Chain Tensor Network: Instability and how to deal with it

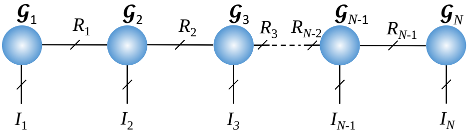
Anh-Huy Phan
(joint work with Petr Tichavský)

Center for Computational and Data-Intensive Science and Engineering (CDISE)
Skolkowo Institute of Science and Technology (SKOLTECH), Moscow, Russia

May 4, 2021

Outlines

- Looped Tensor network
- Relation between TC and Tucker, CP decompositions
- Sensitivity
- Algorithms



Model a tensor $\mathcal{Y}$ of size $I_1 \times I_2 \times \cdots \times I_N$ by N interconnected core tensors $\mathcal{G}_n$ of size $R_n \times I_n \times R_{n+1}$

$$\mathcal{Y} = \sum_{r_1=1}^{R_1} \sum_{r_2=1}^{R_2} \cdots \sum_{r_{N-1}=1}^{R_{N-1}} \mathcal{G}_1(:, r_1) \circ \mathcal{G}_2(r_1, :, r_2) \circ \cdots \circ \mathcal{G}_N(r_{N-1}, :),$$

or

$$\mathcal{Y} = \mathcal{G}_1 \bullet \mathcal{G}_2 \bullet \cdots \bullet \mathcal{G}_{N-1} \bullet \mathcal{G}_N$$

$(R_1, R_2, \ldots, R_{N-1})$ represents the TT-rank of $\mathcal{Y}$.

MPS/Tensor Train - Straight Tensor Network II

- TT/MPS decomposition can be computed by tool of SVDs Vidal (2003); Oseledets and Tyrtyshnikov (2009); Phan et al. (2020a)
- TT/MPS decomposition is very suited to higher-order tensors.
- Applications: solving a huge system of linear equations or eigenvalue decomposition of large-scale data Holtz et al. (2012); Kressner et al. (2014), PDE, data completion, modelling in system identification, deep learning.

Major problem: Intermediate TT-ranks is often very high

- TT decomposition with bound constraint often exhibit badly unbalaced TT-ranks.
- The ranks grow dramatically with the dimensions of the tensor.

Chain tensor network - Tensor chain I

Figure: Looped tensor networks.

- Khoromskij (2009,2011) introduced the looped Tensor chain as an extension of TT
- Since there are no first and last core tensors, TC is expected to overcome imbalance rank issue in TT decomposition.

Chain tensor network - Tensor chain II

Similar to TT, core tensors $\mathcal{G}_n$ of size $R_{n-1} \times I_n \times R_n$, $R_0 = R_N$, the TC model is written as

$$\mathcal{Y} = \sum_{r_1=1}^{R_1} \sum_{r_2=1}^{R_2} \cdots \sum_{r_N=1}^{R_N} \mathcal{G}_1(r_N, :, r_1) \circ \mathcal{G}_2(r_1, :, r_2) \circ \cdots \circ \mathcal{G}_N(r_{N-1}, :, r_N),$$

$$y_{i_1 i_2 \cdots i_N} = \operatorname{tr}(\mathbf{G}_1(:, i_1, :)\mathbf{G}_2(:, i_2, :) \cdots \mathbf{G}_3(:, i_N, :))$$

Shorthand notation for TC

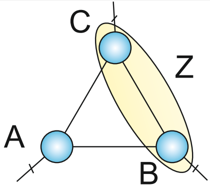
$$\mathcal{Y} = \langle\!\langle \mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_N \rangle\!\rangle$$

When looped network opens

When one of the core tensors, e.g., $\mathcal{G}_N$, in the TC model $\mathcal{Y}$ is a diagonal tensor of 1s, the TC becomes TT model

$$\mathcal{Y} = \mathcal{G}_1 \bullet \mathcal{G}_2 \bullet \cdots \bullet \mathcal{G}_{N-1}$$

ALS Algorithm I



- Consider an order-3 TC decomposition

$$\min \quad \|\mathcal{Y} \approx \llbracket \mathcal{A}, \mathcal{B}, \mathcal{C} \rrbracket\|_F^2$$

- In order to update the core tensor $\mathcal{A}$, we minimize the Frobenius norm Espig et al. (2011, 2012)

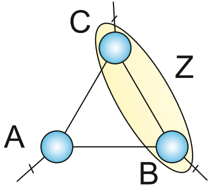
$$\min \quad \|\mathbf{Y}_{(1)} - \mathbf{A}_{(2)}\mathbf{Z}\|_F^2$$

  where $\mathbf{Y}_{(1)}$ is mode-1 unfolding of $\mathcal{Y}$, $\mathbf{A}_{(2)}$ is mode-2 unfolding of $\mathcal{A}$, $\mathbf{Z}$ is mode-(1,3) unfolding of $\mathcal{B} \bullet \mathcal{C}$,

- The update rule reads

$$\mathbf{A}_{(2)} = \mathbf{Y}_{(1)}\mathbf{Z}^{\dagger}$$

ALS Algorithm I

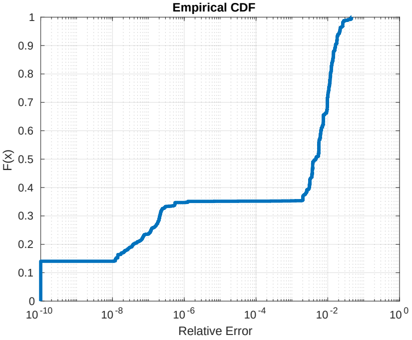


- ALS update rule works efficiently like ALS for other tensor decompositions
- Density-Matrix Renormalization Group (Espig et al. (2011)) and similar ALS algorithms have recently been reinvented or proposed for decomposition of incomplete data

Major problem: Instability

- Unfortunately, loop in TC may lead to severe numerical instability in finding the best TC model, especially when $R_n R_{n+1} > I_n$ (Landsberg (2012) and Handschuh (2015))

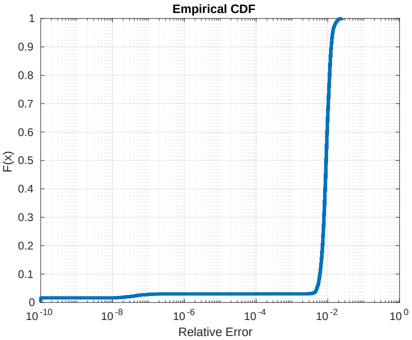
First Example



Decompose synthetic tensor of size $4 \times 4 \times 4$**, rank-(2-2-2).**

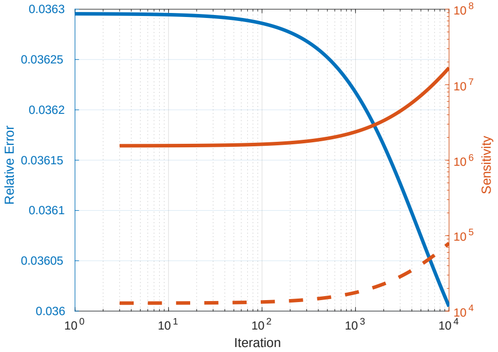
- Tensors are composed from 3 core tensors of size $2 \times 4 \times 2$, generated randomly
- Decomposed using ALS in 5000 iterations with parameters initialized randomly
- Success rate is only 36%.

Second Example (Much harder) I



- Noise free synthetic tensor of size $7 \times 7 \times 7$ with rank-(3-3-3), randomly generated.
- Tensors are decomposed using ALS within 5000 iterations with parameters initialized randomly
- Success rate is much worse, less than 3%.

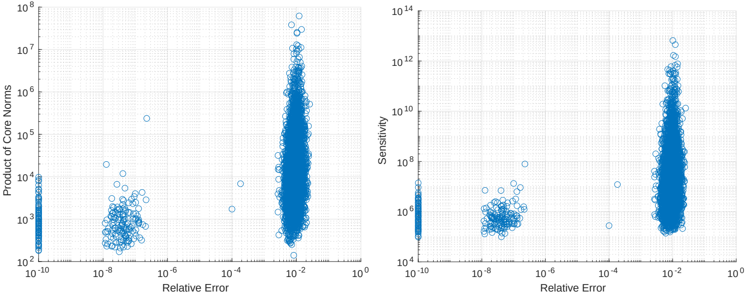
Second Example (Much harder) II



Why ALS and many other algorithms for TC fail

First observation: norms of core tensors quickly increase after several thousand iterations
Approximation gets stuck in false local minima.

Second Example (Much harder) III

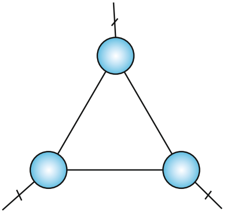


Why ALS and many other algorithms for TC fail

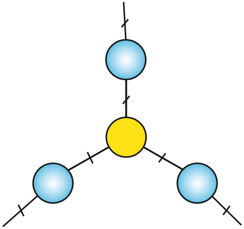
- Instability, especially when the ranks $R_{n-1}R_n > I_n$
- Sensitivity of the estimated model is significantly high, prevents the algorithm from converging to the exact model.

(Relative error vs Sensitivity over 10000 independent runs (100 tensors x 100 decompositions))

TC and Structured Tucker model I



(a) TC-3



(b) Tucker-3

- *Matrix-multiplication tensor* $\mathcal{M}$ obeys the relation

$$\operatorname{vec}(\mathbf{EF}) = \mathcal{M} \times_1 \operatorname{vec}(\mathbf{E})^T \times_2 \operatorname{vec}(\mathbf{F})^T \tag{1}$$

for all matrices **E** and **F** of the size $R_3 \times R_1$ and $R_1 \times R_2$.

TC and Structured Tucker model II

- Applying this relation to order-3 TC tensor, $\mathcal{Y} = \langle\!\langle \mathcal{A}, \mathcal{B}, \mathcal{C} \rangle\!\rangle$ gives

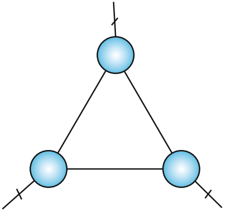
$$\begin{aligned}
y_{ijk} &= \mathrm{tr}(\mathbf{A}_i \mathbf{B}_j \mathbf{C}_k) \\
&= \mathrm{vec}(\mathbf{C}_k)^T \, \mathrm{vec}(\mathbf{A}_i \mathbf{B}_j) \\
&= \mathrm{vec}(\mathbf{C}_k)^T \, (\mathcal{M} \times_1 \mathrm{vec}(\mathbf{A}_i)^T \times_2 \mathrm{vec}(\mathbf{B}_j)^T) \\
&= \mathcal{M} \times_1 \mathrm{vec}(\mathbf{A}_i)^T \times_2 \mathrm{vec}(\mathbf{B}_j)^T \times_3 \mathrm{vec}(\mathbf{C}_k)^T .
\end{aligned}$$

It follows that

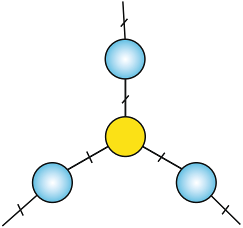
$$\begin{aligned}
\mathcal{Y} &= \mathcal{M} \times_1 \mathbf{A}_{(2)} \times_1 \mathbf{B}_{(2)} \times_3 \mathbf{C}_{(2)} \\
&= [\![\mathcal{M}; \mathbf{A}_{(2)}, \mathbf{B}_{(2)}, \mathbf{C}_{(2)}]\!]
\end{aligned}$$

where $\mathbf{A}_{(2)}, \mathbf{B}_{(2)}, \mathbf{C}_{(2)}$ are mode-2 unfoldings of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, respectively.

TC and Structured Tucker model III



(c) TC-3



(d) Tucker-3

Order-3 TC is a structured Tucker decomposition with fixed known core tensor.

Similar relation can be established for higher order tensors.

- CANonical DEcomposition with LINear Constraints Carroll et al. (1970) PARallel FACtor with LINear DepedencesBro et al. (2009)

$$\mathcal{Y} = [\![\mathbf{AU}, \mathbf{BV}, \mathbf{CW}]\!]$$

  where **U**, **U** and **W** are known dependence matrices.
- Note that multiplication tensor, $\mathcal{M}$, can be represented by a Canonical tensor model with binary factor matrices
- For example, $\mathcal{M}$ of size $4 \times 4 \times 4$ has rank-7

$$\mathbf{M}_{(1)} = \left[\begin{array}{cccc|cccc|cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\mathcal{M} = [\![\mathbf{U}, \mathbf{V}, \mathbf{W}]\!]$$

where

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{bmatrix},$$

$$\mathbf{V} = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

TC and PARallel FACtor with LINear Depedences III

TC is PARALIND

Given the CP decomposition of $\mathcal{M} = [\![\mathbf{U}, \mathbf{V}, \mathbf{W}]\!]$, TC decomposition of $\mathcal{Y}$ becomes

$$\begin{aligned} \mathcal{Y} &= \langle\!\langle \mathcal{A}, \mathcal{B}, \mathcal{C} \rangle\!\rangle && \text{(TC model)} \\ &= [\![\mathcal{M}; \mathbf{A}_{(2)}, \mathbf{B}_{(2)}, \mathbf{C}_{(2)}]\!] && \text{(Tucker model)} \\ &= [\![\mathbf{A}_{(2)}\mathbf{U}, \mathbf{B}_{(2)}\mathbf{V}, \mathbf{C}_{(2)}\mathbf{W}]\!] && (\textit{PARALIND}) \end{aligned}$$

Fitting a TC/TT model is not much different from seeking a structured Tucker or PARALIND model.

Sensitivity for TC I

Definition (Sensitivity)

Given a TC model $\mathcal{Y} = \llbracket \mathcal{A}, \mathcal{B}, \mathcal{C} \rrbracket$. Denote by $\delta\mathcal{A}, \delta\mathcal{B}, \delta\mathcal{C}$ random Gaussian distributed perturbations with element distributed independently with zero mean and variance σ^2.
Sensitivity of the model $\llbracket \mathcal{A}, \mathcal{B}, \mathcal{C} \rrbracket$ is defined as

$$ss(\theta) = \lim_{\sigma^2 \to 0} \frac{1}{\sigma^2} E\left\{ \|\mathcal{Y} - \llbracket \mathcal{A} + \delta\mathcal{A}, \mathcal{B} + \delta\mathcal{B}, \mathcal{C} + \delta\mathcal{C} \rrbracket\|_F^2 \right\}$$

Sensitivity for TC can be derived from that for structured Tucker decomposition

Tichavsky and Phan and Cichocki, Krylov-Levenberg-Marquardt algorithm for Structured Tucker Tensor Decompositions, IEEE Journal of Selected Topics in Signal Processing, 2021

Sensitivity of Kruskal model (for CPD) was proposed in Tichavský et al. (2019)

Tichavsky and Phan and Cichocki, Sensitivity , IEEE Signal processing letters, 2019

Sensitivity for TC II

Consider the error tensor

$$\begin{aligned} & \llbracket \mathcal{A} + \delta\mathcal{A}, \mathcal{B} + \delta\mathcal{B}, \mathcal{C} + \delta\mathcal{C} \rrbracket - \llbracket \mathcal{A}, \mathcal{B}, \mathcal{C} \rrbracket \\ = & \llbracket \delta\mathcal{A}, \mathcal{B}, \mathcal{C} \rrbracket + \llbracket \mathcal{A}, \delta\mathcal{B}, \mathcal{C} \rrbracket + \llbracket \mathcal{A}, \mathcal{B}, \delta\mathcal{C} \rrbracket + \llbracket \delta\mathcal{A}, \delta\mathcal{B}, \mathcal{C} \rrbracket \\ & + \llbracket \delta\mathcal{A}, \mathcal{B}, \delta\mathcal{C} \rrbracket + \llbracket \mathcal{A}, \delta\mathcal{B}, \delta\mathcal{C} \rrbracket + \llbracket \delta\mathcal{A}, \delta\mathcal{B}, \delta\mathcal{C} \rrbracket \end{aligned}$$

Since these TC terms are uncorrelated and expectation of the terms consisting of two or three $\delta\mathbf{A}$, $\delta\mathbf{B}$ and $\delta\mathbf{C}$ are negligible

$$\begin{aligned} & E\{\|\mathcal{Y} - \llbracket \mathcal{A} + \delta\mathcal{A}, \mathcal{B} + \delta\mathcal{B}, \mathcal{C} + \delta\mathcal{C} \rrbracket\|_F^2\} \\ & = E\{\|\llbracket \delta\mathcal{A}, \mathcal{B}, \mathcal{C} \rrbracket\|_F^2\} + E\{\|\llbracket \mathcal{A}, \delta\mathcal{B}, \mathcal{C} \rrbracket\|_F^2\} + E\{\|\llbracket \mathcal{A}, \mathcal{B}, \delta\mathcal{C} \rrbracket\|_F^2\}. \end{aligned}$$

Let $\mathcal{Z} = \mathcal{B} \bullet \mathcal{C}$ be a TT tensor of two cores $\mathcal{B}$ and $\mathcal{C}$. Then $\mathcal{Y} = \llbracket \mathcal{A}, \mathcal{Z} \rrbracket$.

We expand the Frobenius norm $\|\llbracket \delta\mathcal{A}, \mathcal{B}, \mathcal{C} \rrbracket\|_F^2$

$$\begin{aligned}
E\{\|\llbracket \delta\mathcal{A}, \mathcal{B}, \mathcal{C} \rrbracket\|_F^2\} &= E\{\|\delta \mathbf{A}_{(2)} \mathbf{Z}_{(1,4)}\|_F^2\} \\
&= E\{\mathrm{tr}((\delta \mathbf{A}_{(2)}^T \delta \mathbf{A}_{(2)})(\mathbf{Z}_{(1,4)} \mathbf{Z}_{(1,4)}^T)\} \\
&= \sigma^2 I_1 \, \mathrm{tr}(\mathbf{Z}_{(1,4)} \mathbf{Z}_{(1,4)}^T) \\
&= \sigma^2 I_1 \|\mathcal{B} \bullet \mathcal{C}\|_F^2 \\
E\{\|\llbracket \mathcal{A}, \delta\mathcal{B}, \mathcal{C} \rrbracket\|_F^2\} &= \sigma^2 I_2 \|\mathcal{C} \bullet \mathcal{A}\|_F^2 \\
E\{\|\llbracket \mathcal{A}, \mathcal{B}, \delta\mathcal{C} \rrbracket\|_F^2\} &= \sigma^2 I_3 \|\mathcal{A} \bullet \mathcal{B}\|_F^2
\end{aligned}$$

Sensitivity

Sensitivity of a TC model $\llbracket \mathcal{A}, \mathcal{B}, \mathcal{C} \rrbracket$ is computed as

$$ss(\mathcal{A}, \mathcal{B}, \mathcal{C}) = I_1 \|\mathcal{B} \bullet \mathcal{C}\|_F^2 + I_2 \|\mathcal{C} \bullet \mathcal{A}\|_F^2 + I_3 \|\mathcal{A} \bullet \mathcal{B}\|_F^2$$

Balanced norm for Minimal sensitivity I

Sensitivity

$$ss(\mathcal{A}, \mathcal{B}, \mathcal{C}) = l_1 \|\mathcal{B} \bullet \mathcal{C}\|_F^2 + l_2 \|\mathcal{C} \bullet \mathcal{A}\|_F^2 + l_3 \|\mathcal{A} \bullet \mathcal{B}\|_F^2$$

Balanced norm for Minimal sensitivity

A TC model $\llbracket \mathcal{A}, \mathcal{B}, \mathcal{C} \rrbracket$ can be scaled to give a new equivalent model

$$\llbracket \mathcal{A}, \mathcal{B}, \mathcal{C} \rrbracket = \llbracket \alpha_1 \mathcal{A}, \alpha_2 \mathcal{B}, \alpha_3 \mathcal{C} \rrbracket$$

with minimal sensitivity, where

$$\alpha_n = \frac{\beta_n}{\beta}$$

$\beta_1 = \sqrt{l_1} \|\mathcal{B} \bullet \mathcal{C}\|_F$, $\beta_2 = \sqrt{l_2} \|\mathcal{C} \bullet \mathcal{A}\|_F$, $\beta_3 = \sqrt{l_3} \|\mathcal{A} \bullet \mathcal{C}\|_F$ and $\beta = \sqrt[3]{\beta_1 \beta_2 \beta_3}$.

Balanced norm for Minimal sensitivity II

$$\begin{aligned} ss &= \beta_1^2\alpha_2^2\alpha_3^2 + \beta_2^2\alpha_1^2\alpha_3^2 + \beta_3^2\alpha_1^2\alpha_2^2 \\ &\geq 3\sqrt[3]{\beta_1^2\beta_2^2\beta_3^2\alpha_1^4\alpha_2^4\alpha_3^4} \\ &= 3\sqrt[3]{\beta_1^2\beta_2^2\beta_3^2} \end{aligned}$$

Note that $\alpha_1\alpha_2\alpha_3 = 1$.
Equality holds when

$$\frac{\beta_1}{\alpha_1} = \frac{\beta_2}{\alpha_2} = \frac{\beta_3}{\alpha_3}.$$

Hence $\alpha_n = \frac{\beta_n}{\sqrt[3]{\beta_1\beta_2\beta_3}}$.

How to deal with Instability in TC I

- Instability in TC is like degeneracy in Canonical polyadic tensor decomposition, which is hard to avoid
- Instead we propose to correct the unstable estimated model Phan et al. (2018, 2020b)

Error Preserving Correction Method

- Seeking a new tensor, $\hat{\mathcal{Y}}$, which preserves the approximation error but has smaller sensitivity.

$$\begin{array}{ll} \min & ss(\boldsymbol{\theta}) = I_1 \|\mathcal{B} \bullet \mathcal{C}\|_F^2 + I_2 \|\mathcal{C} \bullet \mathcal{A}\|_F^2 + I_3 \|\mathcal{A} \bullet \mathcal{B}\|_F^2 \\ \text{s.t.} & c(\boldsymbol{\theta}) = \|\mathcal{Y} - \hat{\mathcal{Y}}\|_F^2 \leq \delta^2, \end{array}$$

where $\boldsymbol{\theta}$ is vector of parameters and $\hat{\mathcal{Y}} = \langle\!\langle \mathcal{A}, \mathcal{B}, \mathcal{C}, \rangle\!\rangle$

Phan, Tichavsky and Cichocki, Error Preserving Correction: A Method for CP Decomposition at a Target Error Bound, IEEE Transaction on Signal Processing, 2019.

How to deal with Instability in TC II

- Objective and constraint functions are nonlinear in all the factor matrices
- Rewrite the objective function and the constraint function for a single core tensor.
- Then solve the problem using the alternating update scheme

Alternating Sensitivity Correction Method I

$$\min \quad ss(\boldsymbol{\theta}) \quad \text{s.t.} \quad c(\boldsymbol{\theta}) = \|\mathcal{Y} - \hat{\mathcal{Y}}\|_F^2 \le \delta^2,$$

- Rewrite the objective function

$$ss(\boldsymbol{\theta}) = l_1 \|\mathcal{B} \bullet \mathcal{C}\|_F^2 + \operatorname{tr}(\mathbf{Q}\mathbf{A}_{(2)}^T \mathbf{A}_{(2)})$$

where $\mathbf{Q} = l_2(\mathbf{I}_{R_2} \otimes \mathbf{C}_{(3)}\mathbf{C}_{(3)}^T) + l_3(\mathbf{B}_{(1)}\mathbf{B}_{(1)}^T \otimes \mathbf{I}_{R_1})$

- In order to update the core tensor $\mathcal{A}$, we solve the minimization problem

$$\begin{aligned} \min_{\mathbf{X}} \quad & \operatorname{tr}(\mathbf{Q}\mathbf{X}^T\mathbf{X}) \\ \text{s.t.} \quad & \|\mathbf{Y}_{(1)} - \mathbf{X}\mathbf{Z}^T\|_F^2 \le \delta^2 \end{aligned}$$

where $\mathbf{Z}$ is mode-(1,4) unfolding of $\mathcal{B} \bullet \mathcal{C}$,
$\mathbf{X}$ is mode-2 unfolding of the core tensor $\mathcal{A}$, i.e., $\mathbf{A}_{(2)} = \mathbf{X}$.

Alternating Sensitivity Correction Method II

- $\mathcal{A}$ can be found in closed form as Spherical Constrained Quadratic Programming (SCQP) Gander et al. (1989); Phan et al. (2019).

Algorithm 1: `Sensitivity Correction`

Input: Data tensor $\mathcal{Y}$: $(I_1 \times I_2 \times I_3)$, and ranks R and error bound δ

Output: $\hat{\mathcal{Y}} = \langle\!\langle \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3 \rangle\!\rangle$: $\min ss(\hat{\mathcal{Y}}) \quad \text{s.t.} \quad \|\mathcal{Y} - \hat{\mathcal{Y}}\|_F^2 \le \delta^2$

begin
- Initialize $\hat{\mathcal{Y}}$
- **repeat**
  - **for** $n = 1, 2, 3$ **do**
    - $\mathcal{Z} = \mathcal{G}_2 \bullet \mathcal{G}_3$
    - $\mathbf{Q} = I_2(\mathbf{I}_{R_2} \otimes \mathbf{G}_{3,(3)}\mathbf{G}_{3,(3)}^T) + I_3(\mathbf{G}_{2,(1)}\mathbf{G}_{2,(1)}^T \otimes \mathbf{I}_{R_1})$
    - Solve $\mathcal{G}_1 = \arg\min_{\mathbf{X}} \ \mathrm{tr}(\mathbf{X}\mathbf{Q}\mathbf{X}^T) \quad \text{s.t.} \ \|\mathbf{Y}_{(1)} - \mathbf{X}\mathbf{Z}_{(1,4)}\|_F^2 \le \delta^2$
    - Cyclic-shift $\mathcal{Y}$ and $\hat{\mathcal{Y}}$
  - **end**
- **until** *a stopping criterion is met*
- Balance norm of core tensors

end

Sensitivity Correction

Algorithm 2: `Sensitivity Correction Algorithm`

Input: Data tensor $\mathcal{Y}$: $(I_1 \times I_2 \times I_3)$, and ranks R
Output: $\hat{\mathcal{Y}} = \langle\!\langle \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3 \rangle\!\rangle$

```
begin
    repeat
        Run ALS to update the TC tensor Ŷ: min ||Y − Ŷ||_F^2
        if ss(θ) ≤ ss_max then
            Perform sensitivity correction with δ = ||Y − Ŷ||_F^2
        end
    until a stopping criterion is met
end
```

TC Decomposition with Sensitivity control I

- Different from the Sensitivity Correction method, we propose a TC decomposition with bounded sensitivity

$$\min \quad \|\mathcal{Y} - \hat{\mathcal{Y}}\|_F^2 \quad \text{s.t.} \quad ss(\theta) \leq \gamma,$$

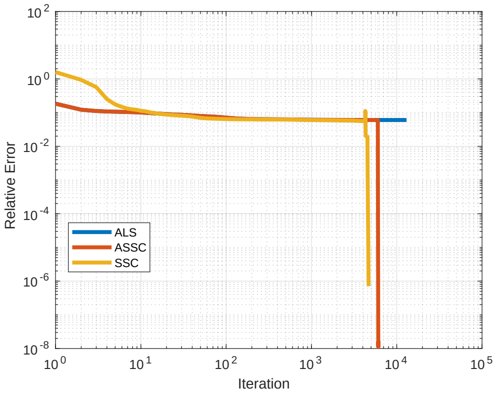
- In order to update the core tensor $\mathcal{A}$, we solve the minimization problem

$$\begin{aligned} \min_{\mathbf{X}} \qquad & \|\mathbf{Y}_{(1)} - \mathbf{X}\mathbf{Z}^T\|_F^2 \\ \text{s.t.} \qquad & \operatorname{tr}(\mathbf{X}\mathbf{Q}\mathbf{X}^T) \leq \gamma_n \end{aligned}$$

where $\gamma_n = \gamma - l_1\|\mathcal{Z}\|_F^2$, $\mathbf{Z}$ is mode-(1,4) unfolding of $\mathcal{Z} = \mathcal{B} \bullet \mathcal{C}$,
$\mathbf{X}$ is mode-2 unfolding of the core tensor $\mathcal{A}$, i.e., $\mathbf{A}_{(2)} = \mathbf{X}$,
$\mathbf{Q} = l_2(\mathbf{I}_{R_2} \otimes \mathbf{C}_{(3)}\mathbf{C}_{(3)}^T) + l_3(\mathbf{B}_{(1)}\mathbf{B}_{(1)}^T \otimes \mathbf{I}_{R_1})$

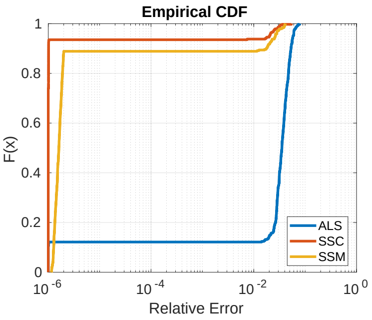
- Similar to the SSC algorithm, $\mathcal{A}$ can be found in closed form.

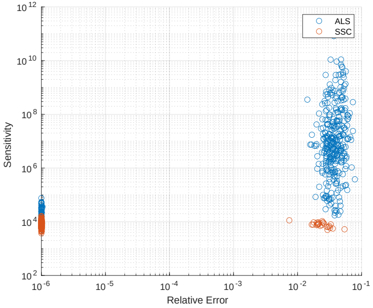
Second Example (cont') I



- Noise-free tensors of size $7 \times 7 \times 7$
- We applied Sensitivity correction after 3000 ALS updates, then continue the decomposition.
- ALS converged quickly after sensitivity correction.

Second Example (cont') II





- In 10000 iterations, ALS succeeded in fitting the tensor in less than 12% of runs.
- With Sensitivity correction (SSC), ALS attained a much higher success rate of 94%.
- Sensitivity minimization (SSM) works well as SSC, with a success rate of 89%.

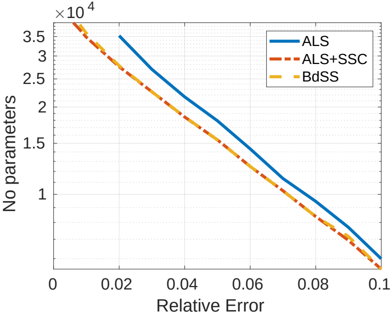
Example: Fitting Images by TC-decomposition I

- Decompose color images of size $128 \times 128 \times 3$ by TC model with ranks $R_1 = R_2$
- For the same approximation bound, we compare three models obtained using ALS, ALS with sensitivity correction, and SSM

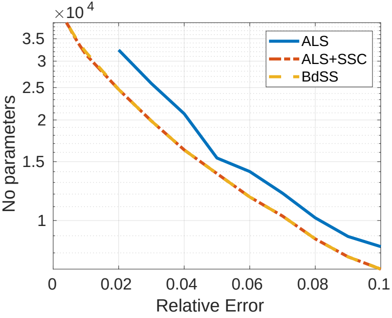
$$||\mathcal{Y} - \mathcal{X}||_F \leq \varepsilon \, ||\mathcal{Y}||_F$$

Sedighin, Cichocki and Phan, Adaptive Rank Selection for Tensor Ring Decomposition, IEEE Journal of Selected Topics in Signal Processing, 2021

Example: Fitting Images by TC-decomposition II

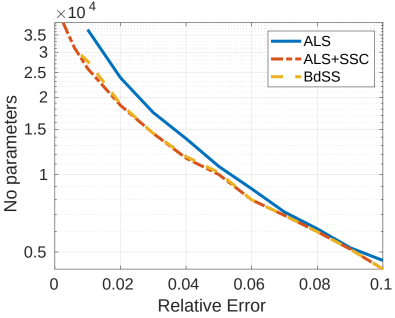


(a) Mandrill

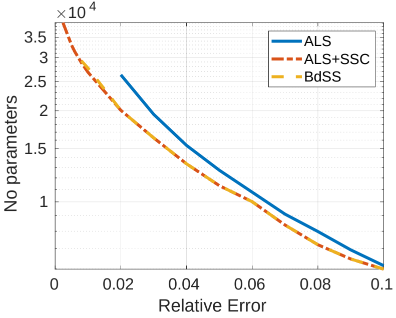


(b) Peppers

Example: Fitting Images by TC-decomposition III

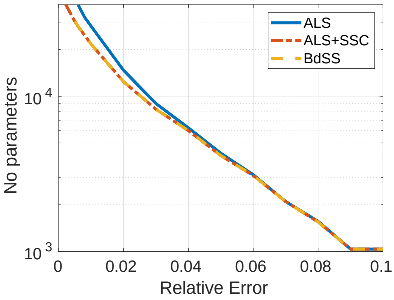


(c) Lena

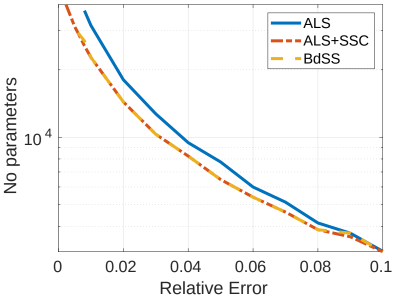


(d) Barbara

Example: Fitting Images by TC-decomposition IV

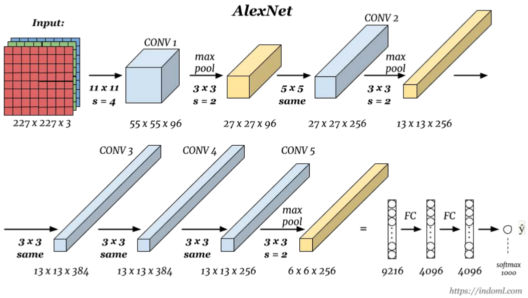


(e) Tiffany



(f) House

Example: Approximation of Convolutional Kernels I



- **AlexNet** one of the first deep convolutional neural networks beating traditional computer vision methodologies, composed of 5 convolutional layers followed by 3 fully connected layers
- Most convolutional neural networks are overparameterized and exhibit high computational cost.

Example: Approximation of Convolutional Kernels II

- Low-rank approximation reduces the number of parameters in convolutional layers.
  Thereby accelerate the inference of the network.
- In Phan et al. (2020b), we show that CPD with sensitivity control can significantly improve compression of widely used CNN including ResNet, VGG, over ordinary CPD methods due to severe degeneracy of the decomposition results.
- By keeping the decomposition at low sensitivity, the compressed CNNs retain their original accuracy (with a minor loss).

Example: Approximation of Convolutional Kernels III

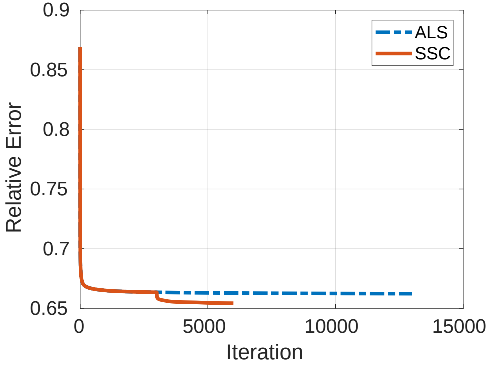


Figure: Decomposition of Layer2 convolutional kernel, rank-(6,10,5). Sensitivity of the estimated TC model after 3000 iterations was 6.5989e+06, and increased to 1.8938e+07 after 13000 iterations. SSC was applied after 3000 ALS updates and yielded a model with sensitivity of 32146.

Example: Approximation of Convolutional Kernels IV

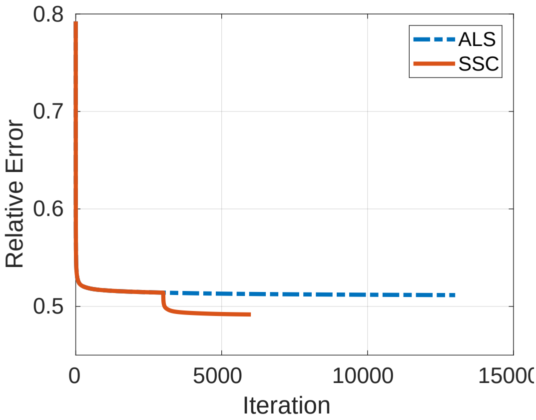


Figure: Decomposition of Layer2 convolutional kernel, rank-(10,10,10). SSC was applied after 3000 ALS updates.

Example: Approximation of Convolutional Kernels V

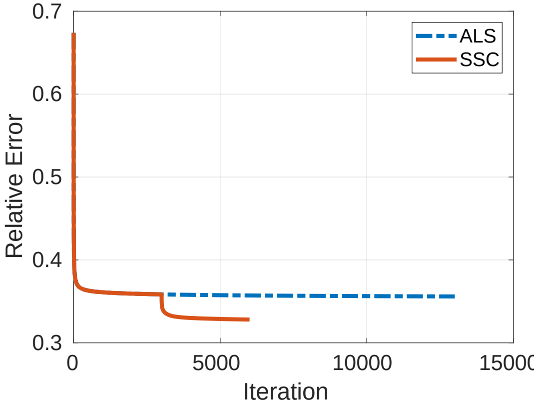


Figure: Decomposition of Layer2 convolutional kernel, rank-(9,18,12). SSC was applied after 3000 ALS updates.

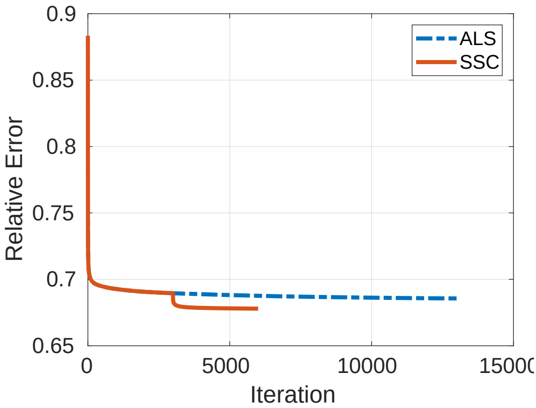


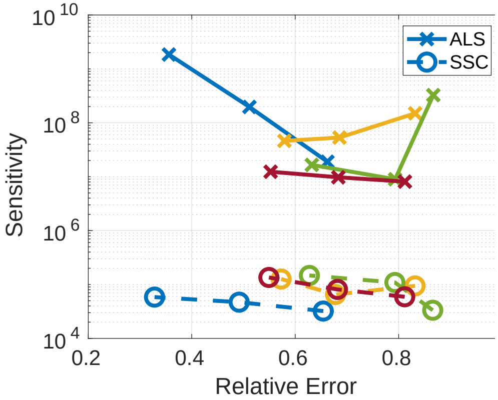
Figure: Decomposition of Layer3 convolutional kernel, rank-(10,10,10). SSC was applied after 3000 ALS updates.

Example: Approximation of Convolutional Kernels VII

Layer	Method	Setting 1	Setting 2	Setting 3
2	ALS	0.6623	0.5116	0.3559
	SSC	**0.6543**	**0.4917**	**0.3281**
3	ALS	0.8319	0.6856	0.5792
	SSC	**0.8317**	**0.6779**	**0.5728**
4	ALS	0.8670	0.7932	0.6321
	SSC	**0.8659**	**0.7928**	**0.6274**
5	ALS	0.8121	0.6833	0.5525
	SSC	**0.8117**	**0.6817**	**0.5492**

Table: Relative approximation errors obtained using ALS and ALS+SSC

Example: Approximation of Convolutional Kernels VIII



Without sensitivity correction or control, TC decomposition often converges to false local minima with very high sensitivity.

TC can encounter severe instability problem, which prevents the compressed CNNs from achieving its original accuracy

References I

Bro, R., Harshman, R. A., Sidiropoulos, N. D., and Lundy, M. E. (2009). Modeling multi-way data with linearly dependent loadings. *Journal of Chemometrics*, 23(7-8):324–340.

Carroll, J., Pruzansky, S., and Kruskal, J. (1970). Candelinc: A general approach to multidimensional analysis of many-way arrays with linear constraints on parameters. *Psychometrika*, 45(1):3–24.

Espig, M., Hackbusch, W., Handschuh, S., and Schneider, R. (2011). Optimization problems in contracted tensor networks. *Comput. Visual. Sci.*, 14(6):271–285.

Espig, M., Naraparaju, K. K., and Schneider, J. (2012). A note on tensor chain approximation. *Computing and Visualization in Science*, 15(6):331–344.

Gander, W., Golub, G. H., and von Matt, U. (1989). A constrained eigenvalue problem. *Special Issue Dedicated to Alan J. Hoffman, Linear Algebra and its Applications*, 114:815 – 839.

Handschuh, S. (2015). *Numerical Methods in Tensor Networks*. PhD thesis, Facualty of Mathematics and Informatics, University Leipzig, Germany, Leipzig, Germany.

Holtz, S., Rohwedder, T., and Schneider, R. (2012). The alternating linear scheme for tensor optimization in the tensor train format. *SIAM J. Scientific Computing*, 34(2).

Kressner, D., Steinlechner, M., and Uschmajew, A. (2014). Low-rank tensor methods with subspace correction for symmetric eigenvalue problems. *SIAM Journal on Scientific Computing*, 36(5):A2346–A2368.

Landsberg, J. M. (2012). *Tensors: Geometry and Applications*, volume 128. American Mathematical Society, Providence, RI, USA.

Oseledets, I. and Tyrtyshnikov, E. (2009). Breaking the curse of dimensionality, or how to use SVD in many dimensions. *SIAM Journal on Scientific Computing*, 31(5):3744–3759.

Phan, A.-H., Cichocki, A., Uschmajew, A., TichavskÃ½, P., Luta, G., and Mandic, D. P. (2020a). Tensor networks for latent variable analysis: Novel algorithms for tensor train approximation. *IEEE Transactions on Neural Networks and Learning Systems*, 31(11):4622–4636.

Phan, A.-H., Sobolev, K., Sozykin, K., Ermilov, D., Gusak, J., Tichavský, P., Glukhov, V., Oseledets, I., and Cichocki, A. (2020b). Stable low-rank tensor decomposition for compression of convolutional neural network. In *Computer Vision – ECCV 2020*, pages 522–539, Cham. Springer International Publishing.

Phan, A.-H., Tichavský, P., and Cichocki, A. (2018). Error preserving correction: A method for CP decomposition at a target error bound. *arXiv preprint*.

Phan, A.-H., Yamagishi, M., and Cichocki, A. (2019). Quadratic programming over ellipsoids and its applications to linear regression and tensor decomposition. *Neural Computing and Applications*.

Tichavský, P., Phan, A.-H., and Cichocki, A. (2019). Sensitivity in tensor decomposition. *IEEE Signal Processing Letters*, 26(11):1653–1657.

Vidal, G. (2003). Efficient classical simulation of slightly entangled quantum computations. *Physical Review Letters*, 91(14):147902.