

Clay Lecture 2:

Border ranks of tensors with symmetry

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The classical substitution method for tensor rank

Prop. Let $T \in A \otimes B \otimes C$. Write $T = \sum_{i=1}^a a_i \otimes M_i$, where $M_i \in B \otimes C$. Let $\mathbf{R}(T) = r$ and $M_1 \neq 0$. Then $\exists \lambda_2, \dots, \lambda_a \in \mathbb{C}$, such that $\mathbf{R}(\tilde{T}) \leq r - 1$ where

$$\tilde{T} := \sum_{j=2}^a a_j \otimes (M_j - \lambda_j M_1) \in \langle a_2, \dots, a_a \rangle \otimes B \otimes C.$$

Proof: $\mathbf{R}(T) = r \Rightarrow \exists X_1, \dots, X_r$ rank one and d_j^i so $M_j = \sum d_j^i X_i$. $M_1 \neq 0 \Rightarrow \text{WLOG } d_1^1 \neq 0$.

Take $\lambda_j = d_j^1 / d_1^1 \Rightarrow \tilde{T}(\langle a_2, \dots, a_a \rangle^*) \subset \langle X_2, \dots, X_r \rangle \Rightarrow \mathbf{R}(\tilde{T}) \leq r - 1$. □

Alexeev-Forbes-Tsimerman: Used to find explicit sequence of tensors with $\mathbf{R}(T^m) \geq 3m - \log(m) + 1$ (best known over $\mathbb{C}$).
Used tensors with nice combinatorial structure

The substitution method rephrase

Prop. Let $T \in A \otimes B \otimes C$ be A -concise. Fix a line $[\alpha] \in \mathbb{P}A^*$. Then

$$\mathbf{R}(T) \geq \min_{A' \subset A^*, \text{hyperplane}, \alpha \notin A'} \mathbf{R}(T|_{A' \otimes B^* \otimes C^*}) + 1$$

Compare: above $[\alpha] = \langle a_2, \dots, a_a \rangle^\perp$

$$\tilde{T} := \sum_{j=2}^a a_j \otimes (M_j - \lambda_j M_1) \in \langle a_2, \dots, a_a \rangle \otimes B \otimes C.$$

The substitution method: Border rank version

Prop. (L-Michalek, Bläser-Lysikov, 2016) Let $T \in A \otimes B \otimes C$ be A -concise.

Then

$$\underline{\mathbf{R}}(T) \geq \min_{A' \subset A^*, \text{hyperplane}} \underline{\mathbf{R}}(T|_{A' \otimes B^* \otimes C^*}) + 1$$

Proof: Say $\underline{\mathbf{R}}(T) = r$ so $T = \lim_{t \rightarrow 0} T_t$.

By rank version, $\exists a(t) \in A$, such that $\underline{\mathbf{R}}(T_t|_{a(t)^\perp \otimes B^* \otimes C^*}) \geq r - 1$
Let $[a] = \lim_{t \rightarrow 0} [a(t)]$ to conclude with $A' = a^\perp$.

How to use? Idea: if T has symmetry, can restrict search of A' 's.

Weights (Generalized eigenvalues)

Fix bases and let $\mathbb{T} \subset GL(V)$ denote the diagonal matrices,

$\mathbb{T}$: maximal torus.

Write $t := \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_v \end{pmatrix}$ then $te_j = t_j e_j$. More generally

$$t(e_1^{\otimes p_1} \otimes e_2^{\otimes p_2} \otimes \cdots \otimes e_v^{\otimes p_v}) = t_1^{p_1} \cdots t_v^{p_v} e_1^{\otimes p_1} \otimes e_2^{\otimes p_2} \otimes \cdots \otimes e_v^{\otimes p_v}$$

Say *weight vector of weight* $(p_1, \dots, p_v)$, := simultaneous eigenvector $\forall t \in \mathbb{T}$. Note $\mathfrak{S}_d$ preserves weight.

Let $\mathbb{B} \subset GL(V)$ upper triangular matrices (Borel subgroup). A weight line $x \in \mathbb{P}V^{\otimes d}$ is a *highest weight line* if $\mathbb{B} \cdot x = x$.

Ex. $W = S_\pi V$, highest weight is π . $\pi = (p_1, \dots, p_v) \Rightarrow \text{proj}_{\pi-\text{def}}(e_1^{\otimes p_1} \otimes e_2^{\otimes p_2} \otimes \cdots \otimes e_v^{\otimes p_v})$ hw vect.

Lie-Kolchin Thm: U G -module $\mathbb{B} \subset G$ Borel, $\exists$ $\mathbb{B}$ -fixed line.

Moreover if U irred. G -module (G reductive) $\Rightarrow \exists!$ hw line.

$\mathbb{B}$ -fixed spaces and border substitution

Borel: More generally X : projective G -variety, $\exists x \in X$, $\mathbb{B}$ -fixed.

In border substitution method, can restrict to Borel fixed points in $G(\mathbf{a} - t, A^*)$:

Prop. (L-Michalek 2016) Let $T \in A \otimes B \otimes C$ be A -concise.

Then

$$\underline{\mathbf{R}}(T) \geq \min_{A' \subset A^*, \text{Borel fixed}, \text{codim } A' = t} \underline{\mathbf{R}}(T|_{A' \otimes B^* \otimes C^*}) + t$$

Next step: Describe Borel fixed subspaces

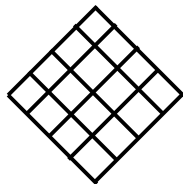
Weight diagrams

W : G -module has a basis consisting of weight vectors.

$\mathbb{B} \rightsquigarrow$ partial order on weights and weight lines. $[w] \leq [u]$ if $[u] \in \overline{\mathbb{B}[w]}$

$G = GL(V)$ get dominance order on sequences of integers $(p_1, \dots, p_v)$. $\rightsquigarrow$ weight diagram.

Ex. $U^* \otimes V$ as $G = GL(U) \times GL(V)$ -module, highest weight vect. $= u^n \otimes v_1$ weight diagram:



Borel fixed subspaces (i.e., Borel fixed points of Grassmannian)

Easy to draw .

Borel subgroups in general

Given G have assoc. Lie algebra $\mathfrak{g} = T_{\text{Id}} G$.

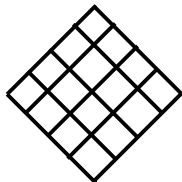
$\mathfrak{g}$ is *solvable* if, $D_1(\mathfrak{g}) := [\mathfrak{g}, \mathfrak{g}]$, $D_k(\mathfrak{g}) := [D_{k-1}(\mathfrak{g}), D_{k-1}(\mathfrak{g})]$ is 0 some k .

G *solvable* if $\mathfrak{g}$ is solvable.

G : reductive, $\exists$ maximal solvable subgroups =: Borel subgroups

Example $M_{\langle n \rangle}$

$$A = U^* \otimes V$$



Computation $\rightsquigarrow$ worst case is along one of outer diagonals.

$\rightsquigarrow$ Thm. (L-Michalek, 2018) $\underline{\mathbf{R}}(M_{\langle n \rangle}) \geq 2\mathbf{n}^2 - \lceil \log_2(\mathbf{n}) \rceil - 1$.

Hay in a haystack: explicit tensors of high border rank

Tension: want $\dim(G_T) > 0$ and T as “generic” as possible

So far best with $G_T = \mathbb{C}^*$:

$\forall k$, define $\lceil \frac{3(2k+1)^2}{4} \rceil$ -dimensional family

$T_k = T_k(p_{ij}) \in \mathbb{C}^{2k+1} \otimes \mathbb{C}^{2k+1} \otimes \mathbb{C}^{2k+1} =: A \otimes B \otimes C$, where $|i|, |j|, |i+j| \leq k$ as follows:

$$T_k = \sum_{i=-k}^k \sum_{j=\max(-k, -i-k)}^{\min(k, -i+k)} p_{ij} a_i \otimes b_j \otimes c_{-i-j},$$

here $(a_{-k}, \dots, a_k), (b_{-k}, \dots, b_k), (c_{-k}, \dots, c_k)$ are bases.

Take p_{ij} distinct primes (computer science explicit) $\leadsto$

$\underline{\mathbf{R}}(T_k) \geq (2.02)(2k+1) = (2.02)m$ once $k \sim 10^8$. (L-Michalek 2019) (Already can beat $2m$ with smallish k .)

Game over?

Both cases limits of known methods. How to go further?

Buczynska-Buczynski idea: use more information limits of *ideals*
 $I_t := \text{ideal of } [T_1(t)] \sqcup \cdots \sqcup [T_r(t)]$

Problem: How to take limits?

First Idea: The Hilbert scheme

Grothendieck: insist on saturated ideals $I \subset \text{Sym}(V^*)$. Then, in a sufficiently high degree D , $I_D \subset S^D V^*$ determines I in all degrees. and sufficiently high can be made precise. Reduced to taking limits in one fixed Grassmannian. “Hilbert Scheme” (Parametrizes saturated ideals with same Hilbert polynomial.)

$I \subset \text{Sym}(V^*)$ any ideal. Let $r_d = \dim(S^d V^* / I_d)$. (Hilbert function $h_I(d) := r_d$)

Fact: (Castelnuovo-Mumford regularity) if fix Hilbert function, $\exists$ explicit $D = D(h_I)$ such that I_D determines $I_{D'}$ for all $D' > D$.

Moreover $h_I(x)$ poly when $x > D \rightsquigarrow$ Hilbert polynomial

Hilbert scheme lives in $G(D, S^D V^*)$.

BB idea

If have border rank decomp. $T = \lim_{t \rightarrow 0} \sum_{j=1}^r T_j(t)$ can study $I_t := \text{ideal of } [T_1(t)] \sqcup \cdots \sqcup [T_r(t)]$

Bad news: Hilbert scheme doesn't work. Consider toy case of 3 points in $\mathbb{P}^2$: $[1, 0, 0], [0, 1, 0], [1, -1, t]$ $t \neq 0$, $(I_t)_1 = 0$ and

$$(I_t)_2 = \langle x_3^2 + t^2 x_1 x_2, x_3^2 - t x_1 x_3, x_1 x_3 + x_2 x_3 \rangle$$

$$\text{But } (I_0)_1 = \langle x_3 \rangle, (I_0)_2 = \langle x_3^2, x_1 x_3 + x_2 x_3 \rangle$$

Ideal of limiting scheme in fixed deg. $\neq$ limit of spans

Limit is taken in Hilbert scheme (i.e. a fixed Grassmannian), loses information important for border rank decomposition.

Solution: Haiman-Sturmfels multi-graded Hilbert scheme.

BB idea

Consider product of Grasmannians

$$G(r_1, V^*) \times G(r_2, S^2 V^*) \times \cdots \times G(r_D, S^D V^*)$$

and map $I \mapsto ([I_1] \times [I_2] \times \cdots \times [I_D])$. For each $\mathbb{Z}_{\geq 0}$ -valued function h , get (possibly empty) subscheme parametrizing all ideals I with *Hilbert function* $h_I = h$.

Rigged such that limit I of ideals has same Hilbert function as ideals I_ϵ .

Key Lemma: In border rank decompositions, WLOG for $t > 0$ points in general position $\leadsto$ const. Hilbert function (as soon as possible)

BB idea: Tensor case

have more information: curves of points on $\text{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$
can think of ideals in

$$\text{Sym}(A \oplus B \oplus C)^* = \bigoplus_{s,t,u} S^s A^* \otimes S^t B^* \otimes S^u C^*, \mathbb{Z}^{\oplus 3}\text{-graded}.$$

$$\text{Hilbert function } h_I(s, t, u) := \dim(S^s A^* \otimes S^t B^* \otimes S^u C^* / I_{s,t,u}).$$

Key Lemma $\Rightarrow$ Know Hilbert function! Namely

$$h_I(s, t, u) = \min\{r, \dim S^s A^* \otimes S^t B^* \otimes S^u C^*\}.$$

BB idea: Tensor case summary

Instead of single curve $E_t \subset G(r, A \otimes B \otimes C)$ limiting to Borel fixed point, for each (i, j, k) get curve in $G(r, S^i A^* \otimes S^j B^* \otimes S^k C^*)$, each limiting to Borel fixed point *and* satisfying compatibility conditions.

Upshot: algorithm that either produces all normalized candidate l_0 's or proves border rank $> r$.

The border apolarity method

If $\underline{\mathbf{R}}(T) \leq r$, there exists a multi-graded ideal I satisfying:

1. I is contained in the annihilator of T . This condition says $I_{110} \subset T(C^*)^\perp$, $I_{101} \subset T(B^*)^\perp$, $I_{011} \subset T(A^*)^\perp$ and $I_{111} \subset T^\perp \subset A^* \otimes B^* \otimes C^*$.
2. For all (ijk) with $i + j + k > 1$, $\text{codim } I_{ijk} = r$.
3. each I_{ijk} is Borel-fixed.
4. I is an ideal, so the multiplication maps $I_{i-1,j,k} \otimes A^* \oplus I_{i,j-1,k} \otimes B^* \oplus I_{i,j,k-1} \otimes C^* \rightarrow S^i A^* \otimes S^j B^* \otimes S^k C^*$ have image contained in I_{ijk} .

Next time: unpack these conditions and apply to matrix multiplication and other tensors.

Thank you for your attention

For more on **tensors**, their geometry and applications, resp. **geometry and complexity**, resp. **recent developments**:

