Subrank, Partition Rank and Slice Rank

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Subrank

- Geometric rank
- Generic subrank
- "Weak submultiplicativity"
- Asymptotic subrank gap

$T \in \mathbb{F}^{n \times n \times n}$

Definition Tensor rank $R(T)$

minimize $\rightarrow$

$$T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$$

equiv.:

$$T = U \otimes V \otimes W \cdot \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$$

Applications

- Matrix multiplication
- Arithmetic complexity [Raz]

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Definition Subrank $Q(T)$

← maximize

$$\sum_{i=1}^{s} e_i \otimes e_i \otimes e_i = U \otimes V \otimes W \cdot T$$

Applications

- Matrix multiplication
- Additive combinatorics

$T \in \mathbb{F}^{n \times n \times n}$

Easy: $Q(T) \leq n$

$$\boxed{\text{Recall: } \sum_{i=1}^{s} e_i \otimes e_i \otimes e_i = U \otimes V \otimes W \cdot T}$$

maximize

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Recall: $\displaystyle\sum_{i=1}^{s} e_i \otimes e_i \otimes e_i = U \otimes V \otimes W \cdot T$ ← maximize

Definition Slice rank $SR(T)$

$$T = \sum_{i=1}^{a} \sum_{j} u_i \otimes v_{ij} \otimes w_{ij} + \sum_{i=1}^{b} \sum_{j} u'_{ij} \otimes v'_i \otimes w'_{ij}$$

$$+ \sum_{i=1}^{c} \sum_{j} u''_{ij} \otimes v''_{ij} \otimes w''_i$$

minimize $a+b+c$

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Recall: $\sum_{i=1}^{s} e_i \otimes e_i \otimes e_i = U \otimes V \otimes W \cdot T$ ($s \leftarrow$ maximize)

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Application $\underbrace{Q(T) \leq SR(T)} \leq n$

Question How far apart can subrank and slice rank be?

Definition Geometric rank $GR(T)$ [KMZ]

$$\operatorname{codim} \{ (x,y) \in \mathbb{F}^n \times \mathbb{F}^n : \forall z \ T(x,y,z) = 0 \}$$

Extends Analytic rank [Gowers, Wolf, Lovett] from pos. char to char. 0.

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<u>Theorem</u> [KMZ]

- invariant under permuting x, y and z
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- for $T = \langle n,n,n \rangle$: $\underline{Q}(T) \overset{!}{=} GR(T) = \lceil \frac{3}{4} n^2 \rceil \overset{!}{<} SR(T) = n^2$

$\boxed{\text{matrix multiplication tensor}}$

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Follow-up [MC] GR and SR are equal up to constant!

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<u>Note</u>
- Surprisingly small, in particular given that generic rank is maximal.
- for generic T, $Q(T) \leq 3n^{2/3}$ while $SR(T) = n$
- Subrank and slice rank very different generically!

Theorem. For generic $T \in \mathbb{F}^{n\times n\times n}$: $Q(T) \le 3n^{2/3}$

$S \subseteq [n] \times [n] \times [n]$

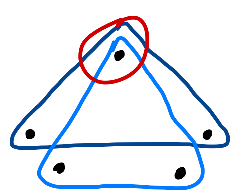
Maximal points: $\mathrm{Max}(S) = \{$ coordinate-wise maximal points in $S \}$

Example:

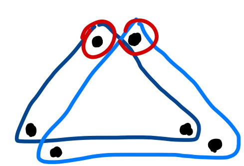
$S = \{(2,1,1), (1,2,1), (1,2,2)\}$, $\mathrm{Max}(S) = \{(2,1,1), (1,2,2)\}$

Cover number: $\mathrm{cov}(S) =$ vertex cover number of S as 3-partite hypergraph.

Example: $S = \{(1,1,1), (1,2,2)\}$ $S = \{(1,1,2), (2,2,1)\}$



1



2

$T \in \mathbb{F}^{n \times n \times n}$

Support: $\mathrm{Supp}(T) \subseteq [n] \times [n] \times [n]$

Action: $g \in GL_n^{\times 3}$, $g \cdot T \in \mathbb{F}^{n \times n \times n}$

Cover number: $\mathrm{cov}(T) := \max\limits_{g \in GL_n^{\times 3}} \mathrm{cov}(\mathrm{Max}(\mathrm{Supp}(g \cdot T)))$

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Example: $T = \sum_{i=1}^{s} e_i \otimes e_i \otimes e_i \in \mathbb{F}^{n \times n \times n} \rightsquigarrow g \cdot T = \sum_{i=1}^{s} e_i \otimes e_i \otimes e_{s-i}$

$\mathrm{Supp}(g \cdot T) = \{(i, i, s-i) : i \in [s]\} \overset{!}{=} \mathrm{Max}(\mathrm{Supp}(T))$

$s \leq \mathrm{cov}(T)$

$\boxed{\underline{\text{Lemma}}: \quad Q(T) \leq \mathrm{cov}(T)}$ Follows from ↑

$\boxed{\text{Lemma 2} \quad \mathrm{cov}(T) \leq 3n^{2/3} \text{ for generic } T \in \mathbb{F}^{n \times n \times n}.}$

Proof sketch

① There is a nonempty open $\mathcal{U} \subseteq \mathbb{F}^{n \times n \times n}$ such that

$$\forall T \in \mathcal{U} \;\; \forall g \in GL_n^{\times 3} \quad |\mathrm{Supp}(g \cdot T)| \geq n^3 - 3n^2 \qquad \text{[Bürgisser]}$$

Lemma 2 $\mathrm{cov}(T) \le 3n^{2/3}$ for generic $T \in \mathbb{F}^{n\times n\times n}$.

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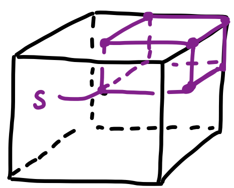

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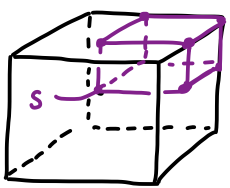

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③ $\mathrm{cov}(\ \ ) \leq 3n^{2/3}$ since $\forall s \ \exists i \ \ s_i \leq n^{2/3}$ $\qquad \approx \square$

"Weak submultiplicativity"

$T \in V_1 \otimes V_2 \otimes V_3 \otimes V_4 = \mathbb{F}^{n \times n \times n \times n}$

	Rank one elements	
Slice rank	$v \otimes w$	$v \in V_i$, $w \in \bigotimes_{j \neq i} V_j$
Partition rank	$v \otimes w$	$v \in \bigotimes_{i \in S} V_i$, $w \in \bigotimes_{j \notin S} V_j$

Weak submultiplicativity: If $SR(T) < n$, then $SR(T^{\otimes m}) \leq c^m$, $c < n$.

Theorem [Lysikov, Lovett, Z]: There are T with $PR(T) = 2$ and

$$PR(T^{\otimes m}) = n^{m - o(m)}$$

Asymptotic subrank gap

$T \in V_1 \otimes \cdots \otimes V_k$

$$\lim_{m \to \infty} SR(T^{\otimes m})^{1/m} = \begin{cases} 0 \\ 1 \\ \geq 2^{h(1/k)} \end{cases}$$

gap

[Costa, Dalai]

$\leftarrow$ binary entropy, e.g., $2^{h(1/3)} \approx 1.88$

Theorem [Christandl, Gesmundo, me]

$$\lim_{m \to \infty} Q(T^{\otimes m})^{1/m} = \begin{cases} 0 \\ 1 \\ \geq 2^{h(1/k)} \end{cases}$$

Generic T, $T \trianglerighteq W_k$