, vector space \mathbf{A}_v at each vertex v , linear map \mathbf{A}_e for each edge e , space of sections $\Gamma(Q; \mathbf{A}_\bullet)$

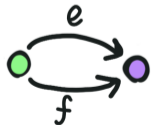


Example (loop)

single vertex v and loop e

$\Gamma(Q; \mathbf{A}_\bullet)$ is the subspace of \mathbf{A}_v fixed by \mathbf{A}_e ,

i.e., the eigenspace of \mathbf{A}_e corresponding to eigenvalue 1.



Example (Kronecker quiver)

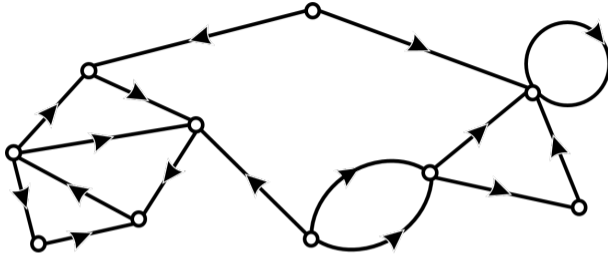
two vertices u, v and edges e, f . $\mathbf{A}_e(\gamma_u) = \mathbf{A}_f(\gamma_u) = \gamma_v$

$$\gamma_u \in \ker(\mathbf{A}_e - \mathbf{A}_f)$$

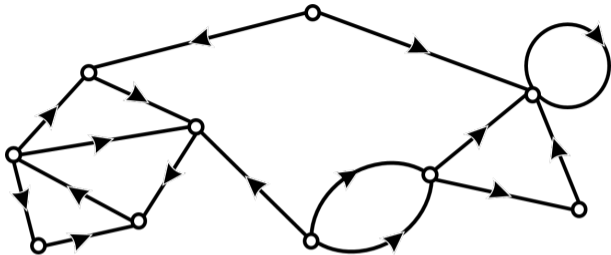
$$\Gamma(Q; \mathbf{A}_\bullet) \simeq \ker(\mathbf{A}_e - \mathbf{A}_f),$$

i.e. the equaliser of the linear maps \mathbf{A}_e and \mathbf{A}_f .

a general quiver



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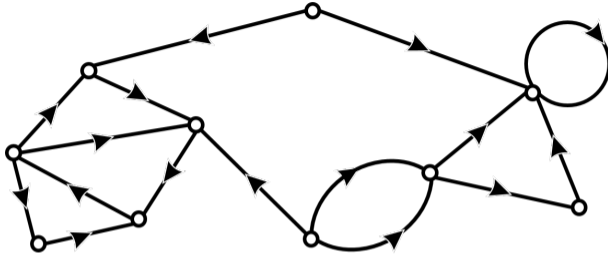


let p be a path in Q

$$u \bullet \xrightarrow{e_1} \bullet \xrightarrow{e_2} \dots \xrightarrow{e_{k-1}} \bullet \xrightarrow{e_k} \bullet v$$

\mathbf{A}_\bullet assigns path p the linear map $\mathbf{A}_p := \mathbf{A}_{e_k} \circ \mathbf{A}_{e_{k-1}} \circ \dots \circ \mathbf{A}_{e_2} \circ \mathbf{A}_{e_1}$.

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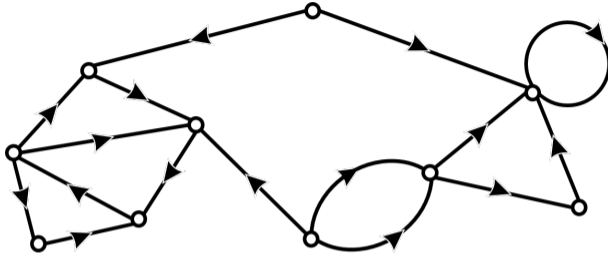
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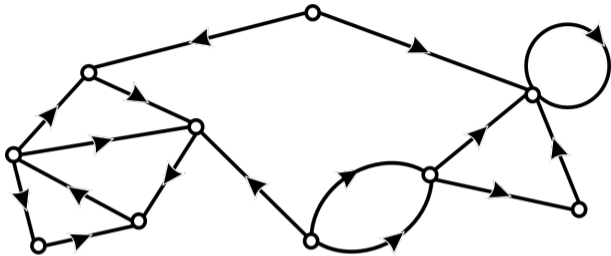
for any paths p, q from u to v , a section $\gamma = (\gamma_v : v \in V)$ has $\mathbf{A}_p \gamma_u = \mathbf{A}_q \gamma_u (= \gamma_v)$,
i.e. $\gamma_u \in \ker(\mathbf{A}_p - \mathbf{A}_q)$

restrictions that preserve space of sections



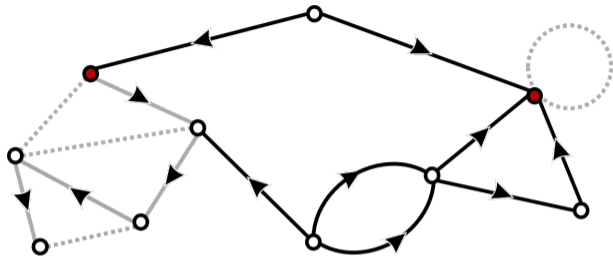
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restrictions that preserve space of sections



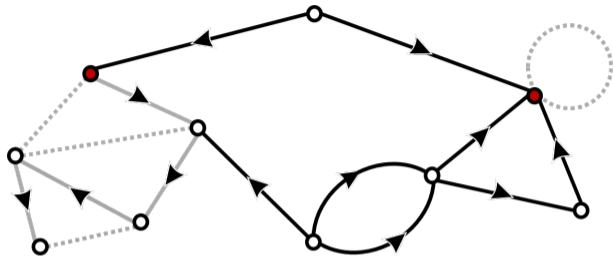
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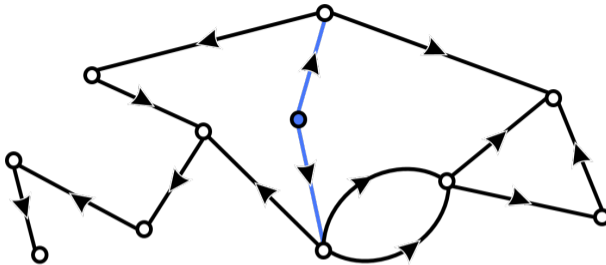
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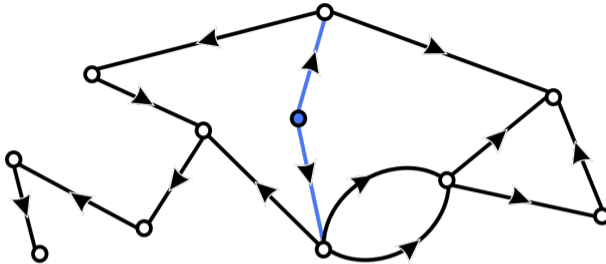
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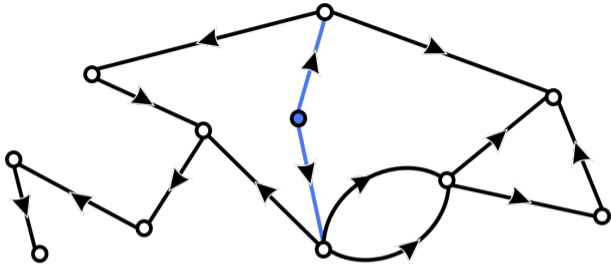
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restrictions that preserve space of sections



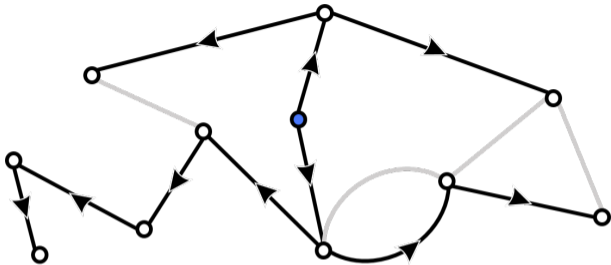
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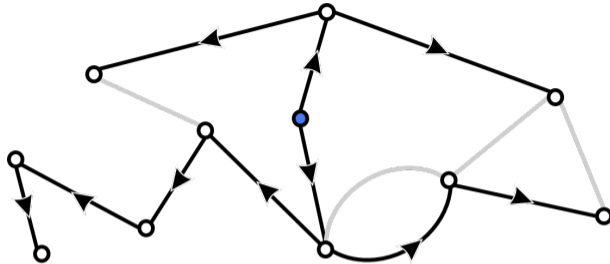
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4. the *arboreal replacement* is a representation \mathbf{A}_\bullet^+ on a spanning arborescence $T^+ \subset Q^+$.

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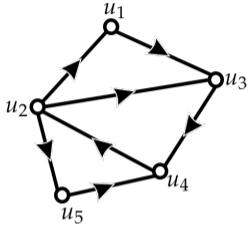
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Theorem

The spaces of sections are all isomorphic: $\Gamma(Q; \mathbf{A}_\bullet) \simeq \Gamma(Q^*; \mathbf{A}_\bullet^*) \simeq \Gamma(T^+; \mathbf{A}_\bullet^+) \simeq \mathbf{A}_r^+$.

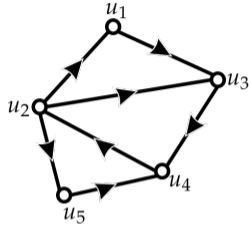
ear decompositions

a quiver is *strongly connected* if there is a path between any two vertices.



ear decompositions

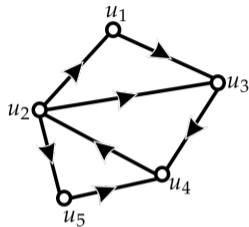
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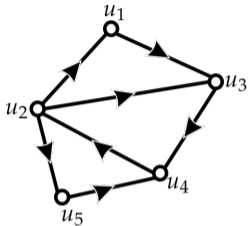
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An **ear decomposition** of a quiver $Q = (V, E)$ is a sequence of subquivers $\{Q_i = (V_i, E_i)\}$ such that

1. the E_i partition E
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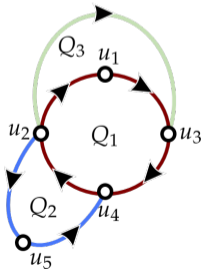


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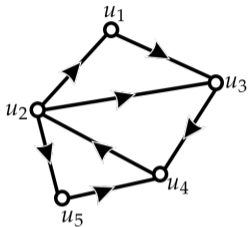
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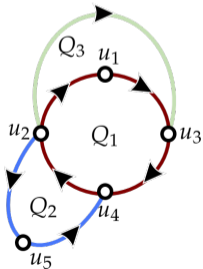


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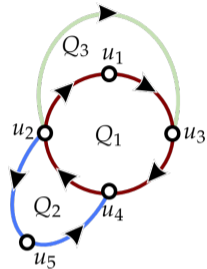
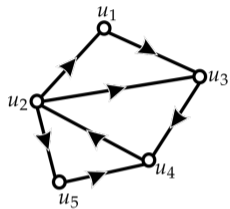
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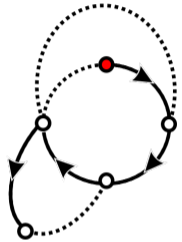
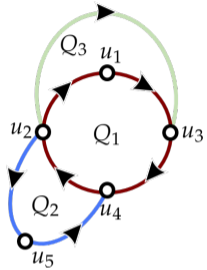
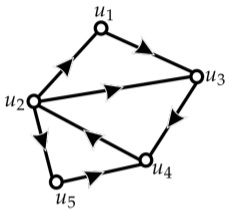
intersecting kernels

...to describe the space of sections of a strongly connected quiver



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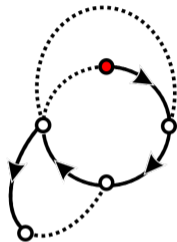
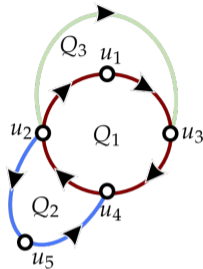
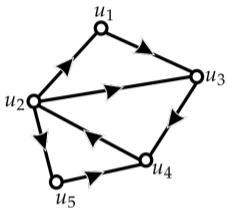
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removing the **last edge of each ear** gives a *spanning arborescence*.

intersecting kernels

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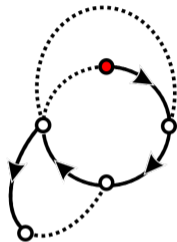
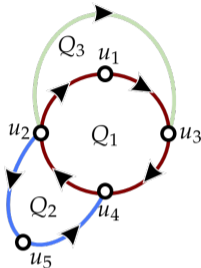
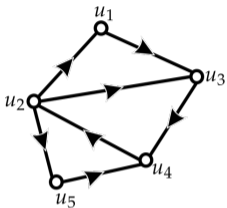


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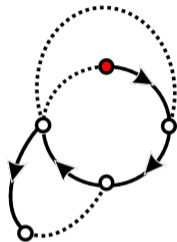
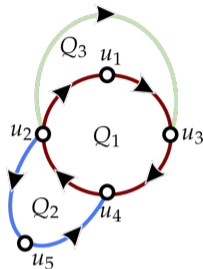
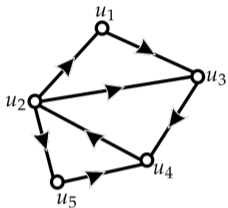
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$$\epsilon = (s(\epsilon) \rightarrow t(\epsilon))$$

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Proposition

the space of sections is $\bigcap_{\epsilon} \ker (\mathbf{A}_{p[t(\epsilon)]} - \mathbf{A}_{\epsilon} \circ \mathbf{A}_{p[s(\epsilon)]})$

$\mathbf{A}_{p[v]}$ is the linear map $\mathbf{A}_r \rightarrow \mathbf{A}_v$ obtained by composing maps along the path $p[v]$

principal components

mean-centred data $\{y_1, \dots, y_N\}$ in \mathbb{R}^n , sample covariance matrix $S = \frac{1}{N} \sum_{i=1}^N y_i y_i^T$,

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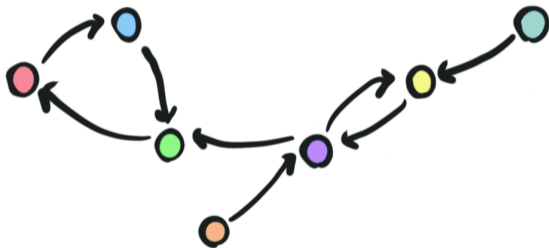
PCs along \mathbf{A}_\bullet : optimisation over image of F can be written in terms of S and F :

Theorem

the r -th principal component along \mathbf{A}_\bullet is spanned by $F u_r$, where u_r is the eigenvector of the matrix pencil $F^\top S F - \lambda(F^\top F)$ corresponding to its r -th largest eigenvalue.

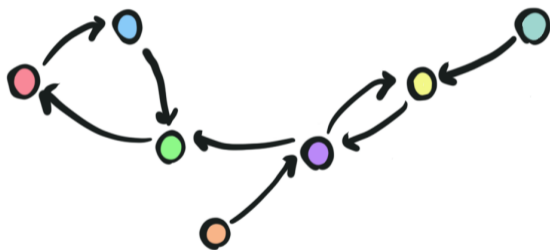
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