Geometry of configurations of points and symmetric rank

Luca Chiantini
Universitá di Siena, Italy

Abstract: Decompositions of symmetric tensors can be viewed as sets of points in a projective space. The geometry of sets of this type is usually studied in terms of a resolution of the associated homogeneous ideal. I will illustrate how one can study problems like the minimality or the uniqueness of a given decomposition by means of algebraic invariants of the corresponding configuration of points.

IPAM - Tensor Methods Weekly Seminars - 2021

Introduction

T = symmetric tensor = homogeneous polynomial (form) $/\mathbb{C}$

Introduction

T = symmetric tensor = homogeneous polynomial (form) $/\mathbb{C}$

(Waring) expression of T:

$$T = \sum_{i=1}^{r} a_i T_i$$

$T_i = L_i^d$, $L_i$linear $\quad d = \deg(T), \quad r =$ length, $\quad a_i \in \mathbb{C}$.

Introduction

T = symmetric tensor = homogeneous polynomial (form) $/\mathbb{C}$

(Waring) expression of T:

$$T = \sum_{i=1}^{r} a_i T_i$$

$T_i = L_i^d$, $L_i$linear $\quad d = \deg(T), \quad r =$ length, $\quad a_i \in \mathbb{C}$.

Coefficients a_i's will become important later in the talk.

Introduction

(Waring) expression of T:

$$T = \sum_{i=1}^{r} a_i T_i$$

$$T_i = L_i^d \qquad d = \deg(T) \quad r = \text{ length}, \quad a_i \in \mathbb{C}.$$

Notation :

Introduction

(Waring) expression of T:

$$T = \sum_{i=1}^{r} a_i T_i$$

$$T_i = L_i^d \qquad d = \deg(T) \quad r = \text{ length}, \quad a_i \in \mathbb{C}.$$

Notation :

- the expression is **non-redundant** if the T_i's are independent and no a_i is zero.

Introduction

(Waring) expression of T:

$$T = \sum_{i=1}^{r} a_i T_i$$

$$T_i = L_i^d \qquad d = \deg(T) \quad r = \text{ length}, \quad a_i \in \mathbb{C}.$$

Notation :

- the expression is **non-redundant** if the T_i's are independent and no a_i is zero.

- the expression is **minimal** if there are no expressions of T of smaller length. I.e. r is the (symmetric) rank.

Introduction

(Waring) expression of T:

$$T = \sum_{i=1}^{r} a_i T_i$$

$$T_i = L_i^d \qquad d = \deg(T) \quad r = \text{ length}, \quad a_i \in \mathbb{C}.$$

Notation :

- the expression is **non-redundant** if the T_i's are independent and no a_i is zero.

- the expression is **minimal** if there are no expressions of T of smaller length. I.e. r is the (symmetric) rank.

- the expression is **unique** if there are no other expressions of T of length $\leq r$, except trivialities. In this case T is **identifiable**.

Introduction

Point of view

For symmetric tensors it is not hard to find **some** expression of T

$$T = \sum_{i=1}^{r} a_i T_i$$

Introduction

Point of view

For symmetric tensors it is not hard to find **some** expression of T

$$T = \sum_{i=1}^{r} a_i T_i$$

- apolar equations;

Introduction

Point of view

For symmetric tensors it is not hard to find **some** expression of T

$$T = \sum_{i=1}^{r} a_i T_i$$

- apolar equations;
- Strassen Additivity Problem for forms $T = T' \oplus T''$;

Introduction

Point of view

For symmetric tensors it is not hard to find **some** expression of T

$$T = \sum_{i=1}^{r} a_i T_i$$

- apolar equations;
- Strassen Additivity Problem for forms $T = T' \oplus T''$;
- matrix multiplication symmetrized.

Introduction

Point of view

For symmetric tensors it is not hard to find **some** expression of T

$$T = \sum_{i=1}^{r} a_i T_i$$

It is hard to find a **minimal** decomposition,

and/or to prove that it is **unique**.

- apolar equations;
- Strassen Additivity Problem for forms $T = T' \oplus T''$;
- matrix multiplication symmetrized.

The geometric (projective) setting

Most problems on tensors are invariant under **rescaling** (= multiplication by a non-zero scalar q).

$$\mathrm{rank}(T) = \mathrm{rank}(qT) \qquad \text{etc.}$$

The geometric (projective) setting

Most problems on tensors are invariant under **rescaling** (= multiplication by a non-zero scalar q).

$$\mathrm{rank}(T) = \mathrm{rank}(qT) \qquad \text{etc.}$$

Factor out the $\mathbb{C}^*$-action

The geometric (projective) setting

Most problems on tensors are invariant under **rescaling** (= multiplication by a non-zero scalar q).

$$\mathrm{rank}(T) = \mathrm{rank}(qT) \qquad \text{etc.}$$

Factor out the $\mathbb{C}^*$-action ⇒ projective geometry.

The geometric (projective) setting

Most problems on tensors are invariant under **rescaling** (= multiplication by a non-zero scalar q).

$$\mathrm{rank}(T) = \mathrm{rank}(qT) \qquad \text{etc.}$$

Factor out the $\mathbb{C}^*$-action $\Rightarrow$ projective geometry.

Advantages

- a geometric insight on the problems;

The geometric (projective) setting

Most problems on tensors are invariant under **rescaling** (= multiplication by a non-zero scalar q).

$$\mathrm{rank}(T) = \mathrm{rank}(qT) \qquad \text{etc.}$$

Factor out the $\mathbb{C}^*$-action $\Rightarrow$ projective geometry.

Advantages

- a geometric insight on the problems;
- access to a huge set of tools from projective geometry.

The geometric (projective) setting

$$T \Rightarrow [T] \in \mathbb{P}(Sym^d(\mathbb{C}^{n+1})).$$

Dictionary

The geometric (projective) setting

$$T \Rightarrow [T] \in \mathbb{P}(Sym^d(\mathbb{C}^{n+1})).$$

Dictionary

linear form L_i $\Rightarrow$ point $P_i = [L_i] \in \mathbb{P}^n$

power L_i^d $\Rightarrow$ image of $[L_i]$ in the Veronese map $v_d : \mathbb{P}^n \to \mathbb{P}^N$, $N = \binom{n+d}{n} - 1$

expression $T = \sum_{i=1}^r a_i L_i^d$ $\Rightarrow$ $T \in$ linear span $\mathbb{P}^{r-1}$ of the $v_d([L_i])$'s

The geometric (projective) setting

$$T \Rightarrow [T] \in \mathbb{P}(Sym^d(\mathbb{C}^{n+1})).$$

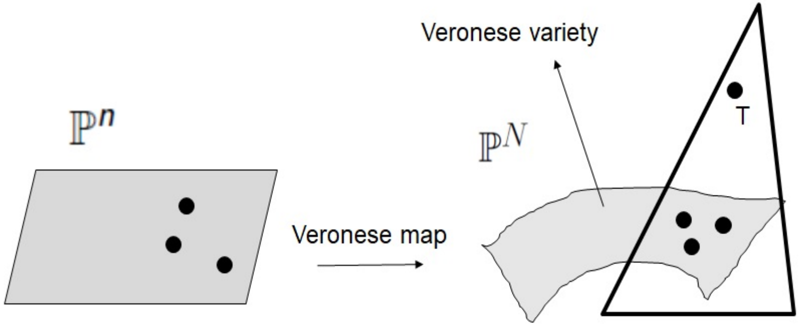
Dictionary

linear form L_i	$\Rightarrow$	point $P_i = [L_i] \in \mathbb{P}^n$
power L_i^d	$\Rightarrow$	image of $[L_i]$ in the Veronese map $v_d : \mathbb{P}^n \to \mathbb{P}^N$, $N = \binom{n+d}{n} - 1$
expression $T = \sum_{i=1}^r a_i L_i^d$	$\Rightarrow$	$T \in$ linear span $\mathbb{P}^{r-1}$ of the $v_d([L_i])$'s

Definition A subset $A = \{P_1, \dots, P_r\} \subset \mathbb{P}^n$ is a (geometric) **decomposition** of T if

$$[T] \in \langle v_d(P_1), \dots, v_d(P_r)\rangle.$$

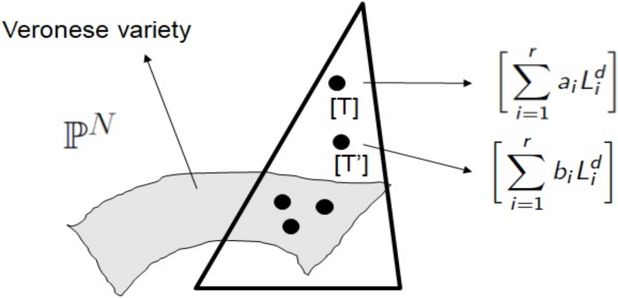
The geometric (projective) setting



The geometric (projective) setting

beware: in general

$$\left[\sum_{i=1}^{r} a_i L_i^d\right] \neq \left[\sum_{i=1}^{r} b_i L_i^d\right].$$



The geometric (projective) setting

Most celebrated criterion for uniqueness: **Kruskal's criterion**.

The geometric (projective) setting

Most celebrated criterion for uniqueness: **Kruskal's criterion**.

Kruskal's rank of a matrix

$k(M) = \max\{i :$ all subsets of i columns of M are linearly independent $\}$.

The geometric (projective) setting

Most celebrated criterion for uniqueness: **Kruskal's criterion**.

Kruskal's rank of a matrix

$k(M) = \max\{i :$ all subsets of i columns of M are linearly independent $\}$.

Kruskal's criterion (symmetric case)

Let $T = \sum_{i=1}^{r} a_i L_i^d$. Let M be the matrix whose i-th column is formed by the coefficients of L_i. If

$$r \leq \frac{dk(M) - d + 1}{2}$$

then the expression is (minimal and) unique.

The geometric (projective) setting

Most celebrated criterion for uniqueness: **Kruskal's criterion**.

Kruskal's rank of a matrix

$k(M) = \max\{i : \text{all subsets of } i \text{ columns of } M \text{ are linearly independent }\}$.

Kruskal's criterion (symmetric case)

Let $T = \sum_{i=1}^{r} a_i L_i^d$. Let M be the matrix whose i-th column is formed by the coefficients of L_i. If

$$r \leq \frac{dk(M) - d + 1}{2}$$

then the expression is (minimal and) unique.

(Original statement was for general 3-way tensors).

The geometric (projective) setting

The geometry of Kruskal's condition.

The geometric (projective) setting

The geometry of Kruskal's condition.

Let M be the matrix whose columns are given by the coefficients of the L_i's, i.e. by projective coordinates for the points of $A = \{[L_1], \dots, [L_r]\}$.

The geometric (projective) setting

The geometry of Kruskal's condition.

Let M be the matrix whose columns are given by the coefficients of the L_i's, i.e. by projective coordinates for the points of $A = \{[L_1], \dots, [L_r]\}$.

Denote with k_A the Kruskal's rank of the matrix M above.

The geometric (projective) setting

The geometry of Kruskal's condition.

Let M be the matrix whose columns are given by the coefficients of the L_i's, i.e. by projective coordinates for the points of $A = \{[L_1], \dots, [L_r]\}$.

Denote with k_A the Kruskal's rank of the matrix M above.

$k_A = \max\{i$: all subsets of A of cardinality i are in **Linear General Position** (no three on a line, no four in a plane, $\dots$)$\}$.

The geometric (projective) setting

The geometry of Kruskal's condition.

Let M be the matrix whose columns are given by the coefficients of the L_i's, i.e. by projective coordinates for the points of $A = \{[L_1], \dots, [L_r]\}$.

Denote with k_A the Kruskal's rank of the matrix M above.

$k_A = \max\{i :$ all subsets of A of cardinality i are in **Linear General Position** (no three on a line, no four in a plane, $\dots)\}$.

Certainly $k_A \leq n + 1$.

The geometric (projective) setting

Bad news for Kruskal's criterion.

The geometric (projective) setting

Bad news for Kruskal's criterion.

Bad.1) $k_A \leq n+1 \Rightarrow$ one cannot hope to apply the criterion when $2r > d(n+1) - d + 1$.

The geometric (projective) setting

Bad news for Kruskal's criterion.

Bad.1) $k_A \leq n+1 \Rightarrow$ one cannot hope to apply the criterion when $2r > d(n+1) - d + 1$.

Bad.2) Kruskal's criterion is sharp. One cannot hope to improve the previous range of application, by using k_A. (**Derksen's examples**).

The geometric (projective) setting

Bad news for Kruskal's criterion.

Bad.1) $k_A \leq n + 1 \Rightarrow$ one cannot hope to apply the criterion when $2r > d(n+1) - d + 1$.

Bad.2) Kruskal's criterion is sharp. One cannot hope to improve the previous range of application, by using k_A. (**Derksen's examples**).

What is so bad about the bad news?

The geometric (projective) setting

Abstract secant variety (incidence variety):

$$A\sigma_r = \{(T, A) : A \text{ is a decomposition for } T\}.$$

$$\dim(A\sigma_r) = r(n+1) - 1.$$

The geometric (projective) setting

Abstract secant variety (incidence variety):

$$A\sigma_r = \{(T, A) : A \text{ is a decomposition for } T\}.$$

$$\dim(A\sigma_r) = r(n+1) - 1.$$

Consequence

No hope for uniqueness if $r(n+1) - 1 \geq N$, i.e. if

$$r \geq r_g = \frac{\binom{n+d}{n}}{n+1},$$

The geometric (projective) setting

Abstract secant variety (incidence variety):

$$A\sigma_r = \{(T, A) : A \text{ is a decomposition for } T\}.$$

$$\dim(A\sigma_r) = r(n+1) - 1.$$

Consequence

No hope for uniqueness if $r(n+1) - 1 \geq N$, i.e. if

$$r \geq r_g = \frac{\binom{n+d}{n}}{n+1},$$

The natural range for identifiability is $r < r_g$ (**subgeneric rank**)

The geometric (projective) setting

The natural range for identifiability is $r < r_g$ (**subgeneric range**)

Theorem (Ottaviani-Vannieuwenhoven-C)

If $r < r_g$ then a 'generic Waring expression' (in the Zariski sense) is minimal and unique, except for a finite, small list of values of n, d.

The geometric (projective) setting

The natural range for identifiability is $r < r_g$ (**subgeneric range**)

Theorem (Ottaviani-Vannieuwenhoven-C)

If $r < r_g$ then a 'generic Waring expression' (in the Zariski sense) is minimal and unique, except for a finite, small list of values of n, d.

Based on the works of Ciliberto-C, Ballico, Brambilla-Ottaviani, Ranestad-Voisin.

The geometric (projective) setting

The natural range for identifiability is $r < r_g$ (**subgeneric range**)

Theorem (Ottaviani-Vannieuwenhoven-C)

If $r < r_g$ then a 'generic Waring expression' (in the Zariski sense) is minimal and unique, except for a finite, small list of values of n, d.

Based on the works of Ciliberto-C, Ballico, Brambilla-Ottaviani, Ranestad-Voisin.

The Kruskal's range $r \leq (dn+1)/2$ is **much smaller** than the range $r < r_g$ in which the uniqueness is expected (and known to hold generically).

The geometric (projective) setting

The natural range for identifiability is $r < r_g$ (**subgeneric range**)

Theorem (Ottaviani-Vannieuwenhoven-C)

If $r < r_g$ then a 'generic Waring expression' (in the Zariski sense) is minimal and unique, except for a finite, small list of values of n, d.

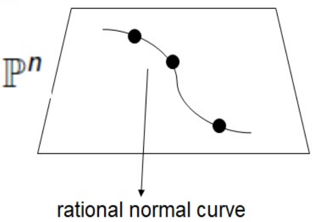
Based on the works of Ciliberto-C, Ballico, Brambilla-Ottaviani, Ranestad-Voisin.

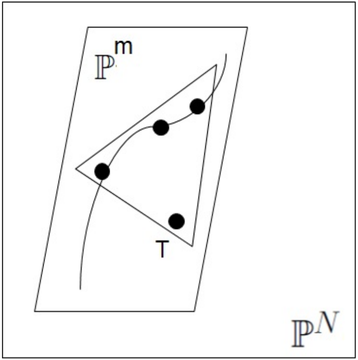
The Kruskal's range $r \leq (dn+1)/2$ is **much smaller** than the range $r < r_g$ in which the uniqueness is expected (and known to hold generically).

Further analysis needed.

The geometric (projective) setting

The geometry of Derksen's example.





The Game

To overcome the problem: use deeper invariants, e.g. **reshaped Kruskal**.

The Game

To overcome the problem: use deeper invariants, e.g. **reshaped Kruskal**. Take a partition $d = d_1 + d_2 + d_3$, and write $L_i^d = (L_i^{d_1}) \cdot (L_i^{d_2}) \cdot (L_i^{d_3})$.

The Game

To overcome the problem: use deeper invariants, e.g. **reshaped Kruskal**.
Take a partition $d = d_1 + d_2 + d_3$, and write $L_i^d = (L_i^{d_1}) \cdot (L_i^{d_2}) \cdot (L_i^{d_3})$.

Definition

For a finite set $A \subset \mathbb{P}^n$ set:
Hilbert i-th rank $h_A(i) =$ (affine) dimension of the linear span of $v_i(A)$;
Kruskal i-th rank $k_A(i) =$ the Kruskal's rank of $v_i(A)$.

The Game

To overcome the problem: use deeper invariants, e.g. **reshaped Kruskal**. Take a partition $d = d_1 + d_2 + d_3$, and write $L_i^d = (L_i^{d_1}) \cdot (L_i^{d_2}) \cdot (L_i^{d_3})$.

Definition

For a finite set $A \subset \mathbb{P}^n$ set:
Hilbert i-th rank $h_A(i) =$ (affine) dimension of the linear span of $v_i(A)$;
Kruskal i-th rank $k_A(i) =$ the Kruskal's rank of $v_i(A)$.

Reshaped Kruskal's Theorem

A decomposition A of length r is minimal and unique when

$$r \leq \frac{k_A(d_1) + k_A(d_2) + k_A(d_3) - 2}{2}.$$

The Game

To overcome the problem: use deeper invariants, e.g. **reshaped Kruskal**. Take a partition $d = d_1 + d_2 + d_3$, and write $L_i^d = (L_i^{d_1}) \cdot (L_i^{d_2}) \cdot (L_i^{d_3})$.

Definition

For a finite set $A \subset \mathbb{P}^n$ set:
Hilbert i-th rank $h_A(i) =$ (affine) dimension of the linear span of $v_i(A)$;
Kruskal i-th rank $k_A(i) =$ the Kruskal's rank of $v_i(A)$.

Reshaped Kruskal's Theorem

A decomposition A of length r is minimal and unique when

$$r \leq \frac{k_A(d_1) + k_A(d_2) + k_A(d_3) - 2}{2}.$$

Bad.3) Even if the $k_A(d_i)$'s are maximal $(= \binom{d_i+n}{n})$, the range of application remains much lower than the range $r \leq r_g$.

The Game

To overcome the problem: use deeper invariants, e.g. **reshaped Kruskal**. Take a partition $d = d_1 + d_2 + d_3$, and write $L_i^d = (L_i^{d_1}) \cdot (L_i^{d_2}) \cdot (L_i^{d_3})$.

Definition

For a finite set $A \subset \mathbb{P}^n$ set:
Hilbert *i*-th rank $h_A(i) =$ (affine) dimension of the linear span of $v_i(A)$;
Kruskal *i*-th rank $k_A(i) =$ the Kruskal's rank of $v_i(A)$.

Reshaped Kruskal's Theorem

A decomposition A of length r is minimal and unique when

$$r \leq \frac{k_A(d_1) + k_A(d_2) + k_A(d_3) - 2}{2}.$$ **not sharp!**

Bad.3) Even if the $k_A(d_i)$'s are maximal $(= \binom{d_i+n}{n})$, the range of application remains much lower than the range $r < r_g$.

The Game

Definition

For a finite set $A \subset \mathbb{P}^n$ define

Hilbert function:	$i \mapsto h_A(i)$;
Kruskal function:	$i \mapsto k_A(i)$:

$(h_A(0) = k_A(0) = 1)$.

Difference Hilbert function:	$Dh_A(i) = h_A(i) - h_A(i-1)$;
Difference Kruskal function:	$Dk_A(i) = k_A(i) - k_A(i-1)$.

The Game

Definition

For a finite set $A \subset \mathbb{P}^n$ define

Hilbert function: $i \mapsto h_A(i)$;

Kruskal function: $i \mapsto k_A(i)$:

$(h_A(0) = k_A(0) = 1)$.

Difference Hilbert function: $Dh_A(i) = h_A(i) - h_A(i-1)$;

Difference Kruskal function: $Dk_A(i) = k_A(i) - k_A(i-1)$.

If A is general enough $h_A(i) = k_A(i)$.

The Game

Many properties are known for the Hilbert function h_A.

- h_A is increasing from 1 to the maximum $r = \#A$,
- $\sum_{i=1}^{\infty} Dh_A(i) = r$;
- if $Dh_A(i) \leq i$ then $Dh_A(i+1) \leq Dh_A(i)$;
- Cayley-Bacharach properties $CB(i)$;
- ...

LC, Hilbert functions and tensor analysis, Lecture Notes of the Unione Matematica Italiana v.25, Springer 2019, 125-151. arXiv:1807.00642

The Game

Most important properties.

Let A, B be two decompositions of T. Set $Z = A \cup B$.

- $h_Z(d) < \#Z$;
- $D_Z(d+1) > 0$;

The Game

Most important properties.

Let A, B be two decompositions of T. Set $Z = A \cup B$.

- $h_Z(d) < \#Z$;
- $D_Z(d+1) > 0$;
- if A, B are disjoint, then

$$\dim(\langle v_d(A)\rangle \cap \langle v_d(B)\rangle) = \#Z - h_Z(d) = \sum_{i=d+1}^{\infty} Dh_Z(i).$$

LC, Hilbert functions and tensor analysis, Lecture Notes of the Unione Matematica Italiana v.25, Springer 2019, 125-151. arXiv:1807.00642

The Game

By means of properties of the Hilbert function and the Kruskal function of A, one can improve the Reshaped Kruskal's Theorem.

The Game

By means of properties of the Hilbert function and the Kruskal function of A, one can improve the Reshaped Kruskal's Theorem.

Game

Player 1 (You) provides a beautiful set A of r points, decomposition of T;

The Game

By means of properties of the Hilbert function and the Kruskal function of A, one can improve the Reshaped Kruskal's Theorem.

Game

Player 1 (You) provides a beautiful set A of r points, decomposition of T;
Player 2 (the Rival) determines a set B of cardinality $\leq r$, which gives for $Z = A \cup B$ the inequality $Dh_Z(d+1) > 0$

The Game

By means of properties of the Hilbert function and the Kruskal function of A, one can improve the Reshaped Kruskal's Theorem.

Game

Player 1 (You) provides a beautiful set A of r points, decomposition of T;
Player 2 (the Rival) determines a set B of cardinality $\leq r$, which gives for $Z = A \cup B$ the inequality $Dh_Z(d+1) > 0$

If the Rival cannot find B, you win (i.e. A is minimal, unique, ...)

The Game

Theorem

A decomposition A of cardinality r of a **ternary form** T of degree d is minimal and unique if:

- $d = 2m$ is even, $k_A(m-1) = \min\{\binom{m+1}{2}, r\}$, $h_A(m) = r \leq \binom{m+2}{2} - 2$;
- $d = 2m+1$ is odd, $k_A(m) = \min\{\binom{m+2}{2}, r\}$, $h_A(m+1) = r \leq \binom{m+2}{2} + \lfloor\binom{m}{2}\rfloor$.

Angelini-C, On the identifiability of ternary forms, Lin. Alg. Applic. 599 (2020), 36-65. arXiv:1901.01796.

The Game

Theorem

A decomposition A of cardinality r of a **ternary form** T of degree d is minimal and unique if:

- $d = 2m$ is even, $k_A(m-1) = \min\{\binom{m+1}{2}, r\}$, $h_A(m) = r \leq \binom{m+2}{2} - 2$;
- $d = 2m+1$ is odd, $k_A(m) = \min\{\binom{m+2}{2}, r\}$, $h_A(m+1) = r \leq \binom{m+2}{2} + \lfloor \binom{m}{2} \rfloor$.

sharp!

Angelini-C, On the identifiability of ternary forms, Lin. Alg. Applic. 599 (2020), 36-65. arXiv:1901.01796.

The Game

Theorem

A decomposition A of cardinality r of a **ternary form** T of degree d is minimal and unique if:

- $d = 2m$ is even, $k_A(m-1) = \min\{\binom{m+1}{2}, r\}$, $h_A(m) = r \leq \binom{m+2}{2} - 2$;
- $d = 2m+1$ is odd, $k_A(m) = \min\{\binom{m+2}{2}, r\}$, $h_A(m+1) = r \leq \binom{m+2}{2} + \lfloor\binom{m}{2}\rfloor$.

sharp!

Angelini-C, On the identifiability of ternary forms, Lin. Alg. Applic. 599 (2020), 36-65. arXiv:1901.01796.

There is a (non-sharp) version for forms in any number of variables. Ottaviani-Vannieuwenoven-C, Angelini-C-Mazzon, Ballico, Mourrain-Oneto.

The Game

Theorem

A decomposition A of cardinality r of a **ternary** form T of degree d is minimal and unique if:

- $d = 2m$ is even, $k_A(m-1) = \min\{\binom{m+1}{2}, r\}$, $h_A(m) = r \leq \binom{m+2}{2} - 2$;
- $d = 2m+1$ is odd, $k_A(m) = \min\{\binom{m+2}{2}, r\}$, $h_A(m+1) = r \leq \binom{m+2}{2} + \lfloor\binom{m}{2}\rfloor$.

Bad news.

Bad.4) In any case we are still far (half-way, for ternary forms) from the expected range $r < r_g$.

The Game

Theorem

A decomposition A of cardinality r of a **ternary** form T of degree d is minimal and unique if:

- $d = 2m$ is even, $k_A(m-1) = \min\{\binom{m+1}{2}, r\}$, $h_A(m) = r \leq \binom{m+2}{2} - 2$;
- $d = 2m+1$ is odd, $k_A(m) = \min\{\binom{m+2}{2}, r\}$, $h_A(m+1) = r \leq \binom{m+2}{2} + \lfloor \binom{m}{2} \rfloor$.

Bad news.
Bad.4) In any case we are still far (half-way, for ternary forms) from the expected range $r < r_g$.
(Even if the version for many variables is not sharp).

The fine tuning

The fine tuning

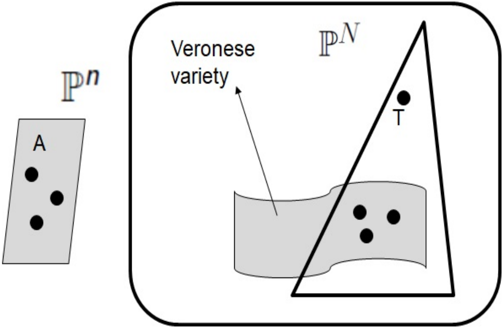
The intrinsic weakness, that makes hopeless to cover the whole range $r < r_g$ with a similar analysis:

The analysis only takes care of properties of the decomposition A.

The fine tuning

The intrinsic weakness, that makes hopeless to cover the whole range $r < r_g$ with a similar analysis:

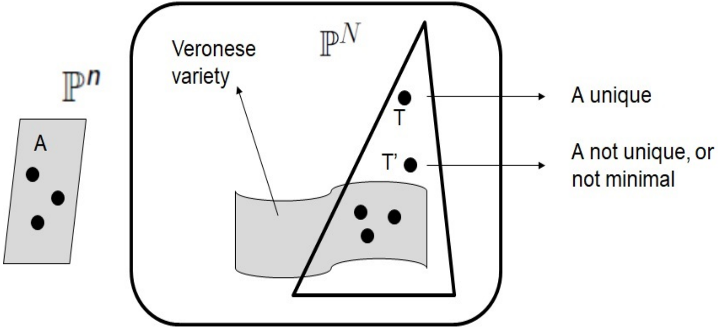
The analysis only takes care of properties of the decomposition A.



The fine tuning

The intrinsic weakness: why one cannot hope to cover the whole range $r < r_g$ with a similar analysis.

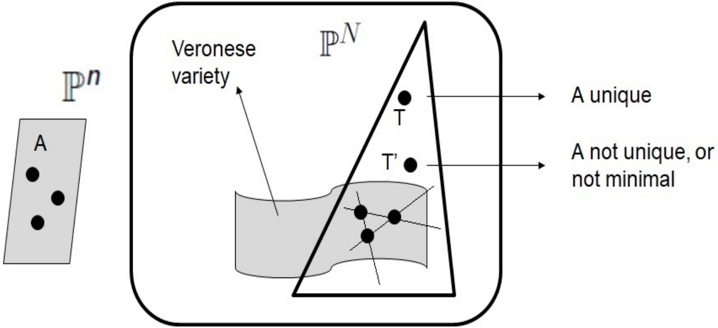
The analysis only takes care of properties of the decomposition A.



The fine tuning

The intrinsic weakness: why one cannot hope to cover the whole range $r < r_g$ with a similar analysis.

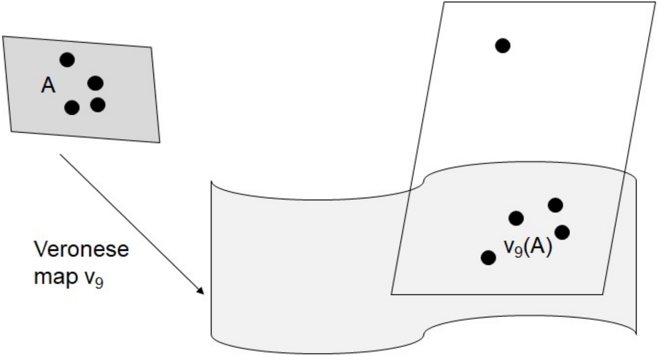
The analysis only takes care of properties of the decomposition A.



The fine tuning

Example

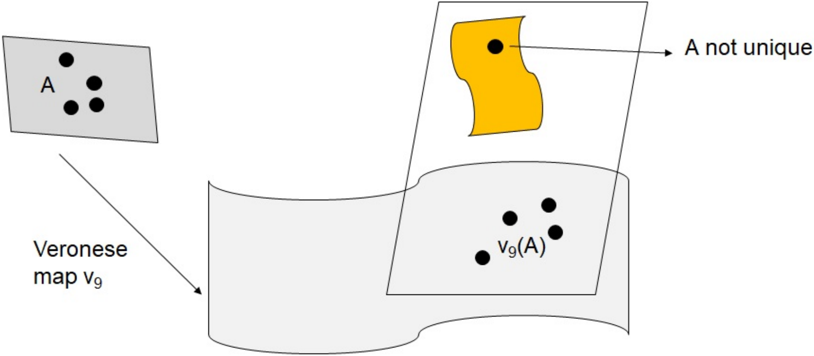
Forms of degree $d = 9$ in $n + 1 = 3$ variables, $r = 18 (= r_g - 1)$.



The fine tuning

Example

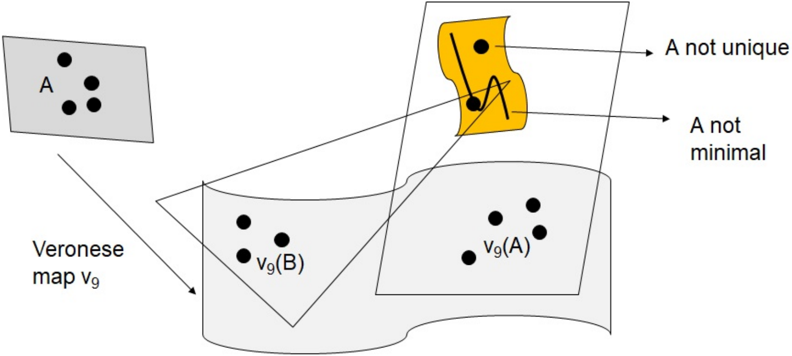
Forms of degree $d = 9$ in $n + 1 = 3$ variables, $r = 18 (= r_g - 1)$.



The fine tuning

Example

Forms of degree $d = 9$ in $n + 1 = 3$ variables, $r = 18 (= r_g - 1)$.
A general.

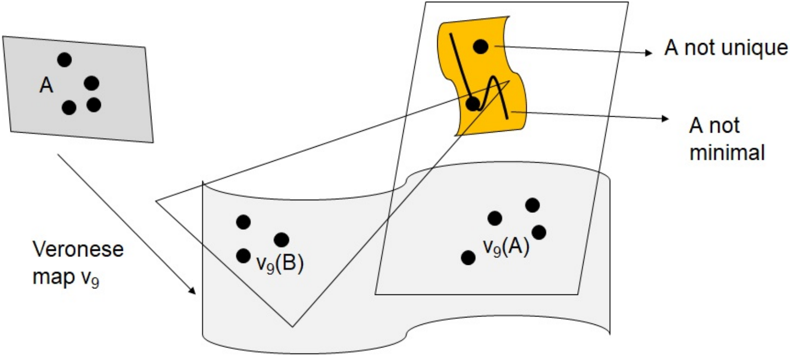


The fine tuning

Problem

Find a strategy to determine whether T lies in

- the bad locus W in which A is not unique; or
- the bad locus W' in which A is not minimal.



The fine tuning

The strategy of fine tuning

- Step 1. Determine parameters of all possible alternative decompositions B for **forms in the span** of $v_d(A)$.

The fine tuning

The strategy of fine tuning

- Step 1. Determine parameters of all possible alternative decompositions B for **forms in the span** of $v_d(A)$.
- Step 2. Compute if the specific T can be obtained from a choice of the parameters.

The fine tuning

The strategy of fine tuning

- Step 1. Determine parameters of all possible alternative decompositions B for **forms in the span** of $v_d(A)$.
- Step 2. Compute if the specific T can be obtained from a choice of the parameters.

Step 1:

Parametric expressions for alternative B can be found by analyzing:

The fine tuning

The strategy of fine tuning

- Step 1. Determine parameters of all possible alternative decompositions B for **forms in the span** of $v_d(A)$.
- Step 2. Compute if the specific T can be obtained from a choice of the parameters.

Step 1:

Parametric expressions for alternative B can be found by analyzing:
- the Hilbert and Kruskal functions of A;

The fine tuning

The strategy of fine tuning

- Step 1. Determine parameters of all possible alternative decompositions B for **forms in the span** of $v_d(A)$.
- Step 2. Compute if the specific T can be obtained from a choice of the parameters.

Step 1:

Parametric expressions for alternative B can be found by analyzing:
- the Hilbert and Kruskal functions of A;
- a free resolution of the homogeneous ideal I_A of A

$$0 \to \mathcal{M}_k \to \mathcal{M}_{k-1} \to \cdots \to \mathcal{M}_1 \to I_A \to 0$$

The fine tuning

Step 1:

Parameters for alternative decompositions B can be found by analyzing:
the Hilbert and Kruskal functions of A;
a free resolution of the homogeneous ideal I_A of A

$$0 \to \mathcal{M}_k \to \mathcal{M}_{k-1} \to \cdots \to \mathcal{M}_1 \to I_A \to 0$$

$\Rightarrow$ one can describe parametrically the homogeneous ideals I_B of all possible candidates for an alternative decomposition of cardinality $s \leq r$ for forms in the span of $v_d(A)$.

The fine tuning

Step 1:

Parameters for alternative decompositions B can be found by analyzing:
the Hilbert and Kruskal functions of A;
a free resolution of the homogeneous ideal I_A of A

$$0 \to \mathcal{M}_k \to \mathcal{M}_{k-1} \to \cdots \to \mathcal{M}_1 \to I_A \to 0$$

$\Rightarrow$ one can describe parametrically the homogeneous ideals I_B of all possible candidates for an alternative decomposition of cardinality $s \leq r$ for forms in the span of $v_d(A)$.

The procedure turns out to be (reasonably) **easy** for ternary forms,

The fine tuning

Step 1:

Parameters for alternative decompositions B can be found by analyzing:
the Hilbert and Kruskal functions of A;
a free resolution of the homogeneous ideal I_A of A

$$0 \to \mathcal{M}_k \to \mathcal{M}_{k-1} \to \cdots \to \mathcal{M}_1 \to I_A \to 0$$

$\Rightarrow$ one can describe parametrically the homogeneous ideals I_B of all possible candidates for an alternative decomposition of cardinality $s \leq r$ for forms in the span of $v_d(A)$.
The procedure turns out to be (reasonably) **easy** for ternary forms, much more complicate for forms in $n+1 \geq 4$ variables.

The fine tuning

Step 1:

Parameters for alternative decompositions B can be found by analyzing:
the Hilbert and Kruskal functions of A;
a free resolution of the homogeneous ideal I_A of A

$$0 \to \mathcal{M}_k \to \mathcal{M}_{k-1} \to \cdots \to \mathcal{M}_1 \to I_A \to 0$$

$\Rightarrow$ one can describe parametrically the homogeneous ideals I_B of all possible candidates for an alternative decomposition of cardinality $s \leq r$ for forms in the span of $v_d(A)$.
The procedure turns out to be (reasonably) **easy** for ternary forms, much more complicate for forms in $n+1 \geq 4$ variables.

Challenge for algebraic geometers: parametrize sets B such that

$$Dh_{A \cup B}(d+1) > 0.$$

The fine tuning

Step 2: the fine tuning

Once we have the ideals of all B's that decompose *some* forms in $\langle v_d(A) \rangle$, then compute if the fixed T in the span can be obtained by some alternative decomposition B.

The fine tuning

Step 2: the fine tuning

Once we have the ideals of all B's that decompose *some* forms in $\langle v_d(A)\rangle$, then compute if the fixed T in the span can be obtained by some alternative decomposition B.

Angelini-C, On the identifiability of ternary forms, Lin. Alg. Applic. 599 (2020), 36-65. arXiv:1901.01796.
Angelini-C, Minimality and uniqueness for decompositions of specific ternary forms. arXiv:2007.10165.

Theorem

T belongs to the intersection of $\langle v_d(A)\rangle$ and $\langle v_d(B)\rangle$ iff

$$(I_A)_d + (I_B)_d \subseteq T^\vee$$

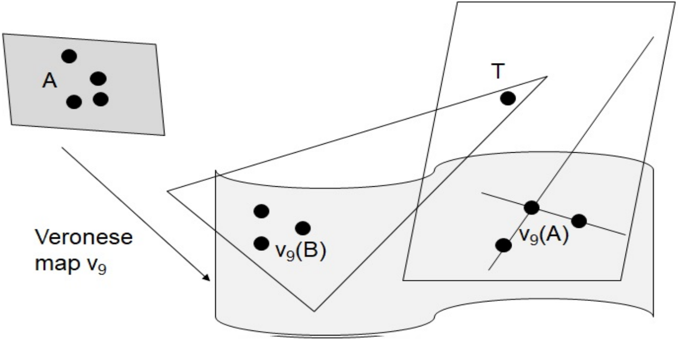
Examples

T ternary form of degree 9, $r = 18$.

Examples

T ternary form of degree 9, $r = 18$.

Even if A is a **general** set of 18 points in $\mathbb{P}^2$, yet the span of $v_9(A)$ contains forms T having (exactly one) second decomposition B, **disjoint from** A, of cardinality 17.

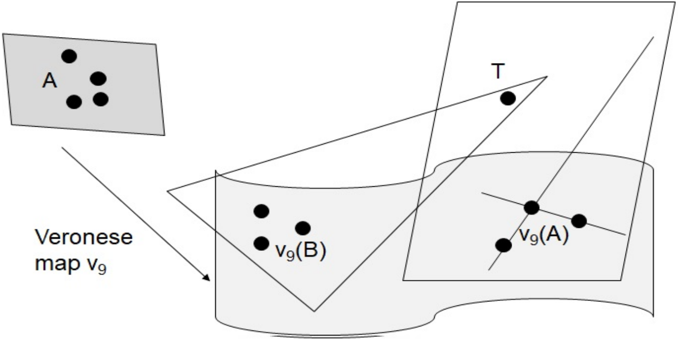


Examples

T ternary form of degree 9, $r = 18$.

Even if A is a **general** set of 18 points in $\mathbb{P}^2$, yet the span of $v_9(A)$ contains forms T having (exactly one) second decomposition B, **disjoint from** A, of cardinality 17.

Thus the rank of T is 17, and the two decompositions A, B are separate.



Examples

T ternary form of degree 9, $r = 18$.

If A is a general set of 18 points in $\mathbb{P}^2$, then the span of $v_9(A)$ contains forms T having (exactly one) second decomposition B, **disjoint from** A, of cardinality 17.

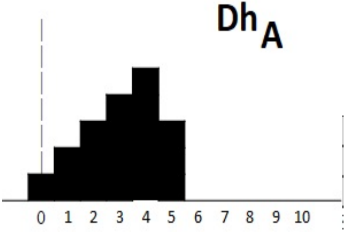
When the Hilbert function of A is generic (a condition easy to test, with elementary algorithms), then the Hilbert function of $A \cup B$ is fixed.

Examples

T ternary form of degree 9, $r = 18$.

If A is a general set of 18 points in $\mathbb{P}^2$, then the span of $v_9(A)$ contains forms T having (exactly one) second decomposition B, **disjoint from** A, of cardinality 17.

When the Hilbert function of A is generic (a condition easy to test, with elementary algorithms), then the Hilbert function of $A \cup B$ is fixed.

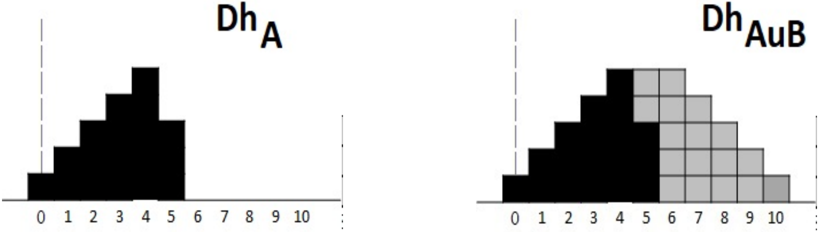


Examples

T ternary form of degree 9, $r = 18$.

If A is a general set of 18 points in $\mathbb{P}^2$, then the span of $v_9(A)$ contains forms T having (exactly one) second decomposition B, **disjoint from** A, of cardinality 17.

When the Hilbert function of A is generic (a condition easy to test, with elementary algorithms), then the Hilbert function of $A \cup B$ is fixed.



Examples

T ternary form of degree 9, $r = 18$.

The shape of $Dh_{A \cup B}$ imples that:

$A \cup B$ is complete intersection of curves of degree $5, 7$ (i.e. A is linked to B by a complete intersection of type $(5, 7)$).

Examples

T ternary form of degree 9, $r = 18$.

The shape of $Dh_{A\cup B}$ imples that:

$A \cup B$ is complete intersection of curves of degree $5, 7$ (i.e. A is linked to B by a complete intersection of type $(5, 7)$).

Thus the admissible B's are parametrized by the choice of a quintic and a septic form in I_A.

Examples

T ternary form of degree 9, $r = 18$.

The shape of $Dh_{A\cup B}$ imples that:

$A \cup B$ is complete intersection of curves of degree $5, 7$ (i.e. A is linked to B by a complete intersection of type $(5, 7)$).

Thus the admissible B's are parametrized by the choice of a quintic and a septic form in I_A.
Moreover I_B can be easily recoverd from I_A and the two forms that determine the linkage. The fine tuning analysis is effective.

Angelini-C, Minimality and uniqueness for decompositions of specific ternary forms. arXiv:2007.10165.

Examples

Concrete case: $T =$

$$
\begin{gathered}
= [9666x_0^9 + 13004x_0^8x_1 + 12463x_0^7x_1^2 - 13235x_0^6x_1^3 - 15442x_0^5x_1^4 + 15509x_0^4x_1^5 + -6311x_0^3x_1^6 + \\
-2390x_0^2x_1^7 + 547x_0x_1^8 - 119x_1^9 - 14916x_0^8x_2 + 1822x_0^7x_1x_2 - 8022x_0^6x_1^2x_2 - 9386x_0^5x_1^3x_2 + \\
-2742x_0^4x_1^4x_2 + 10541x_0^3x_1^5x_2 + 1156x_0^2x_1^6x_2 - 12023x_0x_1^7x_2 + 4417x_1^8x_2 - 11823x_0^7x_3^2 - 737x_0^6x_1x_1^2 + \\
-7616x_0^5x_1^2x_2^2 + 11293x_0^4x_1^3x_2^2 - 8260x_0^3x_1^4x_2^2 - 9332x_0^2x_1^5x_2^2 + 7078x_0x_1^6x_2^2 - 4553x_1^7x_2^2 - 15941x_0^6x_2^3 + \\
+4339x_0^5x_1x_2^3 - 4251x_0^4x_1^2x_2^3 + 9854x_0^3x_1^3x_2^3 - 22x_0^2x_1^4x_2^3 + 8408x_0x_1^5x_2^3 + 11858x_1^6x_2^3 + \\
-9161x_0^5x_2^4 - 9854x_0^4x_1x_2^4 - 13165x_0^3x_1^2x_2^4 - 2105x_0^2x_1^3x_2^4 - 8715x_0x_1^4x_2^4 + 390x_1^5x_2^4 - 9955x_0^4x_2^5 + \\
-11013x_0^3x_1x_2^5 - 10651x_0^2x_1^2x_2^5 - 3850x_0x_1^3x_2^5 + 4029x_1^4x_2^5 - 11735x_0^3x_2^6 - 12427x_0^2x_1x_2^6 + 12255x_0x_1^2x_2^6 + \\
-3686x_1^3x_2^6 - 2271x_0^2x_2^7 + 5939x_0x_1x_2^7 - 3402x_1^2x_2^7 + 13298x_0x_2^8 + 6455x_1x_2^8 + x_2^9].
\end{gathered}
$$

Examples

$$T = \sum_{i=1}^{18} a_i L_i^9$$

$$\begin{aligned}(a_i) = (&10308, -9437, -13956, -12270, 2135, -4854, -2213, 1755, -13629,\\ &7308, -8496, 2940, 11348, -12437, -6712, 4086, -823, -2818)\end{aligned}$$

$$A \Leftrightarrow \begin{pmatrix} 1 & 0 & -1 & 1 & 1 & 2 & 4 & 1 & 5 & 6 & 1 & 1 & 6 & -7 & 3 & 2 & 6 & -7 \\ 1 & 1 & 2 & 2 & -2 & 1 & 2 & 5 & 2 & 2 & 7 & 7 & 5 & 2 & 7 & -5 & 3 & 6 \\ 1 & 2 & 1 & 3 & 0 & 4 & -3 & 1 & 3 & 3 & 7 & 3 & 4 & 3 & 4 & 1 & -4 & 6 \end{pmatrix}$$

Examples

$$T = \sum_{i=1}^{18} a_i L_i^9 = \sum_{i=1}^{17} M_i^9$$

$$B \Leftrightarrow \begin{pmatrix}
1 & 62.6659 & 29.7378 \\
1 & 13.368 + 38.1825\&i & -19.099 + 7.53788\&i \\
1 & 13.368 - 38.1825\&i & -19.099 - 7.53788\&i \\
1 & 35.333 & 40.797 \\
1 & 14.7061 & 27.8538 \\
1 & 10.7119 & 4.95399 \\
1 & -0.796312 & 2.23381 \\
1 & 1.06064 + 0.13583\&i & 1.62951 - 0.563286\&i \\
1 & 1.06064 - 0.13583\&i & 1.62951 + 0.563286\&i \\
1 & 0.737271 & -0.0631582 \\
1 & -0.245331 & -0.76262 \\
1 & -0.187307 & 0.100519 \\
1 & -0.0870499 & -0.126324 \\
1 & 0.00104432 & 0.00164595 \\
1 & 0.306581 + 0.0182712\&i & -0.877193 - 0.031211\&i \\
1 & 0.306581 - 0.0182712\&i & -0.877193 + 0.031211\&i \\
1 & 0.390447 & 0.585521
\end{pmatrix}$$

Examples

$$T = \sum_{i=1}^{18} a_i L_i^9 = \sum_{i=1}^{17} M_i^9$$

$$B \Leftrightarrow \begin{pmatrix}
1 & 62.6659 & 29.7378 \\
1 & 13.368 + 38.1825\&i & -19.099 + 7.53788\&i \\
1 & 13.368 - 38.1825\&i & -19.099 - 7.53788\&i \\
1 & 35.333 & 40.797 \\
1 & 14.7061 & 27.8538 \\
1 & 10.7119 & 4.95399 \\
1 & -0.796312 & 2.23381 \\
1 & 1.06064 + 0.13583\&i & 1.62951 - 0.563286\&i \\
1 & 1.06064 - 0.13583\&i & 1.62951 + 0.563286\&i \\
1 & 0.737271 & -0.0631582 \\
1 & -0.245331 & -0.76262 \\
1 & -0.187307 & 0.100519 \\
1 & -0.0870499 & -0.126324 \\
1 & 0.00104432 & 0.00164595 \\
1 & 0.306581 + 0.0182712\&i & -0.877193 - 0.031211\&i \\
1 & 0.306581 - 0.0182712\&i & -0.877193 + 0.031211\&i \\
1 & 0.390447 & 0.585521
\end{pmatrix}$$

From the **wrong** (= too long) decomposition $A = ([L_1], \dots, [L_{18}])$ one finds the **minimal** decomposition $B = ([M_1], \dots, [M_{17}])$.

Examples

Sextic ternary forms of rank $r = 9$ ($< r_g = 10$).

Examples

Sextic ternary forms of rank $r = 9$ ($< r_g = 10$).

Arbarello-Cornalba. Footnotes to a paper of B.Segre. Math. Ann. 256 (1981), 341-362

A general sextic ternary forms of rank $r = 9$ has exactly **two** minimal decompositions.

Examples

Sextic ternary forms of rank $r = 9$ ($< r_g = 10$).

Arbarello-Cornalba. Footnotes to a paper of B.Segre. Math. Ann. 256 (1981), 341-362

A general sextic ternary forms of rank $r = 9$ has exactly **two** minimal decompositions.

Ottaviani-C. A footnote to a footnote to a paper of B.Segre. arXiv:2103.04659

Examples

Sextic ternary forms of rank $r = 9$ ($< r_g = 10$).

Arbarello-Cornalba. Footnotes to a paper of B.Segre. Math. Ann. 256 (1981), 341-362

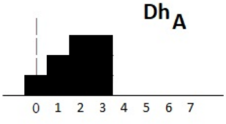
A general sextic ternary forms of rank $r = 9$ has exactly **two** minimal decompositions.

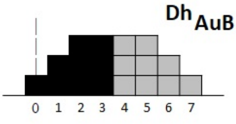
Ottaviani-C. A footnote to a footnote to a paper of B.Segre. arXiv:2103.04659

Problem

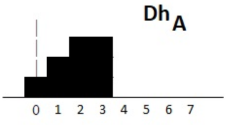
Given one decomposition $T = \sum_{i=1}^{9} L_i^9$, find the second, different decomposition $T = \sum_{i=1}^{9} M_i^9$.

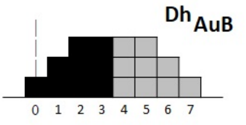
Examples





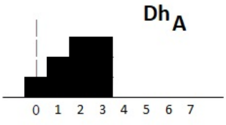
Examples

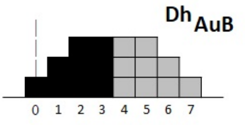




B is linked to A by a complete intersection of type $(3, 6)$.

Examples

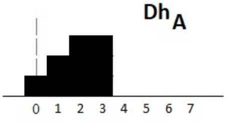


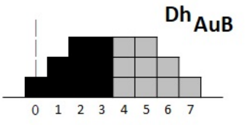


B is linked to A by a complete intersection of type $(3, 6)$.

Fine tuning: Take in I_A a cubic form (unique) and a sextic form (depends on 9 parameters) and compute I_B.

Examples





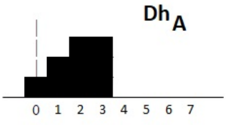
B is linked to A by a complete intersection of type $(3,6)$.

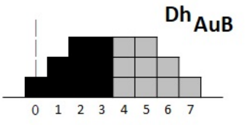
Fine tuning: Take in I_A a cubic form (unique) and a sextic form (depends on 9 parameters) and compute I_B.

$$0 \to R^3(-5) \xrightarrow{M} R^3(-4) \oplus R(-3) \to I_A \to 0$$

$$M = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix}$$

Examples





B is linked to A by a complete intersection of type $(3, 6)$.

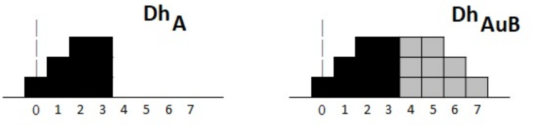
Fine tuning: Take in I_A a cubic form (unique) and a sextic form (depends on 9 parameters) and compute I_B.

$$0 \to R^3(-5) \xrightarrow{M} R^3(-4) \oplus R(-3) \to I_A \to 0$$

$$M = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & x_0\ell & q \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = M'$$

$\ell =$ general linear form, $q =$ general quadratic form (9 parameters).

Examples



B is linked to A by a complete intersection of type $(3,6)$.

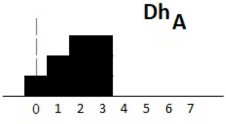
Fine tuning: Take in I_A a cubic form (unique) and a sextic form (depends on 9 parameters) and compute I_B.

$$0 \to R^3(-5) \xrightarrow{M} R^3(-4) \oplus R(-3) \to I_A \to 0$$

$$M = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & x_0\ell & q \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = M'$$

$\ell =$ general linear form, $q =$ general quadratic form (9 parameters).
I_B is generated by the maximal minors of M'.

Examples



B is linked to A by a complete intersection of type $(3,6)$.

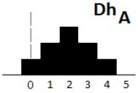
Fine tuning: Take in I_A a cubic form (unique) and a sextic form (depends on 9 parameters) and compute I_B.

$$0 \to R^3(-5) \xrightarrow{M} R^3(-4) \oplus R(-3) \to I_A \to 0$$

$$M = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & x_0\ell & q \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = M'$$

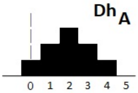
$\ell =$ general linear form, $q =$ general quadratic form (9 parameters).
I_B is generated by the maximal minors of M'.
Then solve the fine tuning equation $T^\vee = (I_A)_6 + (I_B)_6$.

Examples



What if A is not general? E.g. A is complete intersection of two cubics?

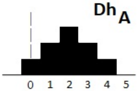
Examples



What if A is not general? E.g. A is complete intersection of two cubics?

(One gets A, **for free**, from the apolar ideal!)

Examples

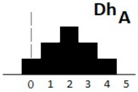


What if A is not general? E.g. A is complete intersection of two cubics?

(One gets A, **for free**, from the apolar ideal!)

Theorem. (Ottaviani-C) *If A is complete intersection of two cubics, then it is unique.*

Examples

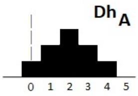


What if A is not general? E.g. A is complete intersection of two cubics?

(One gets A, **for free**, from the apolar ideal!)

Theorem. (Ottaviani-C) *If A is complete intersection of two cubics, then it is unique. These forms T are dense in $W = Sing(\sigma_9(v_6(\mathbb{P}^2)))$.*

Examples



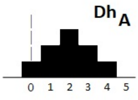
What if A is not general? E.g. A is complete intersection of two cubics?

(One gets A, **for free**, from the apolar ideal!)

Theorem. (Ottaviani-C) *If A is complete intersection of two cubics, then it is unique. These forms T are dense in $W = Sing(\sigma_9(v_6(\mathbb{P}^2)))$.*

	dim	*deg*	eq	
σ_9	26	10	$\det \mathcal{C}^3$	9-secant
$\mathcal{R}$	25	270	$\det \mathcal{C}^3$, H_{27}	ramif. $A\sigma_9 \to \sigma_9$
W	24	165	$\left(\mathcal{C}^3\right)_9$	compl. inters.
σ_8	23	1485	$\left(\mathcal{C}^3\right)_9$, H_{27}	8-secant
W'	21	2640	$\left(\mathcal{C}^3\right)_8$	3 apolar cubics
σ_7	20	11880	$\left(\mathcal{C}^3\right)_8$, H_{27}	7-secant

Examples

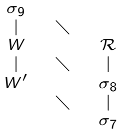


What if A is not general? E.g. A is complete intersection of two cubics?

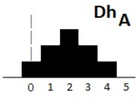
(One gets A, **for free**, from the apolar ideal!)

Theorem. (Ottaviani-C) *If A is complete intersection of two cubics, then it is unique. These forms T are dense in $W = Sing(\sigma_9(v_6(\mathbb{P}^2)))$.*

	dim	*deg*	eq	
σ_9	26	10	$\det \mathcal{C}^3$	9-secant
$\mathcal{R}$	25	270	$\det \mathcal{C}^3$, H_{27}	ramif. $A\sigma_9 \to \sigma_9$
W	24	165	$\left(\mathcal{C}^3\right)_9$	compl. inters.
σ_8	23	1485	$\left(\mathcal{C}^3\right)_9$, H_{27}	8-secant
W'	21	2640	$\left(\mathcal{C}^3\right)_8$	3 apolar cubics
σ_7	20	11880	$\left(\mathcal{C}^3\right)_8$, H_{27}	7-secant



Examples

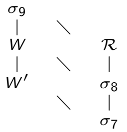


What if A is not general? E.g. A is complete intersection of two cubics?

(One gets A, **for free**, from the apolar ideal!)

Theorem. (Ottaviani-C) *If A is complete intersection of two cubics, then it is unique. These forms T are dense in $W = Sing(\sigma_9(v_6(\mathbb{P}^2)))$.*

	dim	*deg*	eq	
σ_9	26	10	$\det \mathcal{C}^3$	9-secant
$\mathcal{R}$	25	270	$\det \mathcal{C}^3, H_{27}$	ramif. $A\sigma_9 \to \sigma_9$
W	24	165	$\binom{\mathcal{C}^3}{9}$	compl. inters.
σ_8	23	1485	$\binom{\mathcal{C}^3}{9}, H_{27}$	8-secant
W'	21	2640	$\binom{\mathcal{C}^3}{8}$	3 apolar cubics
σ_7	20	11880	$\binom{\mathcal{C}^3}{8}, H_{27}$	7-secant



THANK YOU FOR YOUR ATTENTION