Stochastic (intermittent) Spikes and Strong Noise Limit of SDEs.

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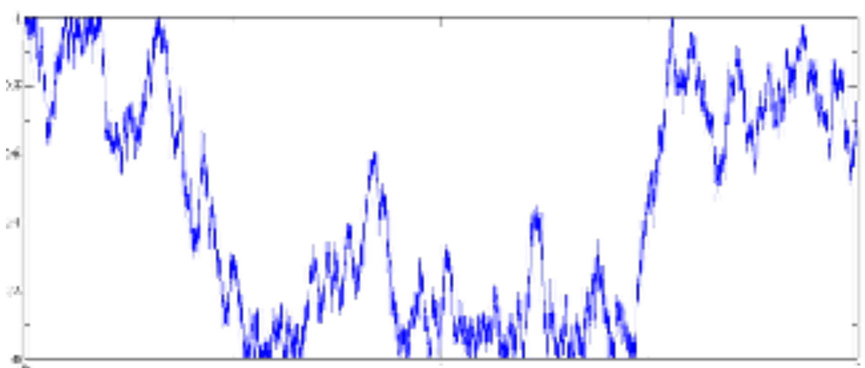
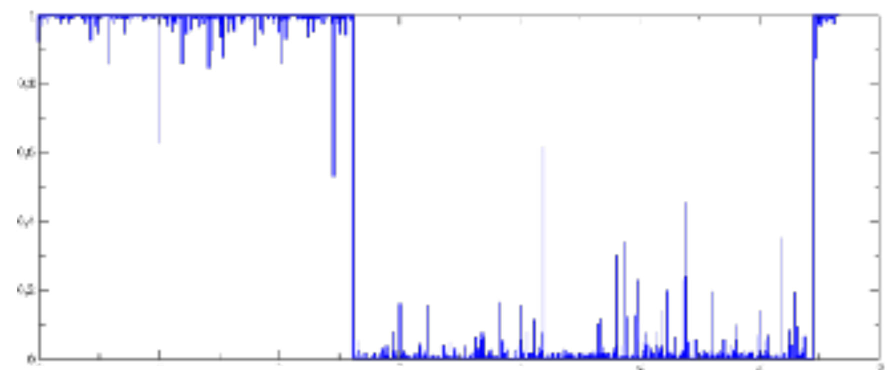
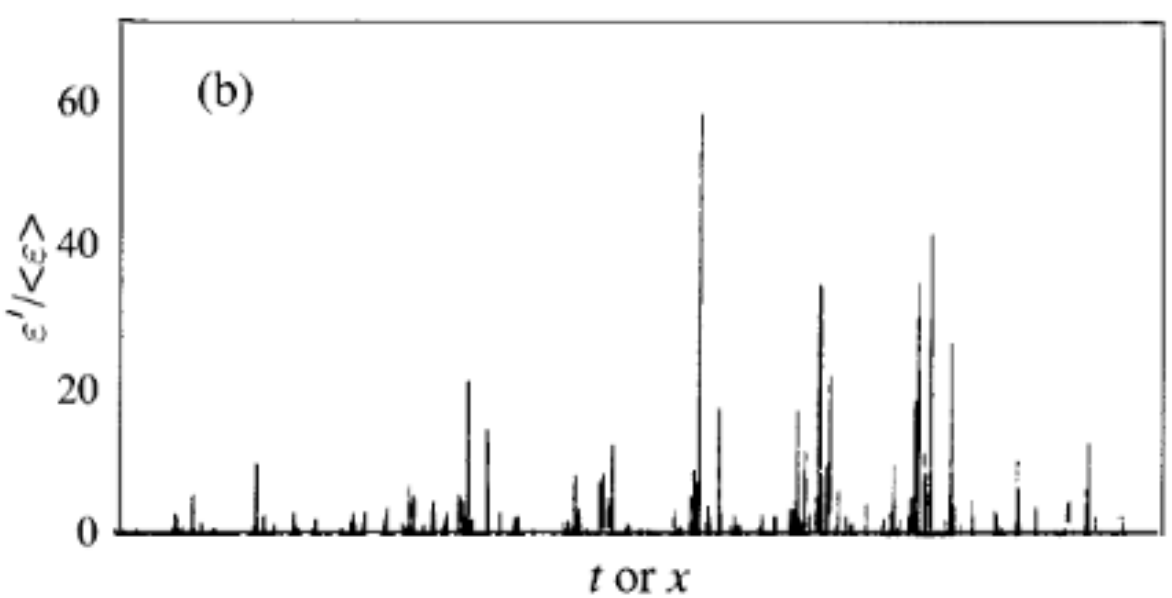






Strong Noise Limit of (some) SDEs

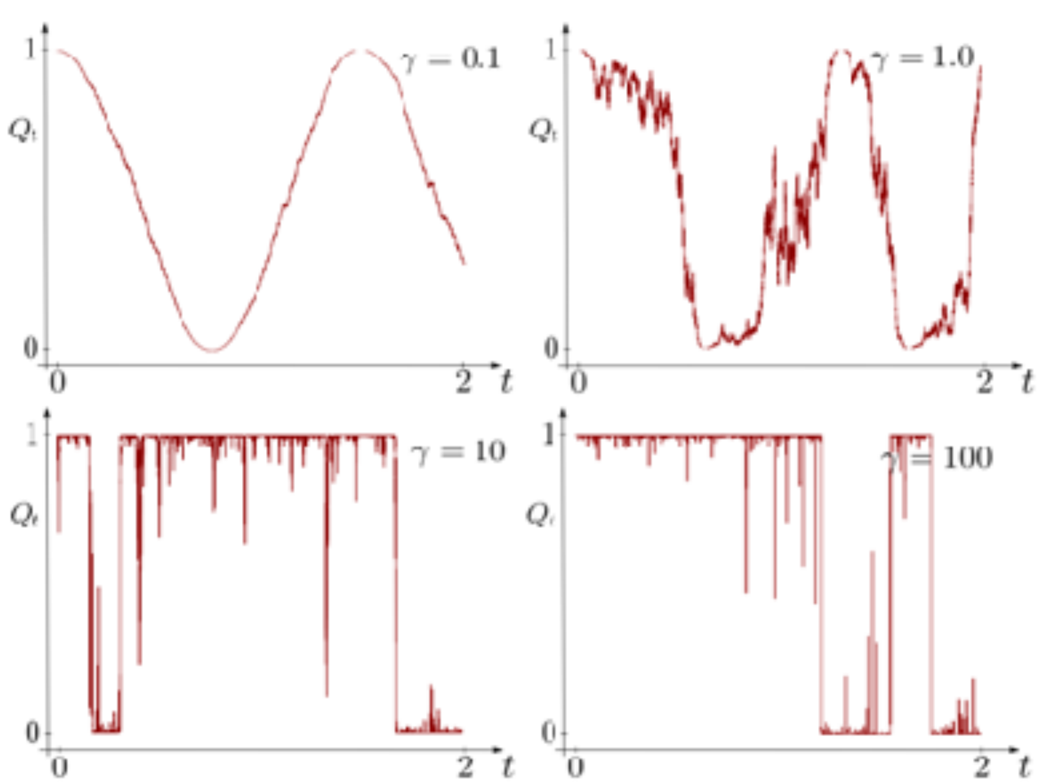
— **Stochastic differential equations** (SDE) of the form :

$$dX_t = \big(a(X_t) - \lambda^2 b(X_t)\big)\,dt + \lambda\, c(X_t)\, dB_t$$

with a(0)>0 and b(0)=0=c(0) with b(x)>0 in a neighbourhood of the origin.

— We are interested in the limit of large lambda (**strong noise limit**) with an appropriate rescaling of a(x).

— **Typical behaviour**
(extracted from monitoring phenomena of quantum systems, here monitored Rabi oscillations)



As the « noise » increases:

- **Jumps** (quantum jumps) emerge.

- A finer structure survives, in between the jumps: the **spikes**.

- **Different from** metastability in the **weak noise** limit of SDE . (Kramer's or Freidlin & Wentzell's like).

Strong Noise Limit of (some) SDEs (II)

— The examples we shall discuss:

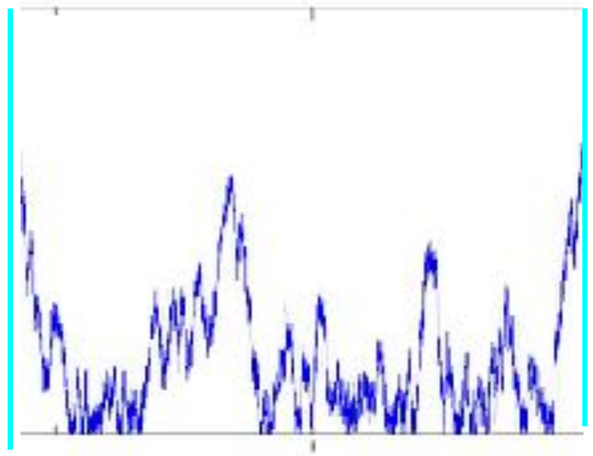
$$dX_t = \frac{\lambda^2}{2}(\epsilon - bX_t)dt + \lambda\, X_t\, dB_t \text{ with } \lambda \to \infty;\ \epsilon \to 0;\ \lambda^2 \epsilon^{b+1} =: J \text{ fixed}$$

— **Claim**: In the scaling limit, the solution of this SDE is a fractional power of a reflected Brownian motion parametrised by its local time:

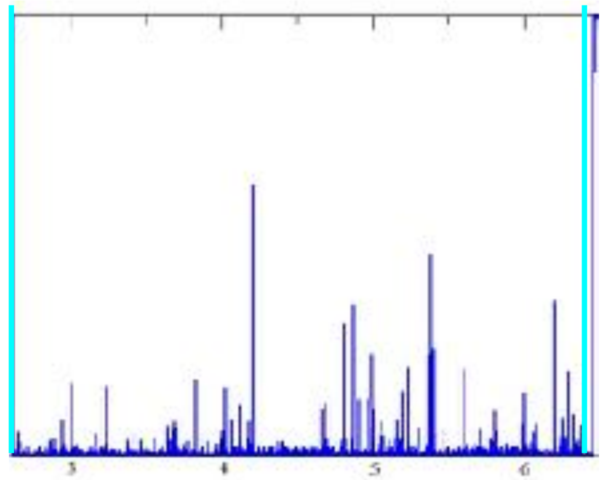
$$X_t = (b+1)^{\frac{1}{b+1}}\ |W_\tau|^{\frac{1}{b+1}} \qquad \text{with} \qquad J\,t = 2\Gamma(b+1)\ L_\tau,$$

with L the local time at the origin of the Brownian motion W.

— A geometrical construction (for b=0):



A sample of a reflected Brownian motion parametrised with its natural time.

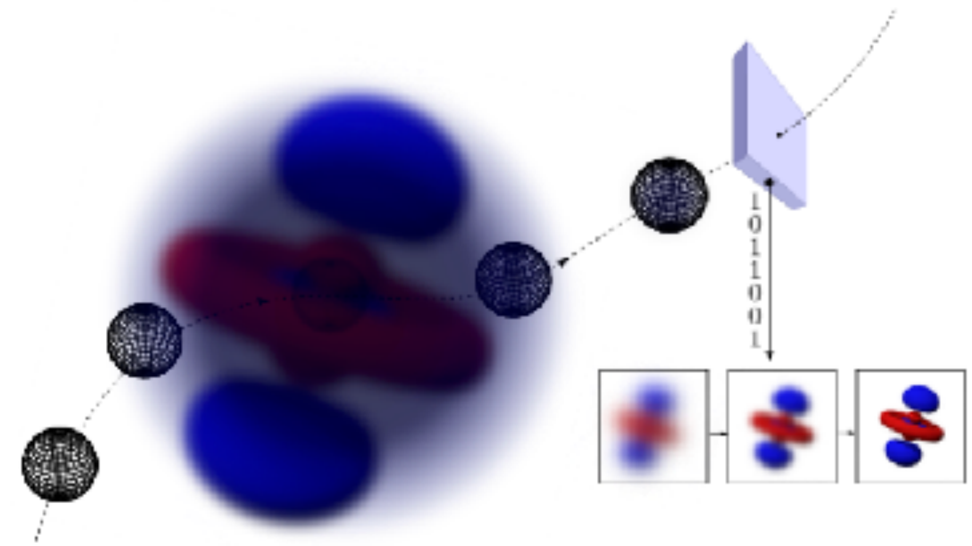


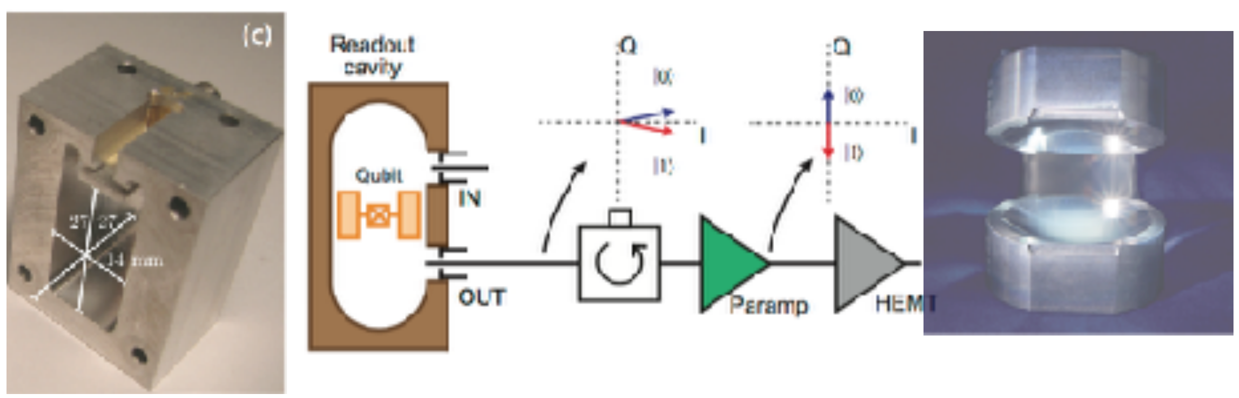
The same example but parametrised by its local time spend at 0.

A Motivation: Quantum Trajectories (I).

— **Continuous monitoring of a quantum system** (say via interaction with series of probes) :

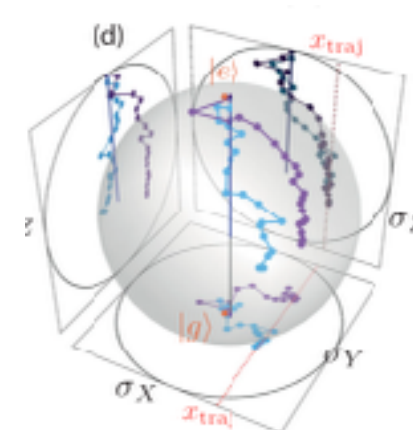
- extract (random) **information**.
- modify (**back-action**) the Q-state.





Many examples/realisations:
QED, circuit QED, Q-dots, etc...

— **Quantum system evolution equation is stochastic** :



$$d\rho_t = -i[H, \rho_t]dt + \lambda^2 L(\rho_t)dt + \lambda D(\rho_t)\, dB_t$$

Shroedinger like deterministic evolution

Measurement back-action scaled with the information rate.

—> Bt is a Brownian motion, **a «informative» noise linked to the readout signal**.

A Motivation: Quantum Trajectories (II)

— An example: **Energy monitoring of a Qu-bit in contact with a thermal bath.**

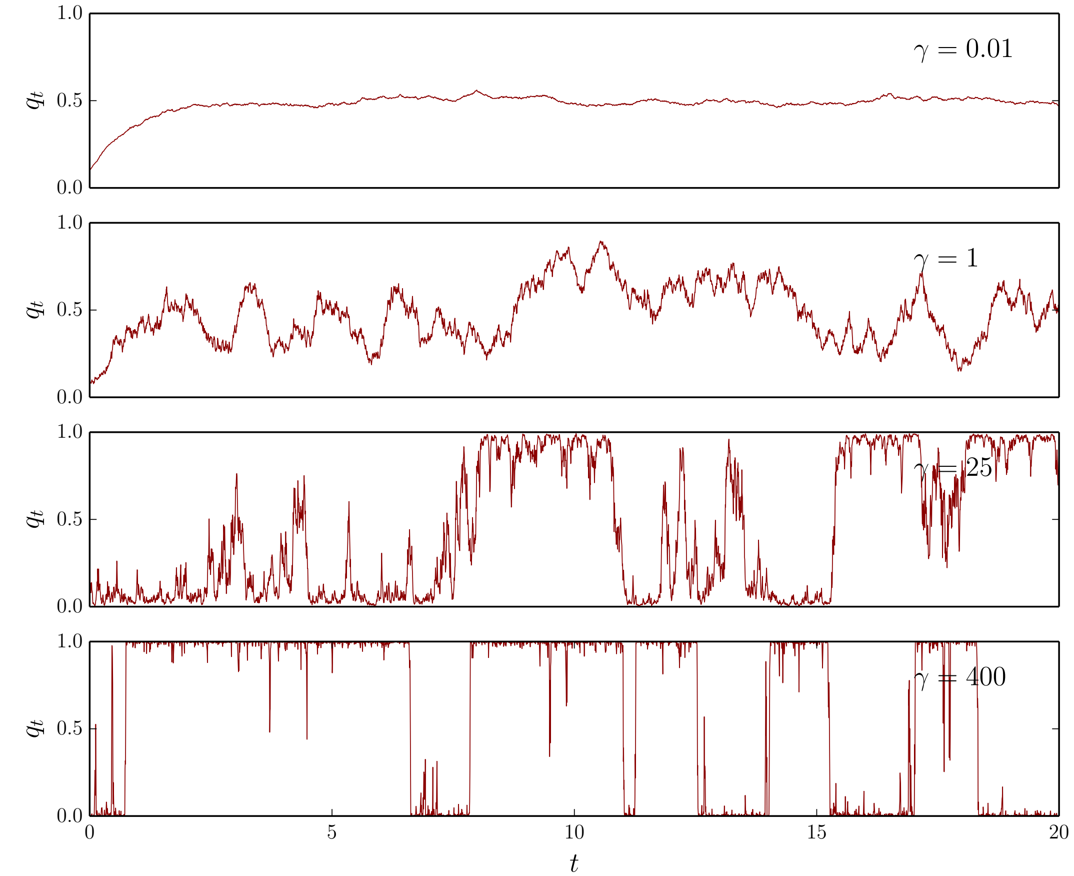
$$dP_t = \eta(p - P_t)dt + \lambda P_t(1 - P_t)dB_t$$

Thermal relaxation

« competing processes »

Effect of monitoring (back-action)

Pt = population of the ground state.



smooth thermal relaxation at small information rate.

jumpy behaviour between energy eigenstates (thermally activated quantum jumps) at high information rate.

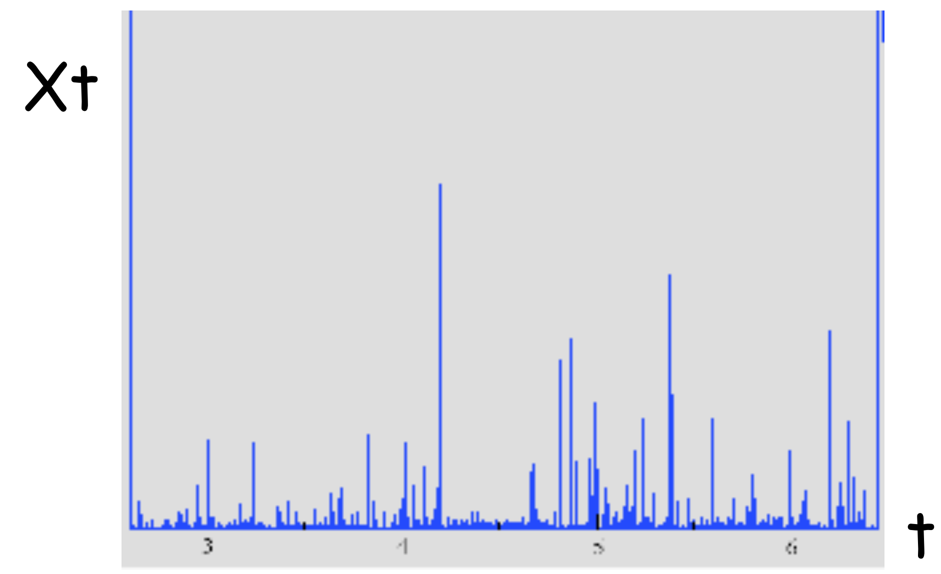
— Linearisation (near P=0) : $dP_t = \eta p\, dt + \lambda\, P_t\, dB_t$ (i.e. the case b=0).

Strong Noise Limit: Strategy of Analysis/Proof

— Recall **the SDEs we are looking at**:

$$dX_t = \frac{\lambda^2}{2}(\epsilon - bX_t)dt + \lambda\, X_t\, dB_t \quad \text{with} \quad \lambda \to \infty;\ \lambda^2 \epsilon^{b+1} =: J \text{ fixed}$$

This can be solved explicitly,
and X —> 0 almost surely (at fixed time t)
but there are `spiky times'.



— **Strategy of analysis (alias proof)**:

(i) Show that the tips of the spikes form a point Poisson process.
(ii) Redefine times (via quadratic variation) to decipher the spike structure.

— Remark: change variable and set $Q_t := \frac{X_t^{b+1}}{b+1}$ then $dQ_t = \frac{\hat{\lambda}^2}{2}\,\hat{\epsilon}\, Q_t^{\frac{b}{b+1}}\, dt + \hat{\lambda}\, Q_t\, dB_t$
Not smooth, but nevertheless trajectories well defined (because X>0).

Passage Times and a Point Poisson Process (I)

— **Two competing effects** in: $dX_t = \frac{\lambda^2}{2}(\epsilon - bX_t)dt + \lambda\, X_t\, dB_t$

— **Why spikes survives in the limit?**

Look at the distribution of the X-maxima in a time interval T:

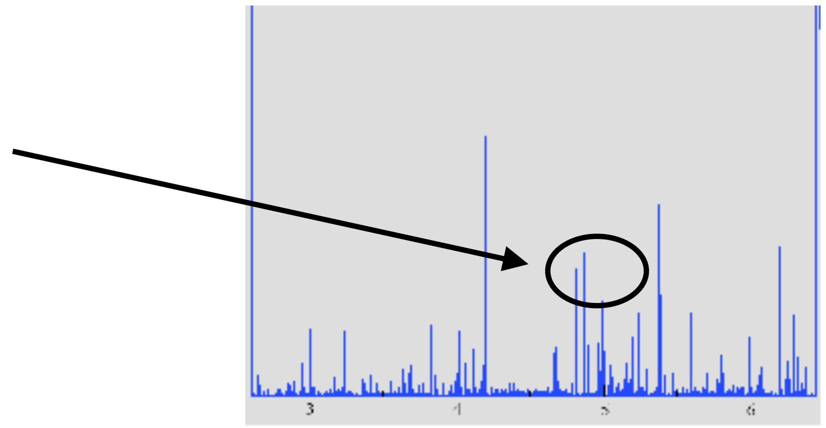
$$\mathbb{P}[X - \max \text{ on } [0,T] < m] \simeq \Big[\int_0^m dx P_{\text{inv}}(x)\Big]^{\lambda^2 T} \simeq \Big[1 - c\,\frac{\epsilon^{b+1}}{m^{b+1}}\Big]^{\lambda^2 T} \simeq e^{-c(\lambda^2 \epsilon^{b+1})\,T/m^{b+1}}.$$

using/assuming independency after time scale of order λ^{-2}.

— **Claim:** The tips of the spikes form a point Poisson process with intensity

$$d\nu = \hat{J}dt\,\frac{dq(x)}{q(x)^2} = \frac{(b+1)^2}{2\Gamma(b+1)}\,Jdt\,\frac{dx}{x^{b+2}}$$

with $q(x) = x^{b+1}/(b+1)$



Passage Times and a Point Poisson Process (II)

— The argument consisting in reconstructing the transition/passage time distribution from the point Poisson process.

— **Passage times** $T_{y\to z}$ from y to z>y, and their generating function:

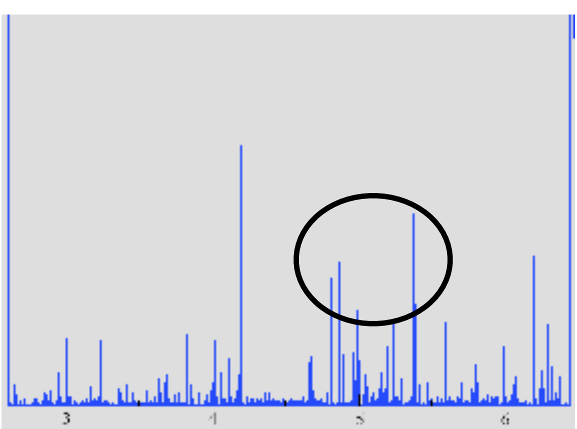
$$\mathbb{E}[e^{-\sigma T_{y\to z}}] = \frac{\psi(y,\sigma)}{\psi(z,\sigma)}$$ (by Markov property)

$\psi(x,\sigma)$ satisfies a 2nd order Schroedinger like equation (by a martingale prop.)

In the scaling limit $\lambda \to \infty;\ \lambda^2 \epsilon^{b+1} =: J$ fixed, $\psi(x,\sigma)$ is linear in sigma:

$$\psi(x,\sigma) = \hat{J} + \sigma\, q(x)$$ with $q(x) = x^{b+1}/(b+1)$

— **Comparison with** spike tips as **a point Poisson process of intensity** $d\nu = dt\, d\hat{\nu}(x)$



$$\mathbb{E}[e^{-\sigma T_{y\to z}}] = \frac{1+\sigma\, \hat{\nu}([y,\infty])^{-1}}{1+\sigma\, \hat{\nu}([z,\infty])^{-1}}$$ (two contributions)

Hence: $d\nu = \hat{J} dt\, \frac{dq(x)}{q^2(x)}$

Effective Time and Local Time (I)

— To decipher the structure of the spikes we may use a time variable which is sensitive to the variation of the process:
Say, **a new clock which clicks only if the process varies in a sensible way**.

— Recall $dQ_t = \frac{\hat\lambda^2}{2}\,\hat\epsilon\, Q_t^{\frac{b}{b+1}}\,dt + \hat\lambda\, Q_t\, dB_t$ for $Q_t := \frac{X_t^{b+1}}{b+1}$

We define the new time as the quadratic variation of Q :

Claim:
Let $d\tau = \hat\lambda^2\, Q_t^2\, dt$
In the scaling limit $\lambda \to \infty;\ \lambda^2 \epsilon^{b+1} =: J$ fixed
$Q_\tau = |\tilde W_\tau|$ with $\tilde W_\tau$ a normalized Brownian motion.

Because Q is a martingale at e=0 and the drift is active only at Q=0.
(to ensure positivity).

— Let $dW_\tau = \hat\lambda\, Q_t\, dB_t = \lambda X_t^{b+1}\, dB_t$, then W is a new Brownian motion,
and $dQ_\tau = dL_\tau + dW_\tau$ with $dL_\tau = \frac{\hat\epsilon}{2} Q_\tau^{-\frac{b+2}{b+1}}\, d\tau$
By the Skorokhod's theorem [and the uniqueness of the decomposition f(x) = g(x)-l(x)] applied to the Brownian motion W.
The equation $dQ_\tau = dL_\tau + dW_\tau$ then reduces to Tanaka's equation.

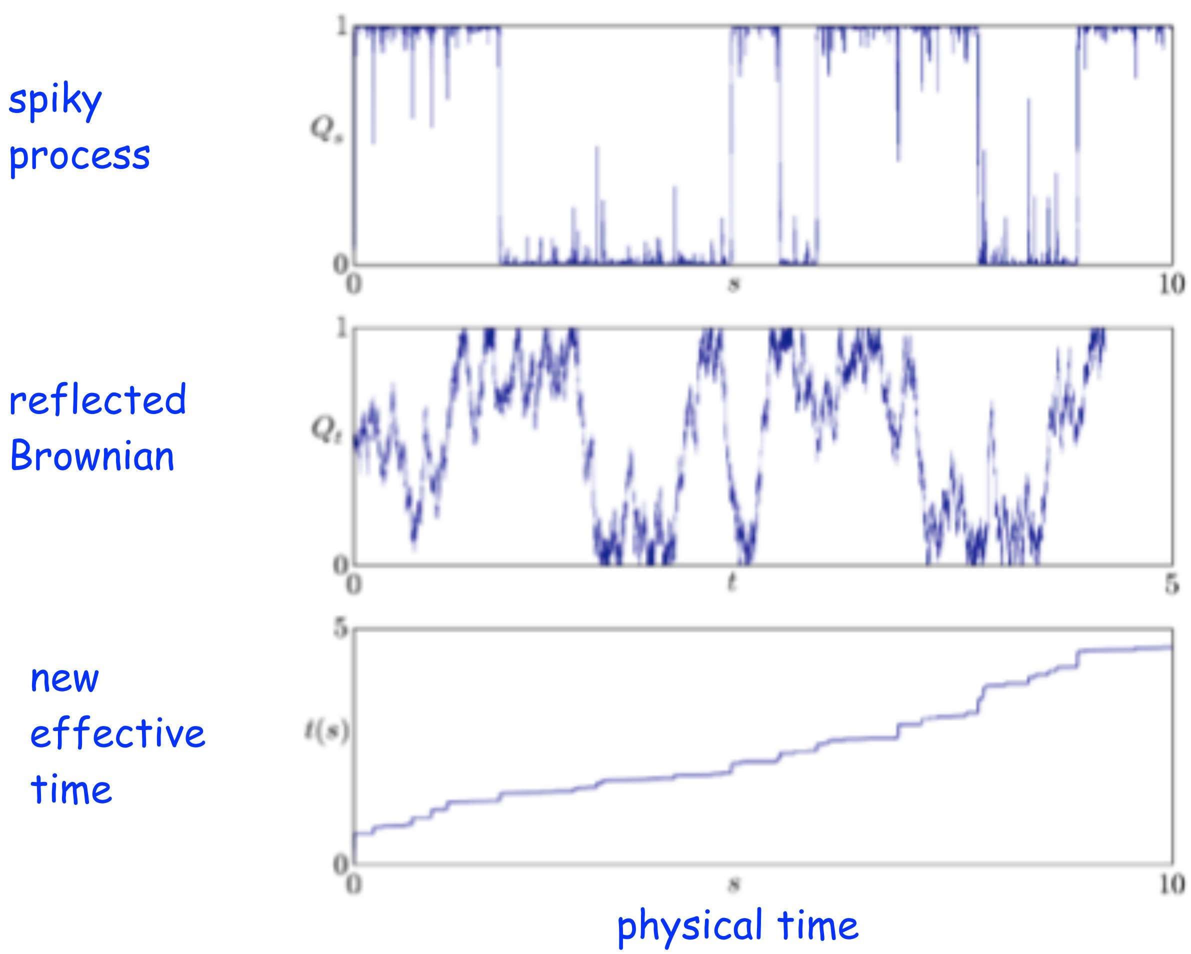
Effective Time and Local Time: Reconstruction.

— To reconstruct we have to determine the relation between the new time tau and the original physical time t (using $dL_\tau = \frac{\lambda^2}{2}\epsilon X_t^b \, dt$ plus ergodicity).

Claim: In the scaling limit $\lambda \to \infty; \; \lambda^2 \epsilon^{b+1} =: J$ fixed

L_τ is the $\tilde{W}_\tau$ local time at the origin and $J\,dt = 2\Gamma(b+1)\, dL_\tau,$

This yields: $X_t = (b+1)^{\frac{1}{b+1}} \, |\tilde{W}_\tau|^{\frac{1}{b+1}}$ parametrised by the local time.



— Since Q is a reflected Brownian motion parametrised by its local time at the origin, its maxima form a point Poisson process with intensity $\hat{J}\, dt\, dq/q^2$

— But **more information** on the internal structure of the jumps (anomalous operators).

Conclusion / Remarks :

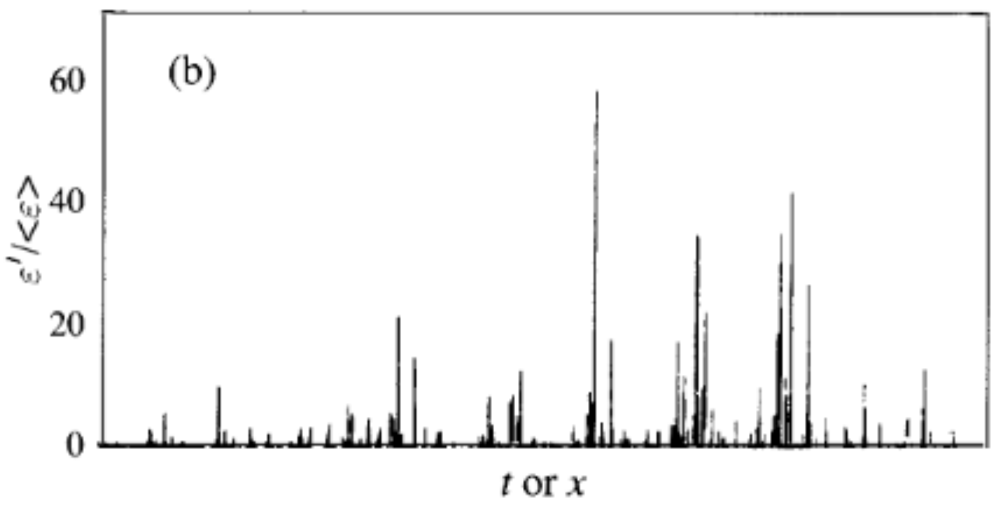
— **Strong noise limit of SDES** jump processes (between unstable states) and spiky processes (as aborted jumps).

— **Generic (enough…) phenomena** but different from weak noise (Kramer's like) behaviour.

— Deciphering the fine/internal structures of stochastic jumps by **parametrising the process with its quadratic variation** instead of its linear natural « clock » time.

— …. **Any application in turbulence ??….** →

… **Dissipation in turbulence:**



Questions: **i)** Is there any (relevant) scaling law in the height distribution?

ii) What is the statistics of the dissipation profil when parametrised by the quadratic variation?

(Naive) Guess?:

— Let P(e) be the integrated P.D.F. of the dissipation spikes;

— Let Q(e) = const./P(e)

— Then Q_v = Q(e_x) parametrized by the dissipation quadratic variation v is (« closed to ») a reflected Brownian motion…

THANK YOU!!

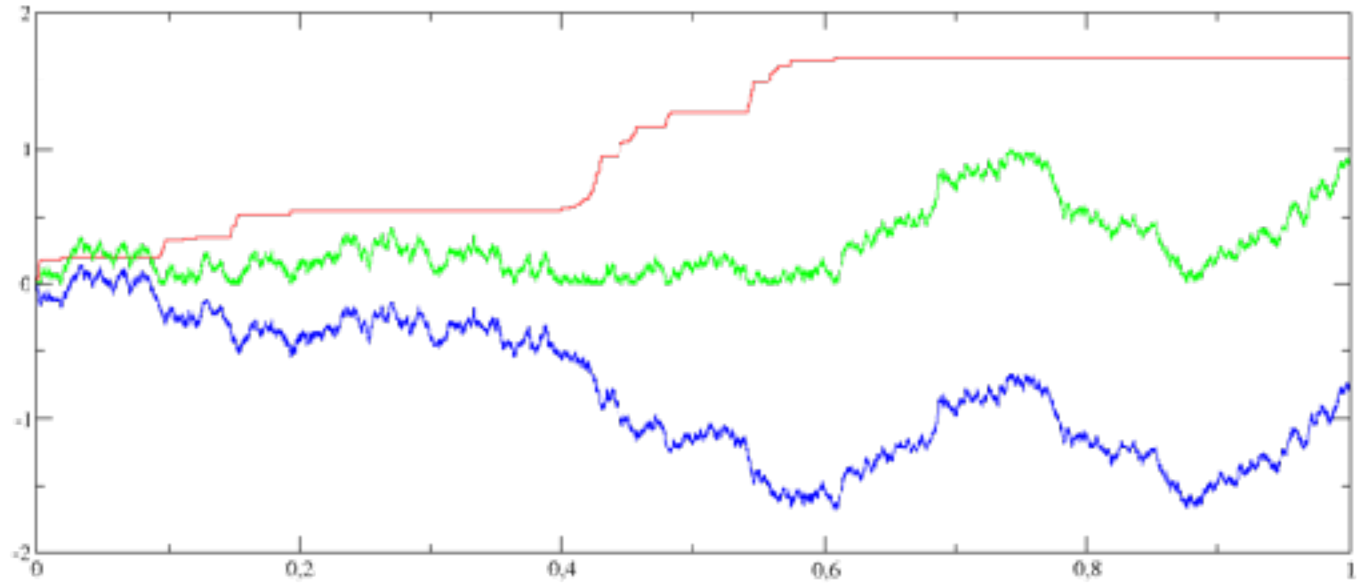
Skorokhod's decomposition:

$$F(x) = A(x) - L(x)$$

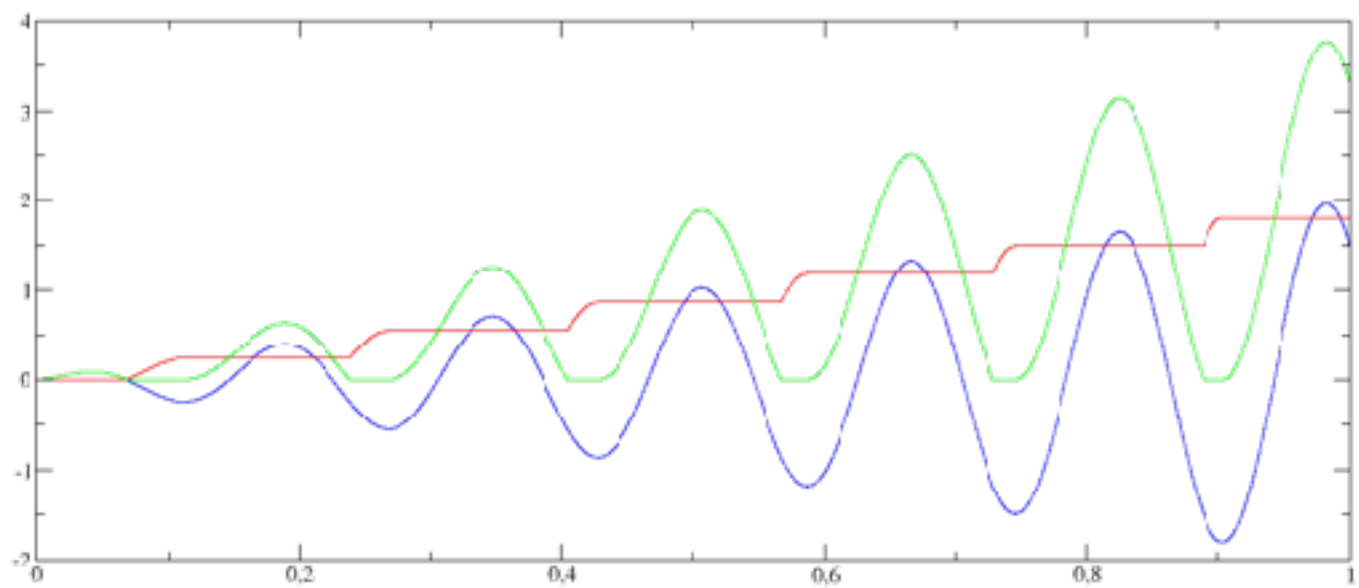
A(x) positive

L(x) >0 & increases only if A=0.

Brownian motion



oscillatory function



Spikes and a Poisson Process.

Consider a point Poisson process in $\mathbb{R} \times \mathbb{R}^+$ with intensity $d\nu = dt\, d\hat{\nu}(x)$. The points of this process are associated to the tips of the spikes as usual. Let $\mathcal{N}_{[t_0,t]\times I}$, with I a Borel set in $\mathbb{R}^+$, be the number of point of the process in $[t_0, t] \times I$. There are two contributions to the distribution of the transition times from $y \to z$:
(i) if the spike going above y at initial time $t = 0$ is also going above z;
(ii) if the spike going above y at initial time $t = 0$ stops before reaching z but, the first a later spike going above z is at time t, up to dt.
The probability of the first event is

$$\lim_{\delta \to 0} \mathbb{P}[\mathcal{N}_{[0,\delta]\times[z,\infty]} = 1 | \mathcal{N}_{[0,\delta]\times[y,\infty]} = 1] = \frac{\hat{\nu}([z,\infty])}{\hat{\nu}([y,\infty])}$$

The probability of the second event is

$$\lim_{\delta \to 0} \mathbb{P}[\mathcal{N}_{[0,\delta]\times[y,z]} = 1, \mathcal{N}_{[0,\delta]\times[z,\infty]} = 0, \mathcal{N}_{[t,t+dt]\times[z,\infty]} = 1 | \mathcal{N}_{[0,\delta]\times[y,\infty]} = 1]$$
$$= \frac{\hat{\nu}([y,z])\, \hat{\nu}([z,\infty])}{\hat{\nu}([y,\infty])} e^{-\hat{\nu}([z,\infty])t}\, dt$$

As a consequence, the probability that the transition time $T_{y\to z}$ be t up to dt is

$$\mathbb{P}[T_{y\to z} \in [t, t+dt]] = \frac{\hat{\nu}([z,\infty])}{\hat{\nu}([y,\infty])} (\delta(t)dt + \hat{\nu}([y,z]) e^{-\hat{\nu}([z,\infty])t} dt)$$

Remark that this distribution is correctly normalized, thanks to $\hat{\nu}([y,\infty]) = \hat{\nu}([y,z]) + \hat{\nu}([z,\infty])$. The corresponding generating function $\mathbb{E}[e^{-\sigma T_{y\to z}}]$ is then computable by inetgratino. We get:

$$\mathbb{E}[e^{-\sigma T_{y\to z}}] = \frac{\hat{\nu}([z,\infty])}{\hat{\nu}([y,z])} \frac{\hat{\nu}([y,z]) + \sigma}{\hat{\nu}([z,\infty]) + \sigma} = \frac{1 + \sigma\, \hat{\nu}([y,z])^{-1}}{1 + \sigma\, \hat{\nu}([z,\infty])^{-1}}$$

What is the strong limit of weak measurement?

$$d\rho = (d\rho)_{sys} + (d\rho)_{meas},$$ - with

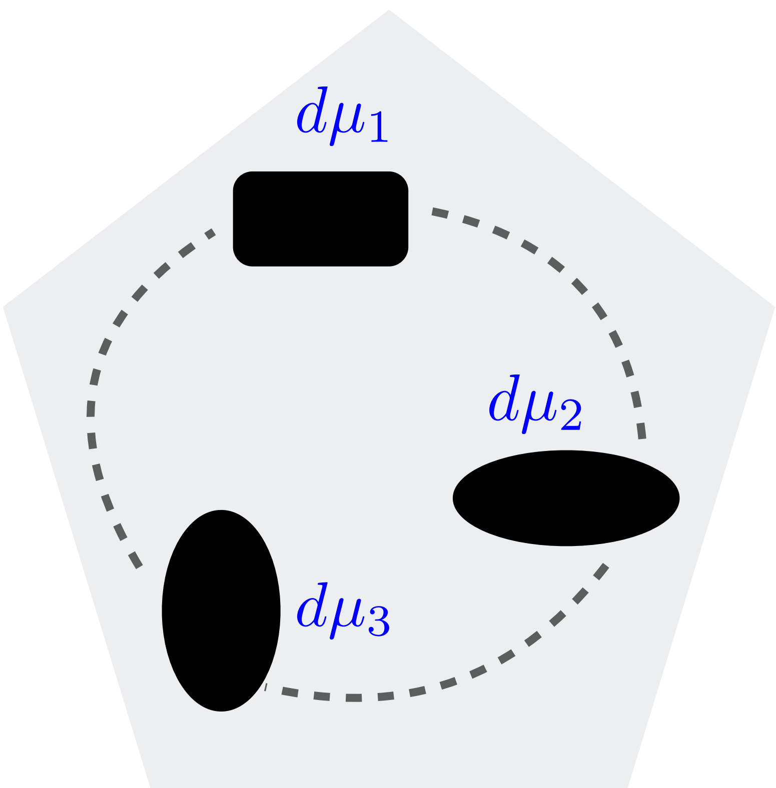
$$(d\rho)_{sys} = \big(-i[H,\rho] + L_{dissip}(\rho)\big)dt$$

$$(d\rho)_{meas} = \sigma^2 L_{meas}(\rho)dt + \sigma\, D_{meas}(\rho)dW_t$$

These are **stochastic differential equations on the space of density matrix**, i.e. for the coordinates 'X' of the quantum states. **Trajectories develop jumps at strong coupling.**

— **Claim**:

- **« At strong coupling these processes converge to (finite state) Markov chains. »**



Hint for a proof (strong noise limit):

— Convergence of the transition kernels:

$$P_t(X_0, dX) \simeq_{\sigma\to\infty} \sum_\alpha f_\alpha(X_0)\,\big[e^{-tM}\big]_{\alpha\beta}\, d\mu_\beta(X)$$

with **functions 'f' and measures 'mu' determined by the measurement process** and **the Markov matrix 'M'** by **the system evolution**.

Comparison with the weak noise limit.

— The **weak noise** limit of SDEs a la Kramer or a la Freidlin & Wentzell:

$$dX_t = -U'(X_t)\,dt + \epsilon\,dB_t$$

with, say, U(x) a double well potential.

— The typical passage time to go up-hill (i.e. U(z)>U(y)):

$$T_{y\to z} \simeq e^{-2(U(z)-U(y))/\epsilon^2}$$

— Scaling limit:
Choose x and fixed $T_{x_*\to x}$ finite.

Then, all other passage times are either 0 or infinite.