

Complexity of computation of tensor rank and best rank one approximation for symmetric tensors

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Outline

Matrices are natural and have many remarkable properties:

They represent in many instances linear operators

Tensors are hard and many natural properties are NP-hard:

Tensors are closely related to system of polynomial equations

In this talk we discuss the complexity of computation of

Tensor rank over \mathbb{C} (or any algebraically closed field and)

best rank one approximation of real or complex symmetric tensor

Notations and facts I

$$[n] := \{1, \dots, n\}, \mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d, [\mathbf{n}] = [n_1] \times \dots \times [n_d]$$

$$\mathbb{F} = \mathbb{R}, \mathbb{C}, \otimes_{j=1}^d \mathbb{F}^{n_j} = \mathbb{F}^{n_1 \times \dots \times n_d} = \mathbb{F}^{\mathbf{n}}$$

$$d\text{-mode tensor } \mathcal{T} = [t_{i_1, \dots, i_d}] \in \mathbb{F}^{n_1 \times \dots \times n_d}, i_j \in [n_j], j \in [d]$$

$$d = 1 \text{ vector: } \mathbf{x} = (x_1, \dots, x_n)^\top; d = 2 \text{ matrix } A = [a_{ij}], d \geq 3 \text{ tensor}$$

$$\text{rank one tensor } \mathcal{T} = [x_{i_1,1} x_{i_2,2} \dots x_{i_d,d}] = \mathbf{x}_1 \otimes \mathbf{x}_2 \dots \otimes \mathbf{x}_d = \otimes_{j=1}^d \mathbf{x}_j \neq 0$$

$$\text{IP: } \langle \mathcal{T}, \mathcal{S} \rangle = \sum_{i_j \in [n_j], j \in [d]} t_{i_1, \dots, i_d} \overline{s_{i_1, \dots, i_d}}, \quad \|\mathcal{T}\|_{HS} := \sqrt{\langle \mathcal{T}, \mathcal{T} \rangle}$$

$$\text{Contractions: } \mathcal{T} \times (\otimes_{j=2}^d \mathbf{x}_j) = \sum_{i_j \in [n_j], j \in \{2, \dots, d\}} t_{i_1, \dots, i_d} x_{i_j, j} \in \mathbb{F}^{n_1}$$

$$\mathcal{T} \times (\otimes_{j=1}^d \mathbf{x}_j) = \sum_{i_j \in [n_j], j \in \{1, \dots, d\}} t_{i_1, \dots, i_d} x_{i_j, j} \in \mathbb{F}$$

$$S(n, \mathbb{F}) := \{\mathbf{x} \in \mathbb{F}^n, \|\mathbf{x}\| = 1\} - \text{unit sphere in } \mathbb{F}^n$$

Rank of tensors - joint paper with M. Aliabadi

Consider equidimensional tensors $\mathcal{T} = [t_{i_1, \dots, i_d}] \in \otimes^d \mathbb{C}^n$

We assume $d \geq 3$

rank \mathcal{T} -minimal r : $\mathcal{T} = \sum_{i=1}^r \otimes_{j=1}^d \mathbf{x}_{j,i}$

Computation of rank \mathcal{T} is NP-hard: Håstad 1990

rank $\mathcal{T} > r$ iff the system of polynomial equations

(1) $\sum_{i=1}^r \otimes_{j=1}^d \mathbf{x}_{j,i} - \mathcal{T} = 0$ is not solvable

Number of variables rdn , number of equations n^d

Hilbert's Nullstellensatz: \exists polynomials g_{i_1, \dots, i_p} :

(2) $\sum g_{i_1, \dots, i_p}(\mathbf{x}_{1,1}, \dots, \mathbf{x}_{d,r}) ((\sum_{i=1}^r \otimes_{j=1}^d \mathbf{x}_{j,i}) - \mathcal{T})_{i_1, \dots, i_d} = 1$

Symmetric tensors I

$S^d \mathbb{C}^n \subset \otimes^d \mathbb{C}^n$ -subspace of d -symmetric tensors

srnk $\mathcal{S} =$ minimal $r: \mathcal{S} = \sum_{i=1}^r \otimes^d \mathbf{x}_i$

srnk $\mathcal{S} \geq$ rank \mathcal{S} and strict inequality may hold Shitov 2018

srnk $\mathcal{S} > r$ iff the system of polynomial equations

(3) $\sum_{i=1}^r \otimes^d \mathbf{x}_i - \mathcal{S} = 0$ is not solvable

Number of variables rn , number of equations $\binom{n+d-1}{d} \leq n^d$

Hilbert NS: (2) $\sum g_{i_1, \dots, i_d}(\mathbf{x}_1, \dots, \mathbf{x}_r) ((\sum_{i=1}^r \otimes^d \mathbf{x}_i) - \mathcal{S})_{i_1, \dots, i_d} = 1$

$1 \leq i_1 \leq \dots \leq i_d \leq n$

Symmetric tensors II

Efficient version of Hilbert Nullstellensatz:

Assuming $rn \leq \binom{n+d-1}{d}$: (# of variables \leq number of equations)

r is at most the generic rank: $r_{gen}(n, d) = \lceil \binom{n+d-1}{d} / n \rceil$

degree of each g_{i_1, \dots, i_d} is at most $\delta = d^{rn-1}$

(3) $\iff \binom{rn+\delta}{\delta}$ linear equations in $\binom{n+d-1}{d} \binom{rn+\delta}{\delta}$ variables solvable

Upper estimates: $n^d d^{(rn-1)rn}$ variables, $d^{(rn-1)rn}$ equations

Flop complexity: $O(d^{3(rn-1)(rn)})$ (Gauss elimination)

As rank decomposition is generically unique for $r < r_{gen}$

Lower bound is $r! \approx r^r \leq \left(\frac{(d+1)^{n-1}}{n!}\right) (d+1)^{n-1} / n!$ for $d \gg 1$

Rank algorithm

1. Find if $\text{srank } \mathcal{S} = 1$:

polynomial time step: verifying if the rank of unfolded tensor is 1

2. Assume that $\text{srank } \mathcal{S} > k$

Check if (2) solvable for $r = k + 1$

If yes set $k = k + 1$ and goto 2.

otherwise $\text{srank } \mathcal{S} = k + 1$

1. Flop complexity: $O(d^{3(\text{srank } \mathcal{S}dn-1)}(\text{srank } \mathcal{S}dn))$

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while elimination method for solving pol. eq. is doubly exponential

Remarks to tensor ranks

1. Can assume \mathcal{S} concise $\Rightarrow r \geq n$
3. $\mathcal{S} \in \mathbb{S}^d \mathbb{F}^n$, $\text{srank } \mathcal{S}$ in algebraic closure of \mathbb{F}

Solve (2) over \mathbb{F}

- (a) \mathbb{F} : real rational, Gaussian rationals
- (b) \mathbb{F} finite field

Can all these results can be much improved?

Spectral norm of tensors I

For $\mathcal{T} \in \otimes_{j=1}^d \mathbb{F}^{n_j}$ the spectral norm is

$$\|\mathcal{T}\|_{\sigma, \mathbb{F}} = \max\{|\mathcal{T} \times (\otimes_{j=1}^d \mathbf{x}_j)|, \mathbf{x}_j \in \mathbb{F}^{n_j}, \|\mathbf{x}_j\| = 1, j \in [d]\}$$

For real tensors $\|\mathcal{T}\|_{\sigma, \mathbb{R}} \leq \|\mathcal{T}\|_{\sigma, \mathbb{C}}$, $d \geq 3$ strict inequality possible

If $\mathcal{T} \geq 0$ (entrywise nonnegative) $\|\mathcal{T}\|_{\sigma, \mathbb{R}} = \|\mathcal{T}\|_{\sigma, \mathbb{C}}$

Maximal value of the spectral norm achieved for nonnegative vectors

Banach's theorem-1938: for symmetric $\mathcal{S} \in \mathbb{S}^d \mathbb{F}^n$

$$\|\mathcal{S}\|_{\sigma, \mathbb{F}} = \max\{|\mathcal{S} \times (\otimes^d \mathbf{x})|, \mathbf{x} \in \mathbb{F}^n, \|\mathbf{x}\| = 1\}$$

Redis.: Hübener-Kleinmann-Wei-González-Guillén-Gühne 2009 (C)

Chen-He- Li-Zhang, Zhang-Ling-Qi 2012, Friedland 2013 (R)

Homogeneous polynomials

$$f(\mathbf{x}) = \sum_{j_k+1 \in [d+1], k \in [n], j_1 + \dots + j_n = d} \frac{d!}{j_1! \dots j_n!} f_{j_1, \dots, j_n} x_1^{j_1} \dots x_n^{j_n} \in \mathcal{P}(d, n, \mathbb{F})$$

$$f(\mathbf{x}) = \sum_{\mathbf{j} \in J(d, n)} c(\mathbf{j}) f_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}, \quad \mathbf{x}^{\mathbf{j}} = x_1^{j_1} \dots x_n^{j_n}, \quad c(\mathbf{j}) = \frac{d!}{j_1! \dots j_n!}$$

$$J(d, n) = \{\mathbf{j} = (j_1, \dots, j_n) \in \mathbb{Z}_+^n, j_1 + \dots + j_n = d\}, \quad |J(d, n)| = \binom{n+d-1}{n-1}$$

$$\dim \mathcal{P}(d, n, \mathbb{F}) = \binom{n+d-1}{d} = \binom{n+d-1}{n-1}, \quad \dim \mathcal{P}(d, 2, \mathbb{F}) = d + 1$$

$$\dim \mathcal{P}(d, n, \mathbb{F}) = O(d^{n-1}) \text{ for a fixed value of } n$$

$$\mathcal{P}(d, n, \mathbb{F}) \sim \mathbb{F}^{J(d, n)} = \{\mathbf{f} = (f_{\mathbf{j}}), \mathbf{j} \in J(d, n)\}$$

$$\langle \mathbf{f}, \mathbf{g} \rangle = \sum_{\mathbf{j} \in J(d, n)} c(\mathbf{j}) f_{\mathbf{j}} \bar{g}_{\mathbf{j}}, \quad \|\mathbf{f}\|_{HS} = \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle} \text{ Hilbert-Schmidt norm}$$

$$\|\mathbf{f}\|_{\sigma, \mathbb{F}} = \|f\|_{\sigma, \mathbb{F}} = \max\left\{\frac{|f(\mathbf{x})|}{\|\mathbf{x}\|^d}, \mathbf{x} \in \mathbb{F} \setminus \{\mathbf{0}\}\right\} = \max\{|f(\mathbf{x})|, \mathbf{x} \in S(n, \mathbb{F})\}$$

Connection to symmetric tensors

$$\mathcal{S} = [s_{i_1, \dots, i_d}] \in \mathbb{S}^d \mathbb{F}^n$$

$$f(\mathbf{x}) = \mathcal{S} \times \mathbf{x}^{\otimes d} = \sum_{i_j, i \in [d]} s_{i_1, \dots, i_d} x_{i_1} \cdots x_{i_d}, \quad f_{j_1, \dots, j_n} = s_{i_1, \dots, i_d}$$

j_l -the number of times $l \in [n]$ appears in $\{i_1, \dots, i_d\}$

$$\|\mathcal{S}\|_{HS} = \|f\|_{HS}, \quad \|\mathcal{S}\|_{\sigma, \mathbb{F}} = \|f\|_{\sigma, \mathbb{F}}$$

Trivial estimate: $\|f\|_{\sigma} \leq \|f\|_{HS}$

THM F 2021: $\lim_{k \rightarrow \infty} \|f^k\|_{HS}^{1/k} = \rho_1(f)$

$$\|f\|_{\sigma, \mathbb{F}} \leq \rho_1(f) \leq \|f^k\|_{HS}^{1/k} \text{ for } k \in \mathbb{N}$$

if f is unitary diagonalizable: $f(U\mathbf{x}) = \sum_{i=1}^n \lambda_i x_i^d$ then $\|f\|_{\sigma, \mathbb{F}} = \rho_1(f)$

For quadratics $\rho_1(f) = \|f\|_{\sigma, \mathbb{F}}$

What is $\rho_1(f)$? for $d \geq 3$?

Measurements of entanglement

A state: $\mathbf{x} \in \mathbb{C}^n$, $\|\mathbf{x}\| = 1$, identified with $\zeta\mathbf{x}$, $\zeta \in \mathbb{C}$, $|\zeta| = 1$

d -partite state $\mathcal{T} \in \otimes_{i=1}^d \mathbb{C}^{n_i}$, $\|\mathcal{T}\|_{HS} = 1$

\mathcal{T} product state, unentangled, if $\mathcal{T} = \otimes_{i=1}^d \mathbf{x}_i$, $\mathbf{x}_i \in \mathbb{C}^{n_i}$, $\|\mathbf{x}_i\| = 1$, $i \in [d]$

State \mathcal{T} is unentangled iff $\|\mathcal{T}\|_{\sigma, \mathbb{C}} = 1$

\mathcal{T} entangled: $\|\mathcal{T}\|_{\sigma, \mathbb{C}} < 1$

$\Pi(\mathbf{n})$, $\mathbf{n} = (n_1, \dots, n_d)$ -the set of product states

Geometric measure of entanglement of state (GMOE) \mathcal{T} :

$$\text{dist}(\mathcal{T}, \Pi(\mathbf{n})) = \sqrt{2(1 - \|\mathcal{T}\|_{\sigma, \mathbb{C}})}$$

Friedland-Wang 2020: GME of symmetric tensor

polynomially computable in d for fixed n

Computability of $\|f\|_{\sigma, \mathbb{F}}, d > 2$

Assume $f \in \mathbb{P}(d, n, \mathbb{F})$, $\mathbf{F}(\mathbf{x}) = \frac{1}{d} \nabla f(\mathbf{x}) = \frac{1}{d} (\partial_1 f, \dots, \partial_n f)^\top$

f nonsingular: $\mathbb{F} = \mathbb{C}$, $\mathbf{F}(\mathbf{x}) = \mathbf{0} \Rightarrow \mathbf{0}$ (hypersurface $f = 0$ nonsingular)

Renormalized critical set of nonzero points of $\Re(f(\mathbf{x})) | \mathbb{S}(n, \mathbb{F})$ ($C(f)$)

(1R) $\mathbf{F}(\mathbf{x}) = \mathbf{x}$: for $\mathbb{F} = \mathbb{R}$, (1C) $\mathbf{F}(\mathbf{x}) = \bar{\mathbf{x}}$ for $\mathbb{F} = \mathbb{C}$

(2C) $\mathbf{H}(\mathbf{x}) = \mathbf{x}$, $\mathbf{H}(\mathbf{x}) = \bar{\mathbf{F}}(\mathbf{F}(\mathbf{x}))$, $\bar{\mathbf{F}}(\mathbf{y}) = \overline{\mathbf{F}(\bar{\mathbf{y}})}$

f nonsingular (1R) has $\frac{(d-1)^{n-1}}{d-2}$ \mathbb{C} -project. rays ($|C(f)| = (d-1)^{n-1}$)

f nonsingular (2C) has $\frac{(d-1)^{2n-1}}{(d-1)^2-1}$ projective rays

f nonsingular (1C) has $\mu(f)$ solutions $\frac{(d-1)^{n-1}}{d} \leq \mu(f) \leq \frac{(d-1)^{2n-1}}{(d-1)^2-1}$

The set of singular $f \in \mathbb{P}(d, n, \mathbb{C})$ is zero of hyperdeterminant

Polynomial-time computability of $\|f\|_{\sigma, \mathbb{F}}$ for fixed n - A

Roots of $\phi(\mathbf{x}) = \sum_{i=0}^D a_i x^{D-i} = 0$, $a_i \in \mathbb{Z}[\mathbf{i}]$ **computable relative prec.** ε

in $O(D^2 \log D(D \log D + \log(1/\varepsilon)))$ **oper., storage** $O(D \log(\sum_{i=0}^D |a_i|))$

$\mathbf{f} \in \mathbb{Z}[\mathbf{i}]^{J(d,n)}$, $|f_j| \leq M$, **size of \mathbf{f} :** $S = \binom{n+d-1}{n-1} 2M = O(d^{n-1} M)$

Assume $f(\mathbf{x})$ nonsingular. (2C) **has at most** $D = (d-1)^{2n}$ **solutions**

coordinate x_i of sol. in X_i , set of all solutions of $\phi_i(t) = 0$, $\deg \phi_i \leq D$

Compute $\phi_i(t)$ by Groebner basis. Complexity polynomial in $\max(S, D)$

$X = X_1 \times \dots \times X_n$, $\|f\|_{\sigma, \mathbb{C}} \approx \max\left\{\frac{|f(\mathbf{x})|}{\|\mathbf{x}\|^d}, \mathbf{x} \in X\right\}$ **$((d-1)^{2n^2}$ evaluations)**

$|||f|||_{\sigma, \mathbb{C}} - L(f)| \leq \delta \|f\|_{\sigma, \mathbb{C}}$ **complexity of computing $L(\mathbf{f})$**

is $(d-1)^{2n^2}$ times a polynomial in $\max(S, (d-1)^{2n}, \log \log(1/\delta))$

Polynom.-time computability of $\|f\|_{\sigma, \mathbb{F}}$ for a fixed n - B

Complexity of approximation $L(f)$ for $\|f\|_{\sigma, \mathbb{R}}$ for nonsingular f

is $(d-1)^{n^2}$ times a polynomial in $\max(S, (d-1)^n, \log \log(1/\delta))$

Complexity of an approximation $L(f)$ to $\|f\|_{\sigma, \mathbb{F}}$ in general

$|\|f\|_{\sigma, \mathbb{F}} - L(f)| \leq \delta \|f\|$ (pay attention to $\|f\|!$)

Complexity of $L(f)$ is polynomial in $\max(\lceil 3/\delta \rceil n(d-1)^{n-1} S, d^{n^2})$

Outline Consider $f_t = tf + (\sum_{i=1}^n x_i^d)$, $t \in A$:

$A = (-\lceil 3/\delta \rceil \lceil (n(d-1)^{n-1} + 1)/2 \rceil] \cup (\lceil 3/\delta \rceil \lceil (n(d-1)^{n-1} + 1)/2 \rceil]$

There is at least one value $t \in A$ where f_t nonsingular

Complexity if f_t singular or not polyn. in $\max(\lceil 3/\delta \rceil n(d-1)^{n-1} S, d^{n^2})$

For first f_t nonsingular use previous results

Polynomial-time computability of $\|f\|_{\sigma, \mathbb{F}}$ for $n = 2$

The dimension space of $S^2\mathbb{F}^2$ is $d + 1$

$\mathcal{S} \in S^d\mathbb{F}^2$ corresponds to $s_0, \dots, s_d \in \mathbb{F}$

To $\phi(z) = \sum_{j=0}^d \binom{d}{j} s_j z^j$ associate $f(\mathbf{x}) = \sum_{j=0}^d \binom{d}{j} s_j x_1^{d-j} x_2^j$

Let $p(z) = \sum_{j=0}^{d-1} \binom{d-1}{j} s_{j+1} z^j$, $q(z) = \sum_{j=0}^{d-1} \binom{d-1}{j} s_j z^j$, $r(z) = \frac{p(z)}{q(z)}$

$\mathbf{F}(\mathbf{x}) = (x_1, x_2)^\top$, $x_1 \neq 0 \iff zq(z) - p(z) = 0$, $q(z) \neq 0$, $z = z_2/z_1$

If $x_1 = 0$. Then $s_{d-1} = 0$ and $s_d \neq 0$.

Polynomial-time computability A

$\mathbf{F}(\mathbf{x}) = (\bar{x}_1, \bar{x}_2)^\top$ yields $\mathbf{H}(\mathbf{x}) = \bar{\mathbf{F}}(\mathbf{F}(\mathbf{x})) = \mathbf{x}$ equivalent to

$$x_1 \neq 0 \Rightarrow zv(z) - u(z) = 0, v(z) \neq 0$$

$$u(z) = \sum_{j=0}^{d-1} \binom{d-1}{j} \bar{s}_{j+1} \left(\sum_{k=0}^{d-1} \binom{d-1}{k} s_{k+1} z^k \right)^j \left(\sum_{k=0}^{d-1} \binom{d-1}{k} s_k z^k \right)^{d-1-j}$$

$$v(z) = \sum_{j=0}^{d-1} \binom{d-1}{j} \bar{s}_j \left(\sum_{k=0}^{d-1} \binom{d-1}{k} s_{k+1} z^k \right)^j \left(\sum_{k=0}^{d-1} \binom{d-1}{k} s_k z^k \right)^{d-1-j}$$

Degree of $zv(z) - u(z)$ at most $(d-1)^2 + 1$

In exceptional cases $zv(z) - z \equiv 0$:

$$(I) \phi(z) = A \binom{d}{k} z^{d-k} \text{ and } \|f\|_{\sigma, \mathbb{C}} = |A| \binom{d}{k} \left(1 - \frac{k}{d}\right)^{\frac{d-k}{2}} \left(\frac{k}{d}\right)^{\frac{k}{2}}$$

Polynomial-time computability of $\|f\|_{\sigma, \mathbb{F}}$ for $n = 2$ - B

$$(II) \phi(z) = A(z + a)^p(z + b)^{d-p}$$

$$a = e^{-\theta i} c, \quad b = -e^{-\theta i} c^{-1}, \quad c \in \mathbb{R} \setminus \{0\}, \quad \theta \in \mathbb{R}, \quad p \in [d - 1]$$

One can approximate well $\|f\|_{\sigma, \mathbb{C}}$ with relative error δ

Non exceptional case: $R_1 = \{z \in \mathbb{C}, zv(z) - u(z) = 0\}$

$$\|f\|_{\sigma, \mathbb{C}} = \max \left\{ |s_d|, \max_{z \in R_1} \left\{ \frac{|\phi(z)|}{(1+|z|^2)^{\frac{d}{2}}} \right\} \right\}$$

For $\mathbf{f} \in \mathbb{Z}[\mathbf{i}]^{d+1}$ we get approximation $|\|f\|_{\sigma, \mathbb{C}} - L(f)| \leq \delta \|f\|_{\sigma, \mathbb{C}}$

Complexity of getting $L(f)$ is $O(d^4 \log d^2 (d^2 \log d^2 + \log(1/\delta)))$

storage: $O(d^3 (\log 3M))$

For $\mathbf{f} \in \mathbb{Z}^{d+1}$ complexity of real approximation $L(f)$ to $\|f\|_{\sigma, \mathbb{R}}$

$O(d^2 \log d (d \log d + \log(1/\delta)))$, storage $O(d(\log dM))$

Iterations of rational self maps of \mathbb{P}^{n-1}

$\mathcal{F} = [f_{i_1, i_2, \dots, i_{p+1}}] \in \mathbb{C}^m \otimes S^p \mathbb{C}^n$ -symmetric in last p -coordinates

$\mathbf{F}(\mathbf{x}) = \mathcal{F} \times (\otimes^p \mathbf{x})$ - polynomial map $\mathbf{F} : \mathbb{C}^n \rightarrow \mathbb{C}^m$

rational map $\tilde{\mathbf{F}} : \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{m-1}$

$$\|\mathbf{F}\|_\sigma := \max\{\|\mathbf{F}(\mathbf{x})\|, \|\mathbf{x}\| = 1\} = \|\mathcal{F}\|_\sigma$$

Assume $m = n$, $\mathcal{F} \in S^{p+1} \mathbb{C}^n \iff \mathbf{F}(\mathbf{x}) = \frac{1}{p+1} \nabla f(\mathbf{x})$

Iterate k times $\mathbf{F}^{\circ k}(\mathbf{x}) = \mathcal{F}^{\circ k} \times (\otimes^{p^k} \mathbf{x})$, $\mathcal{F}^{\circ k} \in \mathbb{C}^n \otimes (S^{p^k} \mathbb{C}^n)$

If \mathcal{F} symmetric $\mathcal{F}^{\circ k}$ usually not symmetric for $p > 1$

for $p = 1$ $\mathbf{F}(\mathbf{x}) = F\mathbf{x}$, $\mathbf{F}^{\circ k}(\mathbf{x}) = F^k\mathbf{x}$, $F \in \mathbb{C}^{n \times n}$

$\lim_{k \rightarrow \infty} \|F^k\|_{HS}^{1/k} = \rho(F)$ -spectral radius of F

Iterations II

THM F 2021: Let $p \in \mathbb{N}$

a. $\|\mathcal{F}^{\circ(k+l)}\|_{HS} \leq \|\mathcal{F}^{\circ k}\|_{HS} \|\mathcal{F}^{\circ l}\|_{HS}^{p^l}$

b. $\rho_2(\mathcal{F}) := \lim_{k \rightarrow \infty} \|\mathcal{F}^{\circ k}\|_{HS}^{(p-1)/(p^k-1)}$

$\rho_2(\mathcal{F}) \leq \|\mathcal{F}^{\circ k}\|_{HS}^{(p-1)/(p^k-1)}$ for each $k \in \mathbb{N}$

c. $\|\mathcal{F}^{\circ(k+l)}\|_{\sigma} \leq \|\mathcal{F}^{\circ k}\|_{\sigma} \|\mathcal{F}^{\circ l}\|_{\sigma}^{p^k}$

$\rho_3(\mathcal{F}) := \lim_{k \rightarrow \infty} \|\mathcal{F}^{\circ k}\|_{\sigma}^{(p-1)/(p^k-1)} (\leq \|\mathcal{F}^{\circ k}\|_{\sigma}^{(p-1)/(p^k-1)})$

d. $\rho_3(\mathcal{F}) \leq \rho_2(\mathcal{F})$

e. If \mathcal{F} symmetric $\|f\|_{\sigma} \leq \rho_2(\mathcal{F})$

What is the dynamics meaning of $\rho_2(\mathcal{F})$ and $\rho_3(\mathcal{F})$?

Meaning of $\rho_3(\mathcal{F})$

In complex dynamics one shows the existence of the limit

Ueda 1994, Fornæss-Siboni 1995





$$G(\mathbf{x}) := \lim_{k \rightarrow \infty} \frac{1}{p^k} \log \|\mathbf{F}^{\circ k}(\mathbf{x})\|$$

The Green function of \mathbf{F} : either $-\infty$ or plurisubharmonic






$$G(\mathbf{F}(\mathbf{x})) = pG(\mathbf{x}), \quad G(t\mathbf{x}) = \log |t| + G(\mathbf{x})$$

$$\max\{G(\mathbf{x}), \|\mathbf{x}\| = 1\} = p \log \rho_3(\mathcal{F})$$




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