The geometry of rank-one tensor completion

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Joint work with Thomas Kahle, Mario Kummer, Zvi Rosen in SIAM Journal on Applied Algebra and Geometry

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Rank-one tensors

Definition

An *n*th-order tensor is *rank-one* if it can be written as the tensor product of n vectors, i.e.

$$T = \begin{pmatrix} \theta_{1,1} \\ \vdots \\ \theta_{1,d_1} \end{pmatrix} \otimes \begin{pmatrix} \theta_{2,1} \\ \vdots \\ \theta_{2,d_2} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} \theta_{n,1} \\ \vdots \\ \theta_{n,d_n} \end{pmatrix}.$$

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This means

$$T_{i_1,i_2,\ldots,i_n} = \theta_{1,i_1}\theta_{2,i_2}\ldots\theta_{n,i_n}.$$

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Example

$$\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Rank-one completion

Given a partial tensor

- can it be completed to a rank-one tensor?
- how many rank-one completions there exist?
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$$\left(\begin{array}{cc|cc} ? & 1 & 1 & ? \\ 1 & ? & ? & -1 \end{array}\right)$$

- This partial tensor can be completed to two rank-one complex tensors, but not to a rank-one real tensor.
- Writing it as $\binom{1}{a} \otimes \binom{1}{b} \otimes \binom{c}{d}$ gives

$$bc = 1, \qquad ac = 1, \qquad d = 1, \qquad abd = -1.$$

Literature

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Main directions in the study of low-rank tensor completion:

- Nuclear norm minimization for matricizations
  [Signoretto et al (2010), Tomioka et al (2011), Gandy et al (2011)]
- Nuclear norm minimization for tensors and its relaxations
  [Yuan and Zhang (2016), Rauhut and Stojanac (2015, 2020), Barak and Moitra (2016)]
- Nonconvex methods
  [Jain and Oh (2014), Kressner at al (2014), Xia and Yuan (2019)]

The ideal of rank-1 tensors

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$D = [2]\times[2]\times[2]$ and $E = \{\{1,2,1\},\{2,1,1\},\{1,1,2\},\{2,2,2\}\}$

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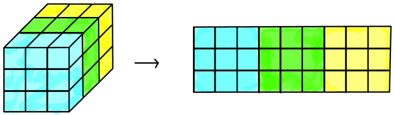
- A flattening of a tensor is a matrix that is obtained from the tensor by partitioning the modes into two nonempty subsets.



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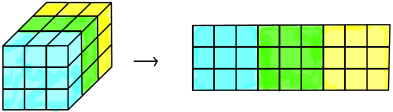

- A tensor $T \in \mathbb{F}^D$ has rank at most one if and only if all (2×2)-minors of all its flattenings vanish.

The ideal of rank-1 tensors

$$\left(\begin{array}{cc|cc} ? & 1 & 1 & ? \\ 1 & ? & ? & -1 \end{array}\right)$$

- Let I be the ideal in $\mathbb{F}[x_i : i \in D]$ generated by (2×2)-minors of all flattenings of the tensor of variables x_i.

The ideal of rank-1 tensors

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- Let I be the ideal in $\mathbb{F}[x_i : i \in D]$ generated by (2×2)-minors of all flattenings of the tensor of variables x_i.

$$\begin{aligned} I =& \langle -x_{112} * x_{211} + x_{111} * x_{212}, -x_{121} * x_{211} + x_{111} * x_{221}, \\ & - x_{121} * x_{212} + x_{112} * x_{221}, -x_{122} * x_{211} + x_{111} * x_{222}, \\ & - x_{122} * x_{212} + x_{112} * x_{222}, -x_{122} * x_{221} + x_{121} * x_{222}, \\ & - x_{112} * x_{121} + x_{111} * x_{122}, -x_{121} * x_{21,1} + x_{111} * x_{221}, \\ & - x_{122} * x_{211} + x_{112} * x_{221}, -x_{121} * x_{212} + x_{111} * x_{222}, \\ & - x_{122} * x_{212} + x_{112} * x_{222}, -x_{212} * x_{221} + x_{211} * x_{222}, \\ & - x_{112} * x_{121} + x_{111} * x_{122}, -x_{112} * x_{211} + x_{111} * x_{212}, \\ & - x_{122} * x_{211} + x_{121} * x_{212}, -x_{112} * x_{221} + x_{111} * x_{222}, \\ & - x_{122} * x_{221} + x_{121} * x_{222}, -x_{212} * x_{221} + x_{211} * x_{222} \rangle \end{aligned}$$

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$$\begin{pmatrix} ? & 1 & \big| & 1 & ? \\ 1 & ? & \big| & ? & -1 \end{pmatrix}$$

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- Let I_E be the intersection of this ideal with $\mathbb{F}[x_i : i \in E]$.

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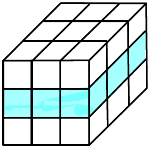
- Let I_E be the intersection of this ideal with $\mathbb{F}[x_i : i \in E]$.

$$I_E = \langle 0 \rangle$$

Rank-one tensor completion over complex numbers

Definition

Fix $j \in [n]$ and $i_j \in [d_j]$. A *maximal slice* of a partial tensor $T \in \mathbb{R}^E$ is the tensor with index set $E \cap [d_1] \times \cdots \times [d_{j-1}] \times \{i_j\} \times [d_{j+1}] \times \cdots \times [d_n]$ which arises from T by fixing the jth index as i_j.



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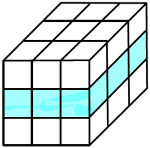

Definition

A partial tensor is *zero-consistent* if every zero entry is contained in a maximal dimensional slice consisting of only zero entries.

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Denote by $V(I_E)$ the set of common zeros of $f \in I_E$.

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Proposition

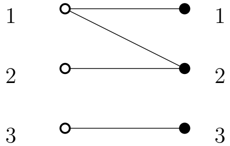
A partial tensor $T_E \in \mathbb{C}^E$ equals the restriction of a rank-one tensor $T \in \mathbb{C}^D$ to E if and only if the following two conditions hold

1. *The partial tensor T_E is zero-consistent.*
2. *The variety $V(I_E)$ contains T_E.*

Rank-one matrix completion

Rank-one matrix completion can be studied combinatorially using graph theory:

$$\begin{pmatrix} M_{1,1} & M_{1,2} & ? \\ ? & M_{2,2} & ? \\ ? & ? & M_{3,3} \end{pmatrix}$$



- A partial matrix satisfies the second condition in the proposition if and only if on every cycle, the product over the edges with even indices equals the product over the edges with odd indices. [1,2]

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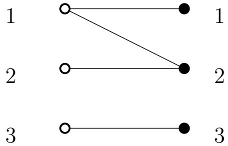

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- This observation follows from the explicit description of the generators of the universal Gröbner basis in terms of cycles on the bipartite graph.

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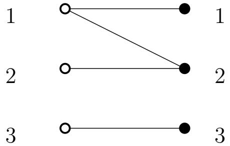

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- In the tensor case the combinatorial interpretations break down, because the universal Gröbner is not square-free.

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Rank-one tensor completion over real numbers

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1. *The partial tensor* T_E *is zero-consistent.*
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Example

The partial tensor

$$\left(\begin{array}{cc|cc} ? & 1 & 1 & ? \\ 1 & ? & ? & -1 \end{array}\right)$$

shows that the previous proposition fails if $\mathbb{C}$ is replaced by $\mathbb{R}$.

Rank-one tensor completion over real numbers

To study the rank-one tensor completion over real numbers, we consider the linear map

$$\mathbb{Z}^D \to \mathbb{Z}^{d_1+\cdots+d_n}, \qquad e_i \mapsto \sum_{j=1}^{n} e_{j,i_j},$$

whose matrix we denote $A \in \{0,1\}^{(d_1+\cdots+d_n)\times D}$.
Let A_E be the matrix whose columns are exactly the columns of A with indices in E.

Example

$$\left(\begin{array}{cc|cc} ? & 1 & 1 & ? \\ 1 & ? & ? & -1 \end{array}\right)$$

$D = [2] \times [2] \times [2]$ and $E = \{\{1,2,1\},\{2,1,1\},\{1,1,2\},\{2,2,2\}\}$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \text{ and } A_E = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Rank-one tensor completion over real numbers

Definition

Let $L \subseteq \mathbb{Z}^m$ be a lattice. The saturation of L is the lattice

$$\{\alpha \in \mathbb{Z}^m : k\alpha \in L \text{ for some } k \in \mathbb{Z}\backslash\{0\}\}.$$

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Example

The sum of the columns of $A_E = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ is $\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$.

The saturation of the lattice spanned by the columns of A_E is spanned by the columns of A_E and $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T$.

Rank-one tensor completion over real numbers

Proposition

For a given subset $E \subseteq D$ the following are equivalent:

(i) *Every real partial tensor $T_E \in \mathbb{R}^E$ with nonzero entries which is completable over the complex numbers is also completable over the real numbers.*

(ii) *The index of the lattice spanned by the columns of A_E in its saturation is odd.*

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Moreover, whether a real partial tensor $T_E \in \mathbb{R}^E$ with nonzero entries which is completable over the complex numbers is also completable over the real numbers depends only on the signs of the entries of T_E.

Finite completability

Example

$$\begin{pmatrix} ? & 1 & \vline & 1 & ? \\ 1 & ? & \vline & ? & -1 \end{pmatrix}$$

- This partial tensor can be completed to two rank-one complex tensors, but not to a rank-one real tensor.
- The index of the lattice spanned by the columns of A_E in its saturation is 2.

Finite completability

Example

$$\left(\begin{matrix} ? & 1 \\ 1 & ? \end{matrix}\;\middle|\;\begin{matrix} 1 & ? \\ ? & -1 \end{matrix}\right)$$

- This partial tensor can be completed to two rank-one complex tensors, but not to a rank-one real tensor.
- The index of the lattice spanned by the columns of A_E in its saturation is 2.
- Writing it as $\left(\begin{smallmatrix} 1 \\ a \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1 \\ b \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right)$ gives

$$bc = 1, \qquad ac = 1, \qquad d = 1, \qquad abd = -1.$$

- The failure of real completability comes from the negative $(2, 2, 2)$ entry.

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If three entries of a rank-one 2×2 matrix are given, the determinant becomes a linear polynomial determining the fourth entry.

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Proposition

For generic T_E, the set of finitely completable entries does not depend on the entries of T_E, but only on E. The entries that are finitely completable from the entries in E are given by a matroid closure $cl(E)$ of the column matroid of the matrix A.

Finite completability

Saky Myllymäki studied in his Bachelor thesis the number of entries required for finite completability:

- There exists E of size $\sum_{i=1}^{n} d_i - n + 1$ such that every generic T_E is finitely completable.
- Observation: Let $d_1 \leq \ldots \leq d_n$. If E is of size at least $(d_n - 1) \prod_{i=1}^{n-1} d_i + 1$, then every generic T_E that is rank-one completable, is finitely completable.

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- $3 \times 3 \times 3$ tensors:
  - If $|E| < 7$, then T_E is not finitely completable.
  - Percentage of finitely completable E:

7	8	9	10	11	12
38.3%	64.9%	80.4%	89.2%	94.2%	97.0%

13	14	15	16	17	18
98.6%	99.4%	99.7%	99.9%	$\approx$ 100%	$\approx$ 100%

  - If $|E| > 18$ and T_E is rank-one completable, then it is finitely completable.

Unique completability

Corollary

(i) *A partial tensor with nonzero entries is uniquely completable to a complex rank-one tensor if and only if it is finitely completable and the lattice spanned by the columns of* A_E *is saturated.*

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(i) *A partial tensor with nonzero entries is uniquely completable to a complex rank-one tensor if and only if it is finitely completable and the lattice spanned by the columns of A_E is saturated.*

(ii) *A real partial tensor with nonzero entries is uniquely completable to a real rank-one tensor if and only if it is finitely completable and the index of the lattice spanned by the columns of A_E in its saturation is odd.*

Summary

- Rank-one tensor completion is an algebraic problem.
- The lattice generated by A_E plays an important role for real rank-one completability and unique completability.
- The challenges with higher rank are
  - the set of tensors of rank at most r is not closed and
  - the ideal is not binomial (or in general even well-understood).

References

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- Hadwin, Don, K. Harrison, and Jo Ward. "Rank-one completions of partial matrices and completely rank-nonincreasing linear functionals." Proceedings of the American Mathematical Society 134, no. 8 (2006): 2169-2178.
- Kahle, Thomas, Kaie Kubjas, Mario Kummer, and Zvi Rosen. "The geometry of rank-one tensor completion." SIAM Journal on Applied Algebra and Geometry 1, no. 1 (2017): 200-221.

Thank you!