# Inner and outer approximations of tropical polytopes and their applications in tropical tensors

Yang Qi

INRIA and CMAP École Polytechnique

(*joint with* Marianne Akian, Stéphane Gaubert, and Omar Saadi)

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- A scalar acts on vector by "$\lambda x$" $= \lambda \odot x = \lambda + x$ entrywise.
- Eigenvalues and eigenvectors of a matrix $A$:

$$A \odot v = \lambda + v$$

# Applications

Tropical tensors arise naturally:

- Optimization
- Game theory
- Approximation theory
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A baby example:

- Given a direct-distances matrix $A = (a_{ij})$, find the longest distances.
- Consider $A^2 = A \odot A$, i.e.,

$$(A^2)_{ij} = \max_k (a_{ik} + a_{kj}),$$

  which gives the longest path between $i$ and $j$ of length 2.
- $A + A^2 + \cdots$ gives longest paths of all lengths.

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- Secret evaluation $0 < f_i \leqslant 1$ (**technical quality**)
- The decision maker minimizes his expected cost:

$$\min_{i \in [n]} p_{ij} f_i^{-1}$$

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will see later: $\forall j \in [q], \quad V_{\cdot j}$ lies in some hyperplane $H_a$

# Tropical data analysis

Given tropical data, we would like to do

- Tropical linear regression:
  Given $m$ points, find a hyperplane which best fits these data.
- Tropical tensor approximation:
  Given a tropical tensor, find a best rank-one approximation.

# Tropical cones

- $C \subseteq (\mathbb{R}_{\max})^n$ is a **tropical (convex) cone** if $\forall x, y \in C, \forall \lambda \in \mathbb{R}_{\max}$:

$$\lambda + x \in C \text{ and } \max(x, y) \in C.$$

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- For a set $\mathcal{V} \subseteq (\mathbb{R}_{\max})^n$, the tropical cone generated by $\mathcal{V}$:
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- Given $A, B \subseteq \mathbb{P}(\mathbb{R}_{\max})^n$, the **one-sided Hausdorff distance**:

$$\mathrm{d}_H(A, B) := \sup_{a \in A} \inf_{b \in B} \|a - b\|_H \ .$$

# Tropical hyperplanes

Given $a \in \mathbb{P}(\mathbb{R}_{\max})^n$, we define the **tropical hyperplane**:

$$H_a := \{x \in (\mathbb{R}_{\max})^n \mid \max_{i \in [n]}(a_i + x_i) \text{ achieved at least twice}\} \ .$$

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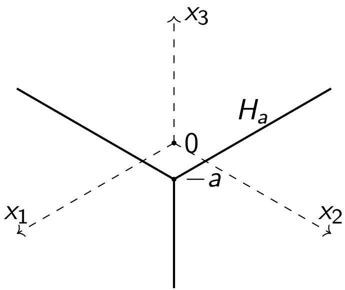


Figure: The hyperplane $H_a$ with $a = (0, 0, 1)^\top$ and in $\mathbb{P}(\mathbb{R}_{\max})^3$.

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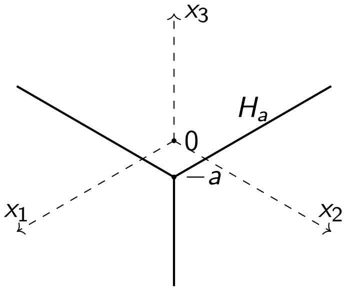


Figure: The hyperplane $H_a$ with $a = (0,0,1)^\top$ and in $\mathbb{P}(\mathbb{R}_{\max})^3$.

Given a set $\mathcal{V} \subseteq \mathbb{R}^n_{\max}$, we solve **the tropical linear regression problem**:

$$\inf_{a \in \mathbb{P}(\mathbb{R}_{\max})^n} \mathrm{d}_H(\mathcal{V}, H_a) \ .$$

# Inner radius

For a subset $\mathcal{V}$ of $(\mathbb{R}_{\max})^n$, the *inner radius* of $\mathcal{V}$ is:

$$\text{in-rad}(\mathcal{V}) := \sup\{r \geqslant 0 \mid \exists b \in \mathbb{R}^n, B(b,r) \subseteq \text{Span}(\mathcal{V})\}.$$

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**Theorem (Akian–Gaubert–Q.–Saadi)**

$$\inf_{a \in \mathbb{P}(\mathbb{R}_{\max})^n} \mathrm{d}_H(\mathcal{V}, H_a) = \text{in-rad}(\mathcal{V}).$$

Question: how to compute in-rad($\mathcal{V}$)?

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After $\ell$ turns, under a strategy $\sigma$ (resp. $\tau$) of Min (resp. Max), if the sequence of actions is $i, k_1, i_1, \cdots, k_\ell, i_\ell$, the total payment:

$$R_i^\ell(\sigma,\tau) = -V_{ik_1} + V_{i_1k_1} - V_{i_1k_2} - \cdots + V_{i_\ell k_\ell}$$

The *value* $v_i^\ell$ of the game at horizon $\ell$ starting from $i$:

$$v_i^\ell := \min_\sigma \max_\tau R_i^\ell(\sigma,\tau) = \max_\tau \min_\sigma R_i^\ell(\sigma,\tau) \ .$$

# Shapley operator

*Shapley operator* $T : (\mathbb{R}_{\max})^n \to (\mathbb{R}_{\max})^n$,

$$T_i(x) = \min_{k \in [p], V_{ik} \neq -\infty} \Big[ -V_{ik} + \max_{j \in [n], j \neq i} (V_{jk} + x_j) \Big], \quad i \in [n] \ ,$$

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- Complexity: NP $\cap$ coNP.
- Zwick and Paterson [1996]: Value iteration (pseudo-polynomial time).
- Our case: $\chi(T) \leqslant 0$.

# Equivalence between tropical linear regression and MPG

The *spectral radius* of $T$ is defined as

$$\rho(T) = \sup\{\lambda \in \mathbb{R} \cup \{-\infty\} \mid \exists u \in (\mathbb{R}_{\max})^n, u \not\equiv -\infty,\ T(u) = \lambda + u\} \ .$$

**Theorem (Akian–Gaubert–Q.–Saadi)**

$$\min_{a \in \mathbb{P}(\mathbb{R}_{\max})^n} \mathrm{d}_H(V, H_a) = -\rho(T) = \text{in-rad}(V)$$

*Moreover,*

- *if $T(a) \geqslant \rho(T) + a$, then $H_a$ is optimal*
- *if $T(b) \leqslant \rho(T) + b$, then $B(-b, -\rho(T))$ is optimal*

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- *if* $T(b) \leqslant \rho(T) + b$, *then* $B(-b, -\rho(T))$ *is optimal*

## Corollary

*The tropical linear regression problem is polynomial-time equivalent to the problem of solving a mean payoff game.*

# Geometric illustration

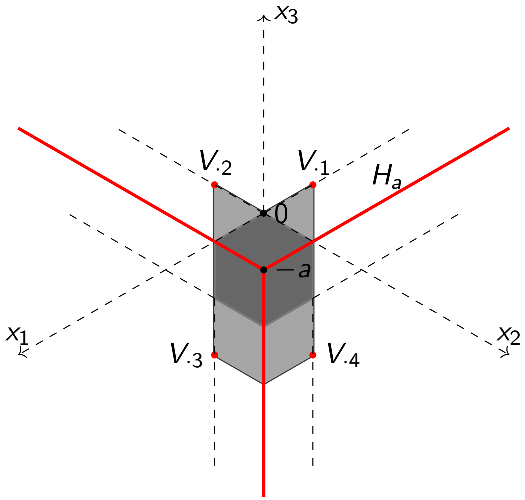



Figure: A tropical cone $\mathrm{Sp}(\mathcal{V})$ with an optimal regression hyperplane $H_a$ and an optimal inner ball $B(-a, 1)$, where $a = (0, 0, 1)^\top$ satisfies $T(a) = -1 + a$.

# Revisit to equilibrium in ITT

Equilibrium:

$$\min_{i\in[q]} p_{ij} f_i^{-1} \text{ is achieved twice at least}$$

By letting $V_{ij} = -\log(p_{ij})$ and $a_i = \log(f_i)$, the equilibrium is:

$$\max_{i\in[n]}(V_{ij} + a_i) \text{ is achieved at least twice,}$$

namely,

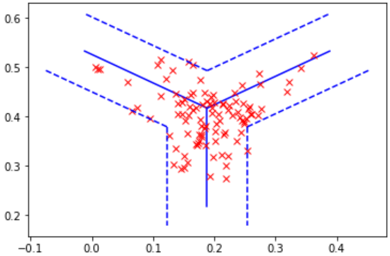
$$\forall j \in [q], \quad V_{\cdot j} \in H_a$$

In practice, we solve the **tropical linear regression problem:**

$$\min_{b\in\mathbb{P}(\mathbb{R}_{\max})^n} \mathrm{d}_H(V, H_b).$$

# Algorithm

- Goal: $T(v) = \rho(T) + v$
- Algorithm: projective Krasnoselkii-Mann value iteration algorithm
  - Given $\epsilon > 0$, start with $v^0 = (0, \cdots, 0)^\top$.
  - If $\| T(v^k) - v^k \| \geqslant \epsilon$, let
    $\tilde{v}^{k+1} = T(v^k) - (\max_{i\in[n]} T(v^k)_i)e$,
    $v^{k+1} = (1-\beta)v^k + \beta\tilde{v}^{k+1}$,
    where $e = (1, \cdots, 1)^\top \in \mathbb{R}^n$ and $\beta \in (0,1)$ fixed.



# $\epsilon$-approximation

## Theorem (Akian–Gaubert–Q.–Saadi)

*Suppose that* $V \in \mathbb{R}^{n\times p}$ *is finite, and let*

$$W := \max_{k\in[p]} \|V_{\cdot,k}\|_H \ .$$

*Then, an* $\epsilon$*-approximation of* in-rad$(V)$*, and vectors* $v, z \in \mathbb{R}^n$ *satisfying*

$$B_H(v, \text{in-rad}(\mathcal{V}) - \epsilon) \subseteq \text{Span}(V)$$

*and*

$$\text{d}_H(\text{Span}(V), H_z) \leqslant \text{in-rad}(\mathcal{V}) + \epsilon$$

*can be obtained in* $O(npW/\epsilon)$ *arithmetic operations.*

# Outer radius

For a subset $\mathcal{V}$ of $(\mathbb{R}_{\max})^n$, the *outer radius* of $\mathcal{V}$ is:

$$\text{out-rad}(\mathcal{V}) := \inf\{r \geqslant 0 \mid \exists b \in \mathbb{R}^n, B(b,r) \supseteq \text{Span}(\mathcal{V})\}.$$

**Theorem (Akian–Gaubert–Q.–Saadi)**

*For any matrix $V \in \mathbb{R}^{n\times p}$, we have*

$$\min_{A\in\mathbb{R}^{n\times p}, \text{rank } A=1} \|V - A\|_\infty = \frac{1}{2}\,\text{out-rad}(\text{Span } V).$$

Question: how to compute out-rad(Span $V$)?

# Reduction to eigenvalue problem

Given $A \in (\mathbb{R}_{\max})^{n\times n}$, define

$$A^* = \mathsf{I} \oplus A \oplus A^2 \oplus \cdots$$

For $V \in \mathbb{R}^{n\times p}$, let $H = V \odot (-V^\top) \in \mathbb{R}^{n\times n}$, where

$$H_{ik} = \max_{j\in[p]}(V_{ij} - V_{kj}), \quad i,k \in [n].$$

**Theorem (Akian–Gaubert–Q.–Saadi)**

*$H$ has a unique eigenvalue, which equals* out-rad(Span $V$)*. Moreover, the set of centers of all Hilbert outer balls of* Span $V$ *is the column space of* $(-\lambda + H)^*$.

# Kernel approximations

$X$: compact metric, $Y$: nonempty, $\mathcal{C}(X)$: the space of continuous functions on $X$, $\mathcal{B}(Y)$: the space of bounded functions on $Y$.

**Theorem (Akian–Gaubert–Q.–Saadi)**

*If $V : X \times Y \to \mathbb{R}$ bounded and $\{V(\cdot, y)\}_{y \in Y}$ equicontinuous, then*

$$\inf_{f \in \mathcal{C}(X),\, g \in \mathcal{B}(Y)} \sup_{x \in X, y \in Y} |V(x,y) - f(x) - g(y)|$$

*achives an optimal solution, which is equal to one half of the tropical eigenvalue of $H$, where*

$$H(x,z) = \sup_{y \in Y} (V(x,y) - V(z,y)) \ ,$$

*and $f$ is a tropical eigenvector of $H$.*

# Conclusion

- Tropical linear regression is equivalent to finding inner radius, which is polynomial-time equivalent to mean payoff game.
- Finding a best tropical rank-one approximation is equivalent to finding outer radius, which is equivalent to eigenvalue problem.

*Thank you for your attention!*