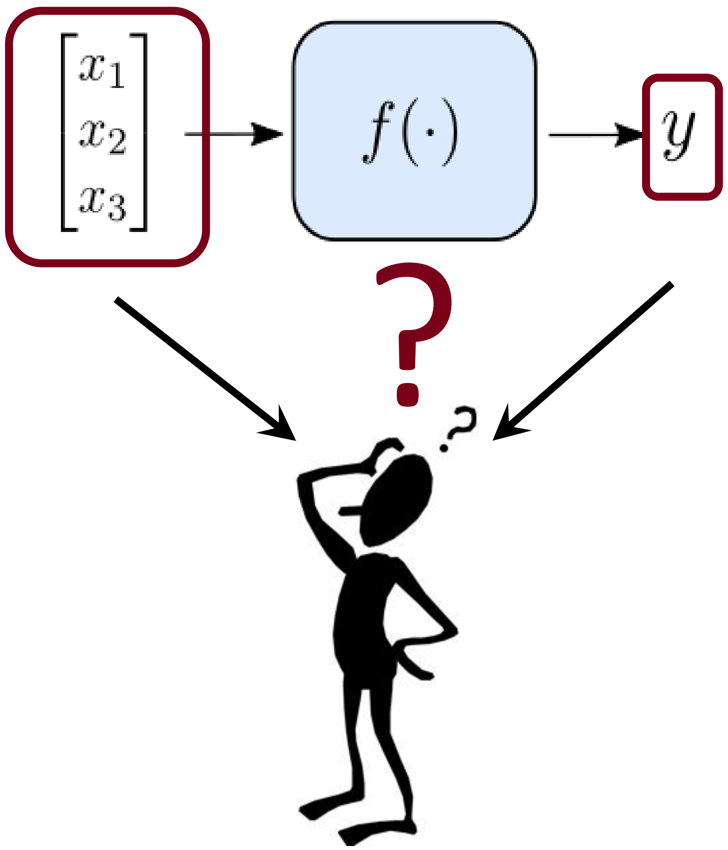
Supervised Learning and Canonical Decomposition of Multivariate Functions

N.D. Sidiropoulos (joint work with N. Kargas)

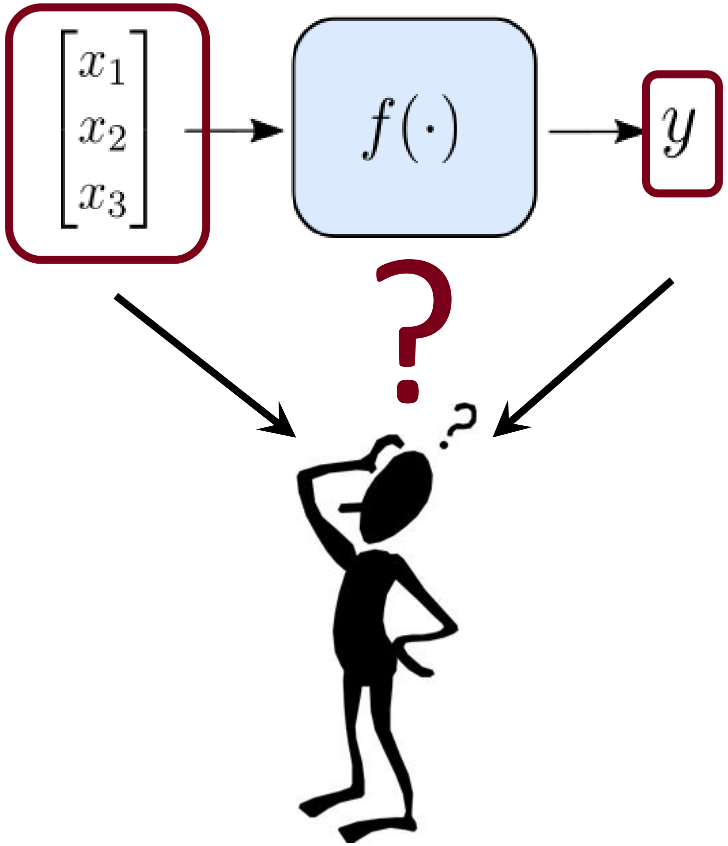
Workshop III: Mathematical Foundations and Algorithms for Tensor Computations, May 5, 2021

The Supervised Learning Problem



Categorical (classification, binary or FA)
Real-valued (prediction, regression)
Complex-valued (channel; MRI k-space)

AKA: I/O (Nonlinear) System Identification



Categorical (classification, binary or FA)
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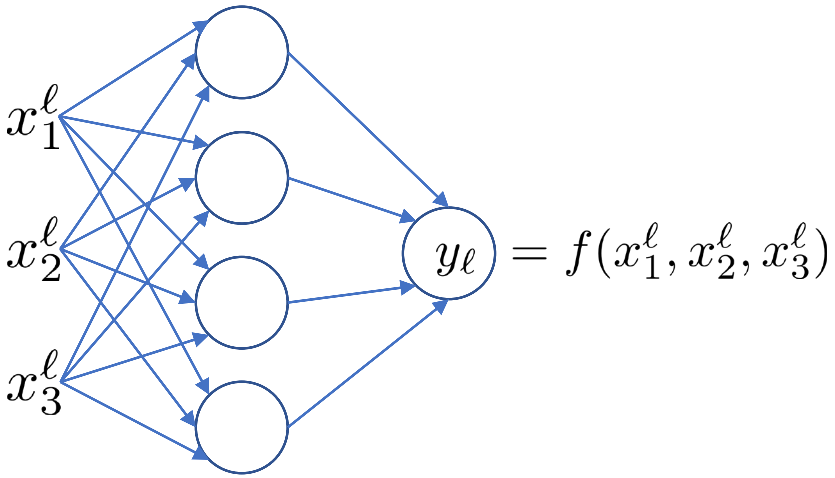
(Deep) Neural Networks



- Most popular method for learning to mimic nonlinear functions
- Some theory … but, for most part …
  - Don't understand why they work so well
  - Choosing architecture is art
  - Hard to interpret
- Against all odds and principles!

(Deep) Neural Networks

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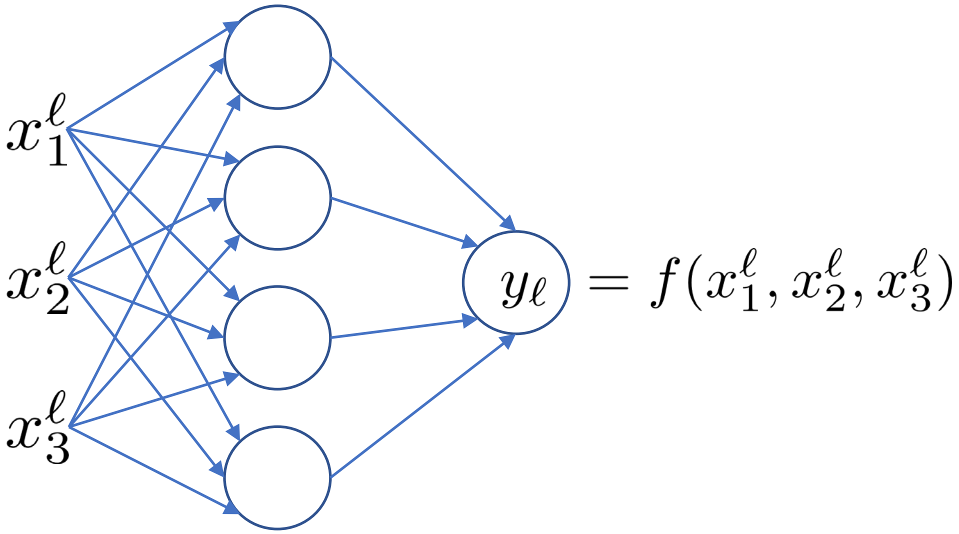
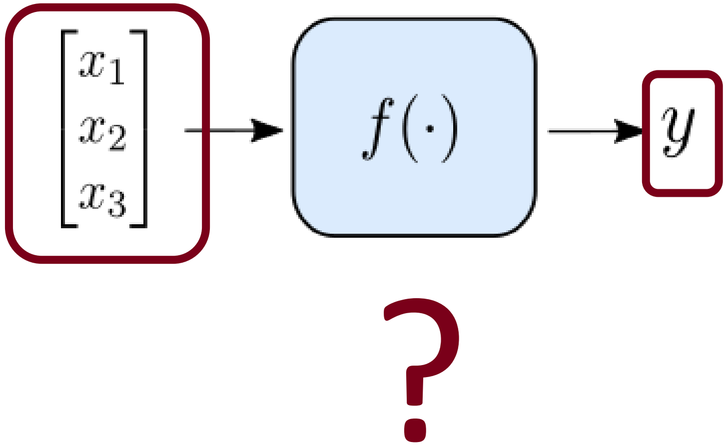


- This talk: principled alternative
- Based on tensor principal components
- Advantages: `universal', intuitive, interpretable, backed by theory
- Works with incomplete input data – important in practice

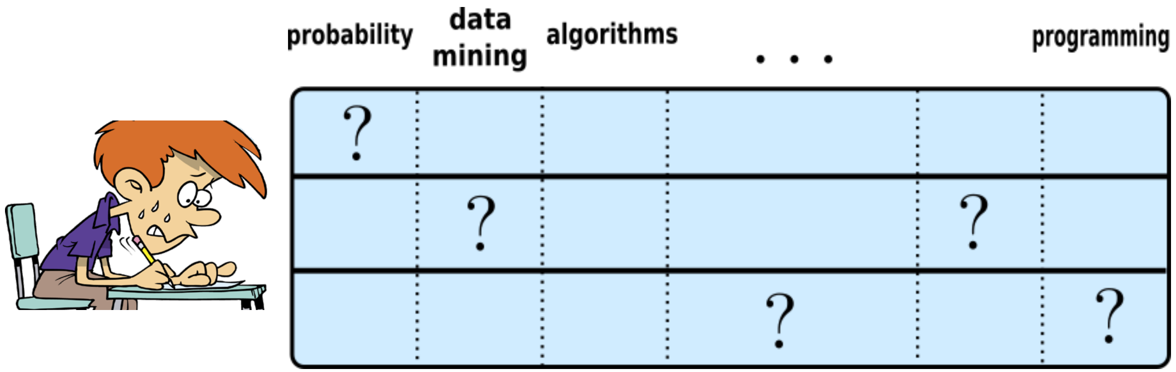
Introduction

- General nonlinear function identification
  - `Supervised' - from input-output data
  - Function approximation problem
  - Identifiability? Performance? Complexity?

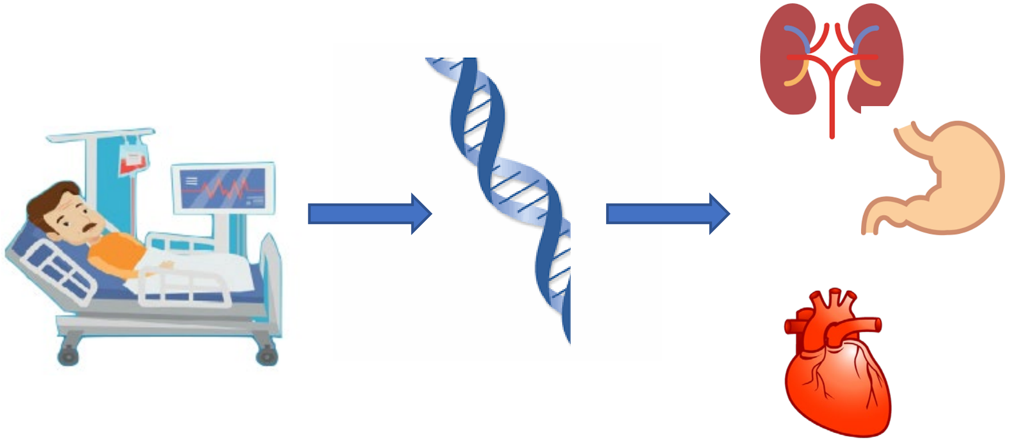
- Applications
  - Machine learning
  - Dynamical system identification and control
  - Communications



Motivation

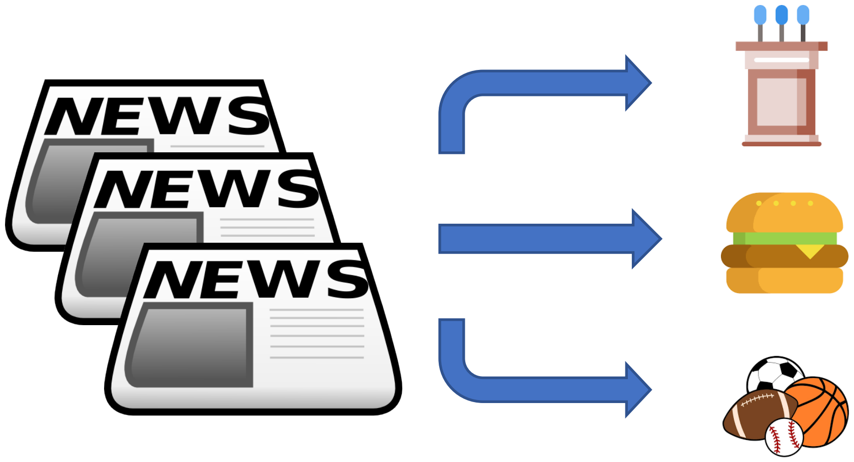


Course grade prediction

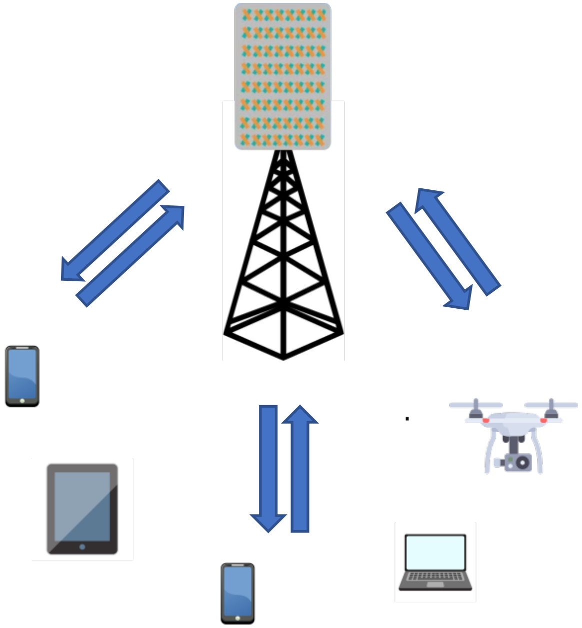


Drug response prediction

Motivation



Text classification

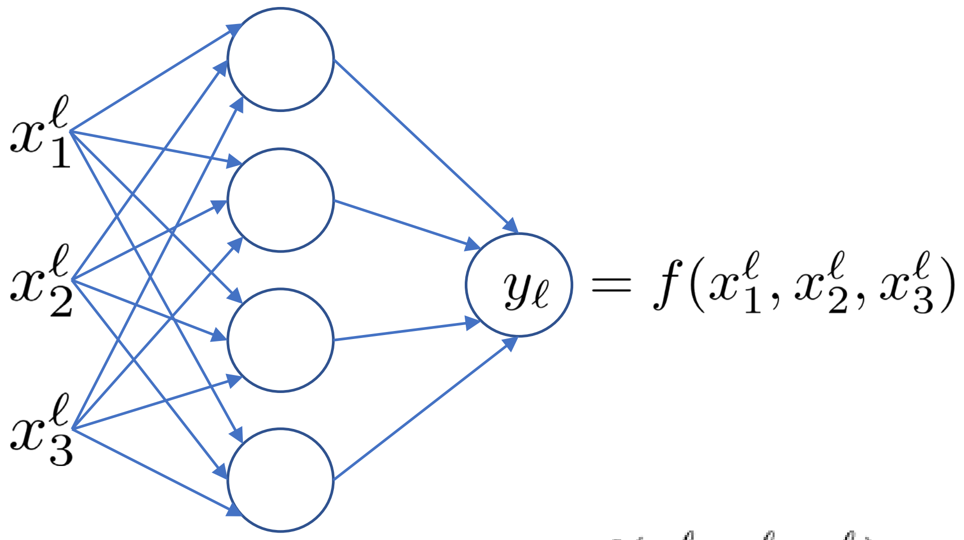


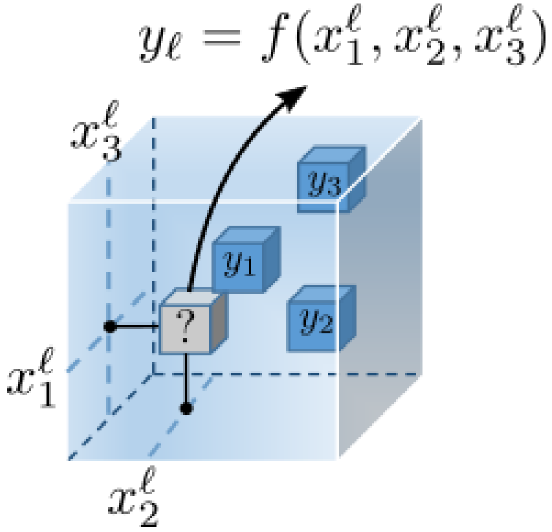
Channel estimation

Sneak preview

- Deep neural networks
  - Work very well in practice
  - Hard to interpret
  - Difficult to tune

- In this work:
  - Simple and elegant alternative
  - Low-rank tensor decomposition
  - Model any nonlinearity
  - Identification guarantees



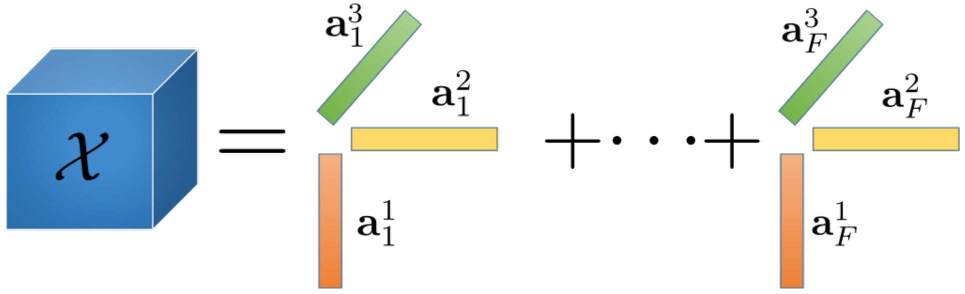


Canonical Polyadic Decomposition (CPD)

- An N-way tensor (multi-way array) admits a decomposition of rank *F* it can be decomposed as a sum of *F* rank-1 tensors

$$\mathcal{X} = \sum_{f=1}^{F} \mathbf{a}_f^1 \circ \mathbf{a}_f^2 \circ \cdots \circ \mathbf{a}_f^N$$

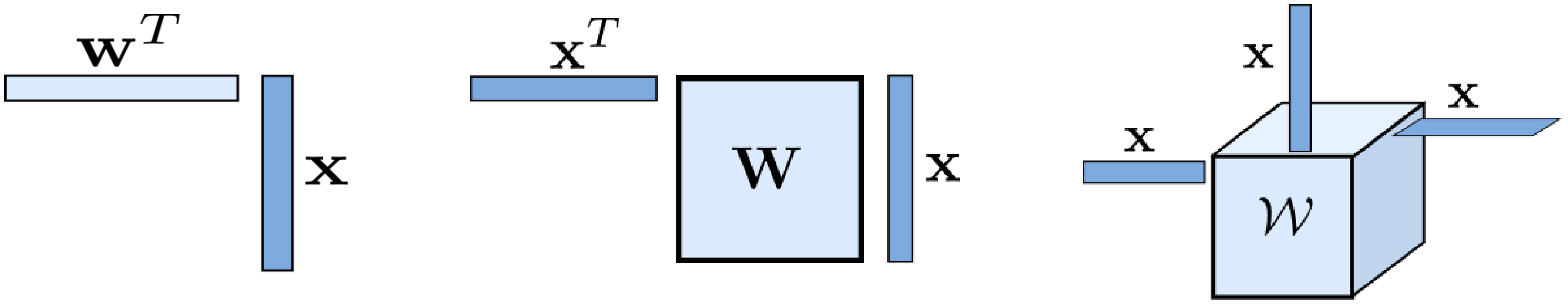
- Tensor rank is smallest *F* for which such decomposition exists → *Canonical*



- Element-wise: $\mathcal{X}(i_1, \ldots, i_N) = \sum_{f=1}^{F} \prod_{n=1}^{N} \mathbf{a}_f^n(i_n)$
- Matrix unfolding: $\mathcal{X}^{(n)} = (\mathbf{A}_N \odot \cdots \odot \mathbf{A}_{n+1} \odot \mathbf{A}_{n-1} \cdots \odot \cdots \mathbf{A}_1)\mathbf{A}_n^T$
- Vector: $\text{vec}(\mathcal{X}) = (\mathbf{A}_N \odot \cdots \odot \mathbf{A}_1)\mathbf{1}$

Prior work

- Tensor modeling of low-order multivariate polynomial systems (Rendle, 2010)
- A multivariate polynomial of order d is represented by a tensor of order d



$y = \mathbf{w}^T \mathbf{x}$ $\quad$ $y = \mathbf{x}^T \mathbf{W} \mathbf{x}$ $\quad$ $y = \mathcal{W} \times_1 \mathbf{x} \times_2 \mathbf{x} \times_3 \mathbf{x}$

Prior work

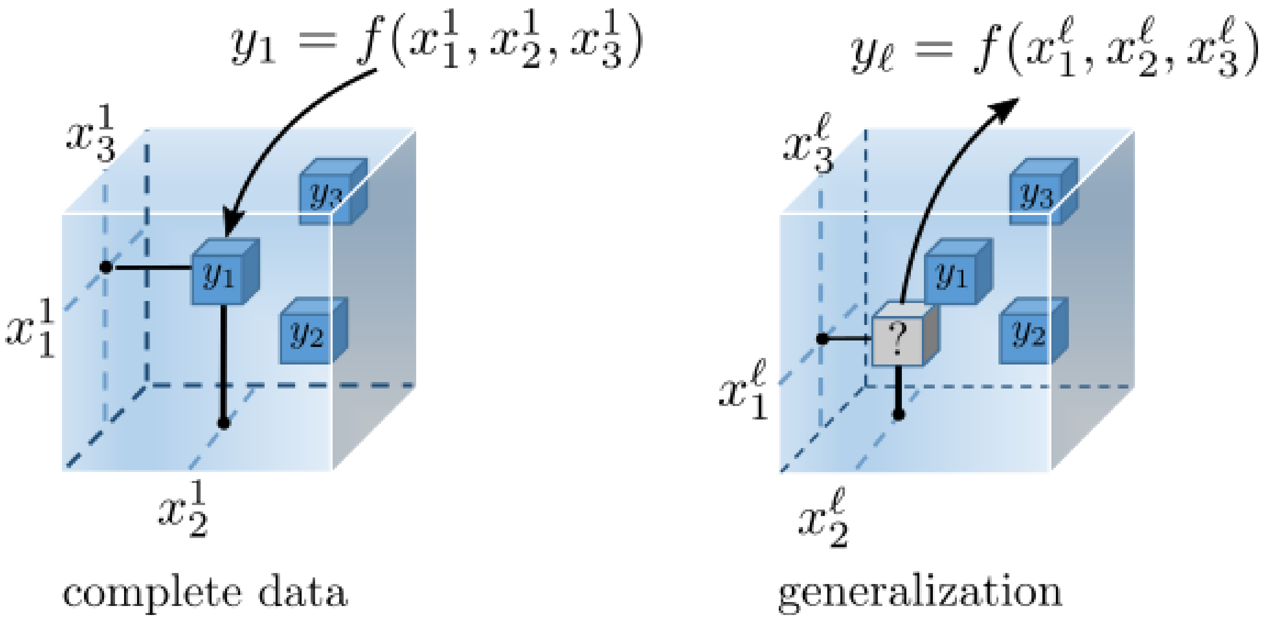
- Number of parameters grows exponentially with the order d

  ⟹ Assume that the coefficient tensor is low-rank

- <u>Drawbacks</u>
  - Require prior knowledge of polynomial order
  - Assuming polynomial of a given degree can be restrictive
  - Simplest rank=1 model ⟹ number of parameters grows linearly with d
  - Cannot model high-degree polynomial functions

Canonical System Identification (CSID)

- We propose:
  - Single high-order tensor for learning a general nonlinear system



Kargas, N., and Sidiropoulos, N. D. "Nonlinear System Identification via Tensor Completion", AAAI 2020, NYC.

Canonical System Identification (CSID)

- <u>Claims:</u>
  - CPD can model *any* nonlinearity (even of ∞ order) for high-enough rank. Even for low ranks, it can model highly nonlinear operators
  - Provably correct nonlinear system identification from limited samples, when the tensor is low rank
  - Even when not low rank ⟹ identification of the principal components!

Rank of generic nonlinear systems?

- Seperable function: $y = f(x_1, \ldots, x_N) = \prod_{n=1}^{N} f_n(x_n)$
  - Rank: 1

    e.g., $f(x_1, \ldots, x_N) = \prod_{n=1}^{N} \mathrm{sign}(x_n)$
- Sum of separable functions: $y = f(x_1, \ldots, x_N) = \sum_{n=1}^{N} f_n(x_n)$
  - Maximal rank: N

    e.g., $f(x_1, \ldots, x_N) = \sum_{n=1}^{N} \mathrm{sign}(x_n)$
- Sum of pairwise functions: $y = f(x_1, \ldots, x_N) = \sum_{i=1}^{N} \sum_{j>i} f_{ij}(x_i, x_j)$
  - Maximal rank: $\frac{IN^2}{2} \ll I^{N-1}$

- Other nonlinear systems?

Problem formulation

- Each input vector $[\mathbf{x}_m(1),\ldots,\mathbf{x}_m(N)]$ is viewed as a cell multi-index and the cell content is the estimated response of the system:

$$\min_{\mathcal{X}} \frac{1}{M}\sum_{m=1}^{M}\left(y_m - \mathcal{X}\left(\mathbf{x}_m(1),\ldots,\mathbf{x}_m(N)\right)\right)^2$$

- We aim for the principal components of the nonlinear operator:

$$\begin{aligned}\min_{\mathcal{X},\{\mathbf{A}_n\}_{n=1}^{N}} \quad & \frac{1}{M}\sum_{m=1}^{M}\left(y_m - \mathcal{X}(\mathbf{x}_m(1),\ldots,\mathbf{x}_m(N))\right)^2 + \sum_{n=1}^{N}\rho\|\mathbf{A}_n\|_F^2 \\ \text{subject to} \quad & \mathcal{X} = \sum_{f=1}^{F}\mathbf{A}_1(:,f)\circledcirc\cdots\circledcirc\mathbf{A}_N(:,f)\end{aligned}$$

Handling ordinal features

- Datasets often contain both categorical and ordinal predictors.

$$\min_{\mathcal{X},\{\mathbf{A}_n\}_{n=1}^N} \frac{1}{M}\left\|\sqrt{\mathcal{W}} \circledast (\mathcal{Y}-\mathcal{X})\right\|_F^2 + \sum_{n=1}^N \rho\|\mathbf{A}_n\|_F^2 + \sum_{n=1}^N \mu_n\|\mathbf{T}_n\mathbf{A}_n\|_F^2$$

$$\text{subject to} \quad \mathcal{X} = \sum_{f=1}^F \mathbf{A}_1(:,f) \odot \cdots \odot \mathbf{A}_N(:,f),$$

where

$$\mathbf{T}_n = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix} \quad \text{or} \quad \mathbf{T}_n = \begin{bmatrix} 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \end{bmatrix}$$

Tensor completion: Identifiability

- Probabilistic results
  - Adaptive sampling (Krishnamurthy and Singh 2013)
  - Random sampling with orthogonal factors (Jain and Oh 2014)
  - Random sampling assuming low mode-n ranks (Huang et al. 2014)

- Deterministic results
  - Fiber sampling (Sorensen and De Lathauwer 2019)
  - Regular sampling (Kanatsoulis et al. 2019)

Tensor completion: Identifiability

- Depends on how the x-samples are generated – randomly or systematically, and if randomly from what distribution
- Practical experience: generic sample complexity for randomly drawn point samples ~ degrees of freedom $O(FNI)$ in the model. Proven for randomly drawn *linear* (generalized, aggregated) samples in
  - M. Bousse, N. Vervliet, I. Domanov, O. Debals, and L. De Lathauwer, "Linear systems with a canonical polyadic decomposition constrained solution: Algorithms and applications", *Numerical Linear Algebra with Applications*, vol. 25, no. 6, Aug. 2018.
- … but not (yet?) for point samples.
- For $F < I$, can show that for uniform random point samples, the sample complexity for our low-rank model is $O(\sqrt{FI^N}\log(N))$, using
  - M. Yuan C. Zhang, "On Tensor Completion via Nuclear Norm Minimization", *Foundations Computational Mathematics*, vol. 16, no. 4, Aug. 2016.

Algorithm

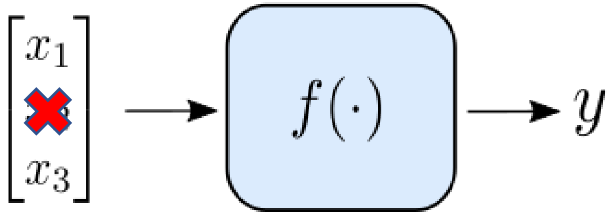
- Alternating minimization
  - Exploit sparsity (Smith and Karypis 2015)
  - Cyclically update variables
  - Lightweight row-wise updates

$$\min_{\mathcal{X},\{\mathbf{A}_n\}_{n=1}^N} \frac{1}{M}\left\|\sqrt{\mathcal{W}} \circledast (\mathcal{Y}-\mathcal{X})\right\|_F^2 + \sum_{n=1}^N \rho\|\mathbf{A}_n\|_F^2 + \sum_{n=1}^N \mu_n\|\mathbf{T}_n\mathbf{A}_n\|_F^2$$

$$\text{subject to} \quad \mathcal{X} = \sum_{f=1}^F \mathbf{A}_1(:,f)\odot\cdots\odot\mathbf{A}_N(:,f),$$

- Large scale problems ⟹ SGD, Block-stochastic GD

Missing data



- Let $\mathcal{O}$ and $\mathcal{M}$ denote the indices of the observed and missing entries of a single observation

$$f(\mathbf{x}_{\mathcal{O}}) = \mathbb{E}_{\mathbf{x}_{\mathcal{M}}|\mathbf{x}_{\mathcal{O}}}[f(\mathbf{x}_{\mathcal{O}}, \mathbf{x}_{\mathcal{M}})] = \sum_{\mathbf{x}_{\mathcal{M}}} P_{X_{\mathcal{M}}|X_{\mathcal{O}}}(\mathbf{x}_{\mathcal{M}}|\mathbf{x}_{\mathcal{O}}) f(\mathbf{x}_{\mathcal{O}}, \mathbf{x}_{\mathcal{M}})$$

- We adopt a simple rank-1 joint PMF model estimated via the empirical one-dimensional marginal distributions (K. Huang, N. D. Sidiropoulos, 2017)

$$f(\mathbf{x}_{\mathcal{O}}) = \mathbb{E}_{\mathbf{x}_{\mathcal{M}}|\mathbf{x}_{\mathcal{O}}}[f(\mathbf{x}_{\mathcal{O}}, \mathbf{x}_{\mathcal{M}})] = \mathcal{X}(i_1, \ldots, i_T, :, \ldots, :) \times_{T+1} \mathbf{p}_{T+1} \cdots \times_{T+L} \mathbf{p}_N$$
$$= \sum_{f=1}^{F} \prod_{n=1}^{T} \mathbf{A}_n(i_n, f) \prod_{n=T+1}^{N} \mathbf{p}_n^T \mathbf{A}_n(:, f)$$

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Multi-output regression

- No correlation between the K output variables ⟹ build K independent models
- Output variables are usually correlated

- <u>Better approach:</u>
  - Build a single model capable of predicting all K outputs $\mathcal{X} = [\![\mathbf{A}_1, \ldots, \mathbf{A}_N, \mathbf{V}]\!]_F$
  - The new tensor model can be described by N+1 factors
  - No modification is needed for the ALS updates
  - Prediction: $\mathcal{X}(i_1, \ldots, i_N, :) = (\mathbf{A}_1(i_1, :) \circledast \cdots \circledast \mathbf{A}_N(i_N, :))\, \mathbf{V}^T$

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Experiments

- Regression task using 9 UCI datasets
- Grade prediction task
  - 20 CS courses selected from University of Minnesota
  - 20 independent models using 34 courses as predictors

- 10 Monte Carlo simulations
- 80% training, 20% test (5-fold cross-validation for parameter selection)
- Evaluate the performance using RMSE

Dataset information

Dataset	N	M	Type	Range
Concrete Compressive Strength	8	1030	Ordinal	$y \in (2, 83)$
SkillCraft Master Table	18	3337	Ordinal	$y \in (1, 7)$
Abalone	8	4177	Mixed	$y \in (1, 29)$
Wine Quality	11	4898	Ordinal	$y \in (3, 9)$
Combined Cycle Power Plant	4	9568	Ordinal	$y \in (420, 496)$
Physicochemical Properties	9	45730	Ordinal	$y \in (0, 21)$
Energy efficiency (2)	8	788	Ordinal	$y_1 \in (6, 44)$ $y_2 \in (10, 49)$
Parkinsons Telemonitoring (2)	19	5875	Mixed	$y_1 \in (5, 40)$ $y_2 \in (7, 55)$
Bike Sharing (2)	12	17379	Mixed	$y_1 \in (0, 367)$ $y_2 \in (0, 886)$

Dataset	N	M	Sparsity
CSCI-1	34	996	0.54
CSCI-2	34	990	0.55
CSCI-3	34	983	0.55
CSCI-4	34	958	0.55
CSCI-5	34	953	0.56
CSCI-6	34	931	0.56
CSCI-7	34	911	0.56
CSCI-8	34	898	0.56
CSCI-9	34	867	0.56
CSCI-10	34	856	0.57

Dataset	N	M	Sparsity
CSCI-11	34	704	0.57
CSCI-12	34	696	0.58
CSCI-13	34	650	0.57
CSCI-14	34	636	0.59
CSCI-15	34	600	0.57
CSCI-16	34	598	0.57
CSCI-17	34	529	0.56
CSCI-18	34	519	0.55
CSCI-19	34	431	0.55
CSCI-20	34	403	0.55

Results: Full data

- Baselines: Ridge Regression (RR), Support Vector Regression (SVR), Decision Tree (DT), Neural network: multilayer perceptron (MLP).

Dataset	RR	SVR (RBF)	SVR (polynomial)	DT	MLP (5 Layer)	CSID
Energy Eff. (1)	2.91±0.17	2.68±0.17	4.09±0.49	0.56±0.03	**0.48±0.06** [**50**]	**0.39±0.05**
Energy Eff. (2)	3.09±0.19	3.03±0.21	4.14±0.44	1.86±0.19	**0.97±0.14** [**50**]	**0.57±0.09**
C. Comp. Strength	10.47±0.42	9.72±0.38	11.30±0.36	6.57±0.82	**4.92±0.63** [**50**]	**4.67±0.50**
SkillCraft Master Table	1.68±1.61	**0.99±0.03**	1.22±0.05	1.03±0.04	1.00±0.03 [10]	**0.91±0.02**
Abalone	2.25±0.10	**2.19±0.08**	3.90±3.43	2.35±0.08	**2.09±0.09** [**10**]	2.23±0.09
Wine Quality	0.76±0.02	**0.69±0.02**	1.01±0.39	0.75±0.03	0.72±0.02 [10]	**0.70±0.02**
Parkinsons Tel. (1)	7.51±0.11	6.66±0.14	7.89±0.88	**2.40±0.26**	3.60±0.18 [100]	**1.33±0.10**
Parkinsons Tel. (2)	9.75±0.15	9.14±0.17	10.04±0.43	**2.60±0.38**	5.01±0.19 [100]	**1.79±0.17**
C. Cycle Power Plant	5.51±0.09	4.13±0.09	8.00±0.19	**3.98±0.13**	4.06±0.11 [50]	**3.76±0.15**
Bike Sharing (1)	36.45±0.46	32.67±0.81	34.93±0.97	18.89±0.36	**14.81±0.44** [**100**]	**15.17±0.44**
Bike Sharing (2)	122.65±2.87	113.18±1.73	117.25±2.01	42.06±2.06	**38.69±1.24** [**100**]	**36.93±1.19**
Phys. Prop.	5.19±0.03	4.91±1.26	6.49±1.15	4.40±0.04	**4.20±0.05** [**100**]	**4.21±0.04**

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Results: Missing data

- Randomly hide 30% of the data
- Mean and mode imputation for baselines

Dataset	RR	SVR (RBF)	SVR (polynomial)	DT	MLP (5 Layer)	CSID
Energy Eff. (1)	3.01±0.15	3.38±0.27	6.88±0.63	2.57±0.49	**2.49±0.48 [10]**	**2.17±0.25**
Energy Eff. (2)	3.26±0.16	3.57±0.30	6.65±0.48	**2.64±0.28**	3.02±0.36 [10]	**2.48±0.22**
C. Comp. Strength	10.33±0.61	11.39±0.48	13.16±1.17	**9.90±1.05**	10.01±0.54 [10]	**9.69±0.79**
SkillCraft Master Table	1.79±1.63	**1.05±0.03**	1.61±0.33	1.08±0.03	1.10±0.04 [10]	**1.05±0.01**
Abalone	**2.27±0.07**	2.31±0.08	3.12±0.79	2.42±0.07	**2.28±0.07 [10]**	2.40±0.13
Wine Quality	0.76±0.02	**0.73±0.02**	0.93±0.21	0.78±0.02	**0.76±0.03 [10]**	0.78±0.02
Parkinsons Tel. (1)	7.52±0.11	6.91±0.13	8.12±0.11	**3.10±0.22**	5.90±0.28 [10]	**4.98±0.12**
Parkinsons Tel. (2)	9.76±0.18	9.38±0.21	10.68±0.23	**3.59±0.81**	7.67±0.18 [10]	**6.58±0.18**
C. Cycle Power Plant	5.51±0.09	6.16±0.15	10.45±0.31	**5.29±0.36**	5.33±0.07 [50]	**5.04±0.12**
Bike Sharing (1)	37.40±0.52	35.50±0.31	36.85±0.38	25.41±1.5	**21.51±0.83± [50]**	**23.89±0.19**
Bike Sharing (2)	123.81±1.26	127.06±1.55	130.20±1.13	**71.93±1.18**	**64.03±1.66 [50]**	75.65±1.51
Phys. Prop.	5.18±0.02	7.53±0.67	7.87±0.83	5.08±0.03	**4.99±0.09 [100]**	**4.70±0.03**

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Grade prediction

- Baselines: Grade Point Average (GPA), Biased Matrix Factorization

Dataset	GPA	BMF	CSID
CSCI-1	0.52 ± 0.02	$\mathbf{0.48\pm0.03}$	$\mathbf{0.48\pm0.03}$
CSCI-2	0.56 ± 0.02	$\mathbf{0.55\pm0.02}$	$\mathbf{0.55\pm0.03}$
CSCI-3	$\mathbf{0.48\pm0.04}$	$\mathbf{0.48\pm0.04}$	$\mathbf{0.48\pm0.05}$
CSCI-4	0.53 ± 0.03	0.52 ± 0.04	$\mathbf{0.51\pm0.03}$
CSCI-5	0.43 ± 0.02	0.43 ± 0.02	$\mathbf{0.42\pm0.02}$
CSCI-6	0.63 ± 0.03	0.58 ± 0.03	$\mathbf{0.57\pm0.03}$
CSCI-7	0.57 ± 0.02	0.58 ± 0.01	$\mathbf{0.56\pm0.02}$
CSCI-8	0.52 ± 0.02	0.49 ± 0.03	$\mathbf{0.47\pm0.02}$
CSCI-9	0.61 ± 0.03	0.60 ± 0.05	$\mathbf{0.57\pm0.03}$
CSCI-10	0.58 ± 0.04	$\mathbf{0.56\pm0.04}$	$\mathbf{0.56\pm0.04}$

Dataset	GPA	BMF	CSID
CSCI-11	0.68 ± 0.06	$\mathbf{0.66\pm0.04}$	0.67 ± 0.03
CSCI-12	0.58 ± 0.04	0.51 ± 0.04	$\mathbf{0.48\pm0.01}$
CSCI-13	0.67 ± 0.03	0.55 ± 0.05	$\mathbf{0.54\pm0.03}$
CSCI-14	0.70 ± 0.06	$\mathbf{0.62\pm0.03}$	0.65 ± 0.07
CSCI-15	0.56 ± 0.03	$\mathbf{0.56\pm0.06}$	0.57 ± 0.03
CSCI-16	0.52 ± 0.03	0.51 ± 0.03	$\mathbf{0.50\pm0.02}$
CSCI-17	0.60 ± 0.02	$\mathbf{0.58\pm0.05}$	0.59 ± 0.05
CSCI-18	0.57 ± 0.03	0.56 ± 0.05	$\mathbf{0.55\pm0.04}$
CSCI-19	0.68 ± 0.04	0.70 ± 0.04	$\mathbf{0.61\pm0.04}$
CSCI-20	0.61 ± 0.06	$\mathbf{0.58\pm0.02}$	0.63 ± 0.04

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Canonical Decomposition of Multivariate Functions

- Recap
  - Nonlinear system identification is tensor completion
  - Low-rank models can model highly nonlinear functions
  - Even if not low-rank: Identification of principal components

- What about continuous inputs?
  - One way to address the problem ⟹ quantization
  - Coarse quantization ⟹ low resolution, poor generalization
  - Fine quantization ⟹ high computational complexity, overfitting
  - Alternative approach?

Fourier Series Representation

- Modeling non-periodic functions with compact support
  - Without loss of generality we may restrict the domain of f on [0,1]
  - Using the even periodic extension of f we have

$$f(x) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k \sqrt{2} \cos(k\pi x),$$

using orthogonal basis functions $\phi_0(x) = 1$ and $\phi_k(x) = \sqrt{2}\cos(k\pi x)$, $k > 0$.

- Every periodic function which has finite energy over a single period has a unique Fourier representation that converges in L_2 to f as $K \rightarrow \infty$

Kargas & Sidiropoulos, IEEE Transactions on Signal Processing, 2021, doi: 10.1109/TSP.2021.3055000

Fourier Series Representation

- Similarly for the multivariate case

$$f(\mathbf{x}) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_N=0}^{\infty} \alpha_{\mathbf{k}} \prod_{n=1}^{N} \phi_{k_n}(\mathbf{x}[n]), \text{ where } \mathbf{k} = (k_1, \ldots, k_N) \in \mathbb{N}^N.$$

- Series estimator
  - Truncated series with cutoffs $K_1, \ldots, K_N$

$$f(\mathbf{x}) = \sum_{k_1=0}^{K_1-1} \cdots \sum_{k_N=0}^{K_N-1} \alpha_{\mathbf{k}} \prod_{n=1}^{N} \phi_{k_n}(\mathbf{x}[n]), \;\; \alpha_{\mathbf{k}} = \int_0^1 \cdots \int_0^1 f(\mathbf{x}) \phi_{\mathbf{k}}(\mathbf{x}) d\mathbf{x}, \qquad (1)$$

where $\phi_{\mathbf{k}}(\mathbf{x}) = \prod_{n=1}^{N} \phi_{k_n}(\mathbf{x}[n])$.

  - Assuming that input samples $\{\mathbf{x}_m\}$ are drawn independently, a natural estimator for the coefficients is

$$\mathcal{X}[k_1, \ldots, k_N] = \widehat{\alpha}_{\mathbf{k}} = \frac{1}{M} \sum_{m=1}^{M} y_m \phi_{\mathbf{k}}(\mathbf{x}_m).$$

Kargas & Sidiropoulos, IEEE Transactions on Signal Processing, 2021, doi: 10.1109/TSP.2021.3055000

Generalized Canonical Polyadic Decomposition

- Challenges
  - Number of Fourier coefficient grows exponentially
  - Suitable only for small N
  - Large variance of the estimates when M is small

- Breaking the curse of dimensionality
  - Low rank model of the coefficient tensor

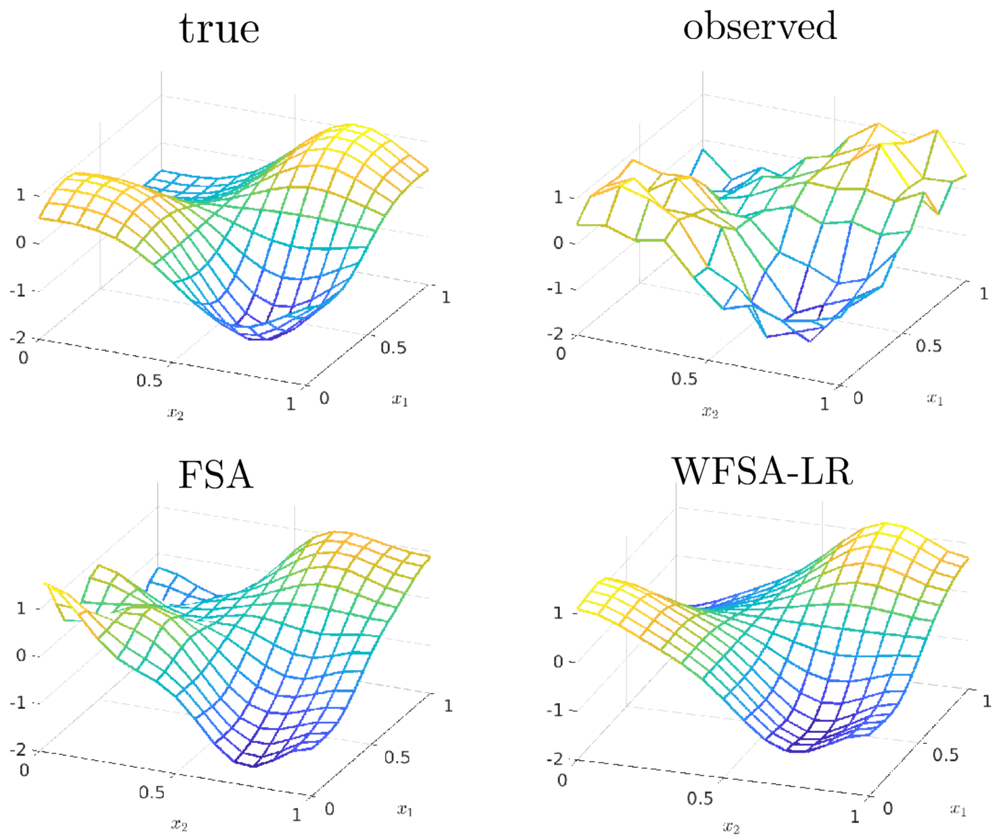
$$\mathcal{X}[k_1,\ldots,k_N] = \widehat{\alpha}_{\mathbf{k}} = \frac{1}{M}\sum_{m=1}^{M} y_m \phi_{\mathbf{k}}(\mathbf{x}_m), \quad \mathcal{X}[k_1,\ldots,k_N] = \sum_{r=1}^{R}\prod_{n=1}^{N} \mathbf{a}_n^r[k_n].$$

  - Substituting back in Equation (1) we have that $f(\mathbf{x}) = \sum_{r=1}^{R} f_{1,r}(\mathbf{x}[1])\cdots f_{N,r}(\mathbf{x}[N]).$
  - Each univariate function has a compact Fourier representation

Kargas & Sidiropoulos, IEEE Transactions on Signal Processing, 2021, doi: 10.1109/TSP.2021.3055000

Generalized Canonical Polyadic Decomposition

- Generalization of the CPD from tensors to functions
  - f is compactly supported in the domain $[0,1]^N$.
  - Differentiable up to a certain order.
  - Low rank Fourier series coefficient tensor.
    - Controls complexity
    - Number of parameters $O(K^N) \to O(KNR)$



- Training the model:
- Direct Optimization
  - Compute the coefficient tensor using sample averages.
  - We refer to this approach as (W)FSA-LR.

- Hidden Tensor Factorization
  - Fit the coefficients directly on the training data.
  - We refer to this approach as FSA-HTF.

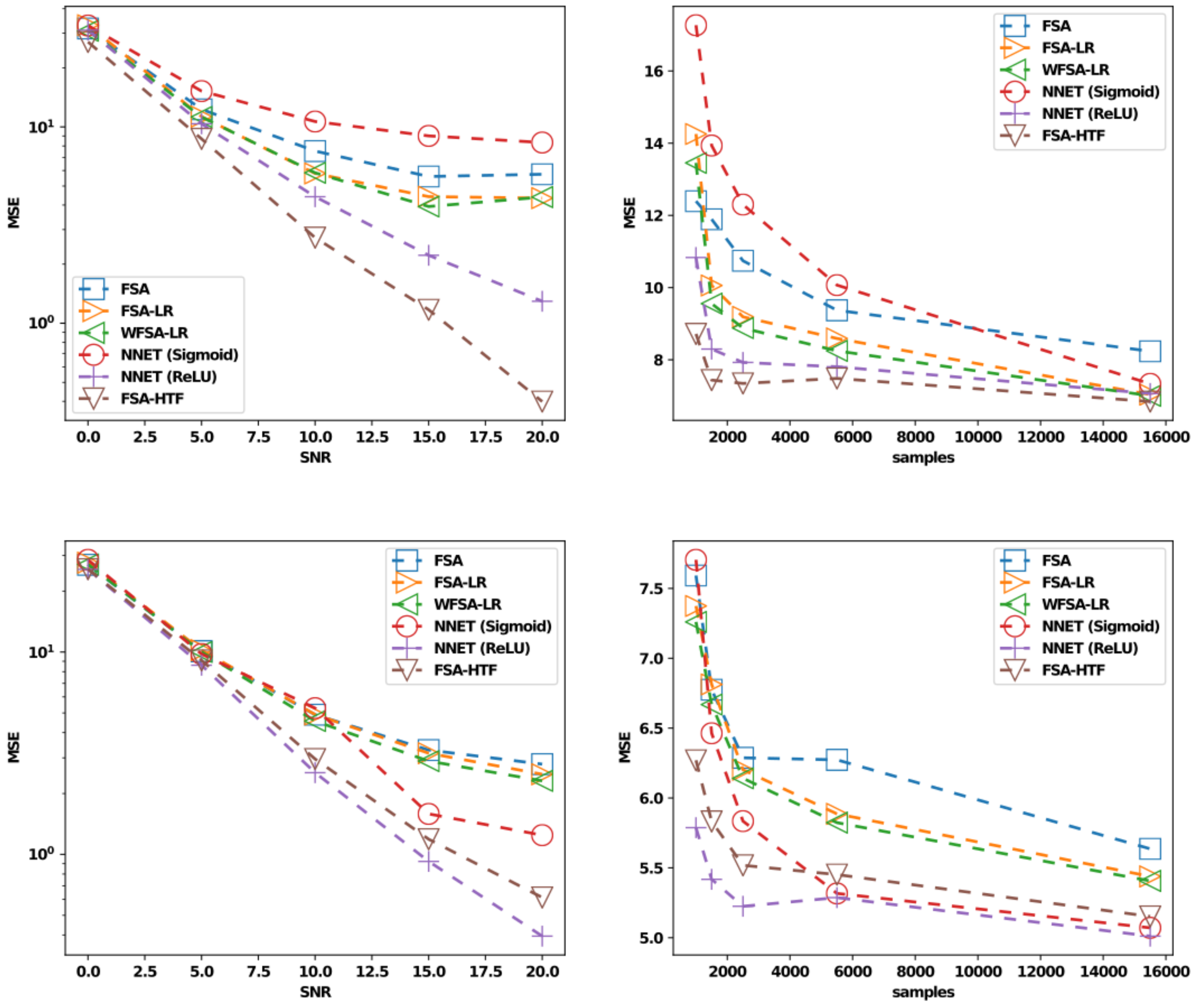
Kargas & Sidiropoulos, IEEE Transactions on Signal Processing, 2021, doi: 10.1109/TSP.2021.3055000

Results (synthetic data)

- Exp. 1
  - True function is low-rank (R=3)
  - Each univariate function is a linear combination of two Gaussians

- Exp. 2
  - True function is a neural network with a single hidden layer

- Number of variables: 10
- Generate 1000 samples, vary SNR
- Set SNR=15, vary number of samples



Kargas & Sidiropoulos, IEEE Transactions on Signal Processing, 2021, doi: 10.1109/TSP.2021.3055000

Results (real data)

Linear Regression (LR), Polynomial Regression (PR), Support Vector Regression (SVR), Neural Network (NNET), Decision Tree (DT), low-rank RKHS model (AMP), Canonical System Identification (CSID).

MSE performance of standard baseline models.

Dataset	LR	PR	SVR (RBF)	NNET	DT
QSAR	1.41 ± 0.31	1.41 ± 0.44	$\mathbf{1.19 \pm 0.30}$	1.42 ± 0.55	1.64 ± 0.37
EE [1]	8.15 ± 1.13	0.76 ± 0.08	4.69 ± 0.68	0.31 ± 0.08	0.30 ± 0.03
EE [2]	10.39 ± 1.84	2.87 ± 0.36	7.36 ± 1.66	2.03 ± 0.71	3.21 ± 0.26
ASN	23.27 ± 1.74	15.95 ± 1.21	11.09 ± 1.60	6.61 ± 1.24	6.45 ± 0.35
SMT	1.03 ± 0.06	1.02 ± 0.09	1.01 ± 0.05	1.03 ± 0.06	1.06 ± 0.04
AB	4.91 ± 0.30	4.54 ± 0.12	4.44 ± 0.21	$\mathbf{4.33 \pm 0.22}$	5.29 ± 0.22
CPP	19.57 ± 0.50	17.18 ± 0.41	15.60 ± 0.51	16.08 ± 0.50	$\mathbf{14.85 \pm 0.97}$
SC	315.93 ± 6.76	171.52 ± 5.41	213.60 ± 6.71	$\mathbf{125.12 \pm 8.13}$	130.56 ± 8.35
PP	27.10 ± 0.27	25.05 ± 1.06	21.24 ± 0.42	17.55 ± 0.38	19.24 ± 0.23

MSE performance of CPD-based models.

Dataset	AMP	CSID	FSA-HTF
QSAR	1.65 ± 0.50	1.48 ± 0.17	1.38 ± 0.53
EE [1]	0.31 ± 0.06	0.18 ± 0.02	$\mathbf{0.16 \pm 0.10}$
EE [2]	0.46 ± 0.10	0.34 ± 0.06	$\mathbf{0.24 \pm 0.09}$
ASN	8.73 ± 1.29	$\mathbf{3.05 \pm 0.53}$	4.01 ± 0.32
SMT	0.99 ± 0.05	0.98 ± 0.06	$\mathbf{0.94 \pm 0.06}$
AB	4.75 ± 0.3	4.95 ± 0.21	4.52 ± 0.30
CPP	N/A	15.25 ± 0.45	15.00 ± 0.66
SC	N/A	N/A	127.76 ± 11.86
PP	N/A	18.21 ± 0.45	$\mathbf{16.77 \pm 0.35}$

Kargas & Sidiropoulos, IEEE Transactions on Signal Processing, 2021, doi: 10.1109/TSP.2021.3055000

Take-home points

- Nonlinear system identification is tensor completion.
- Provably correct system identification is possible under low rank conditions.
- Low-rank models can model highly nonlinear functions.
- Even if not low-rank: Identification of principal components of nonlinear mapping.

- Extension of CPD from tensors to multivariate functions.
  - Compactly supported functions.
  - Differentiable up to certain order.

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