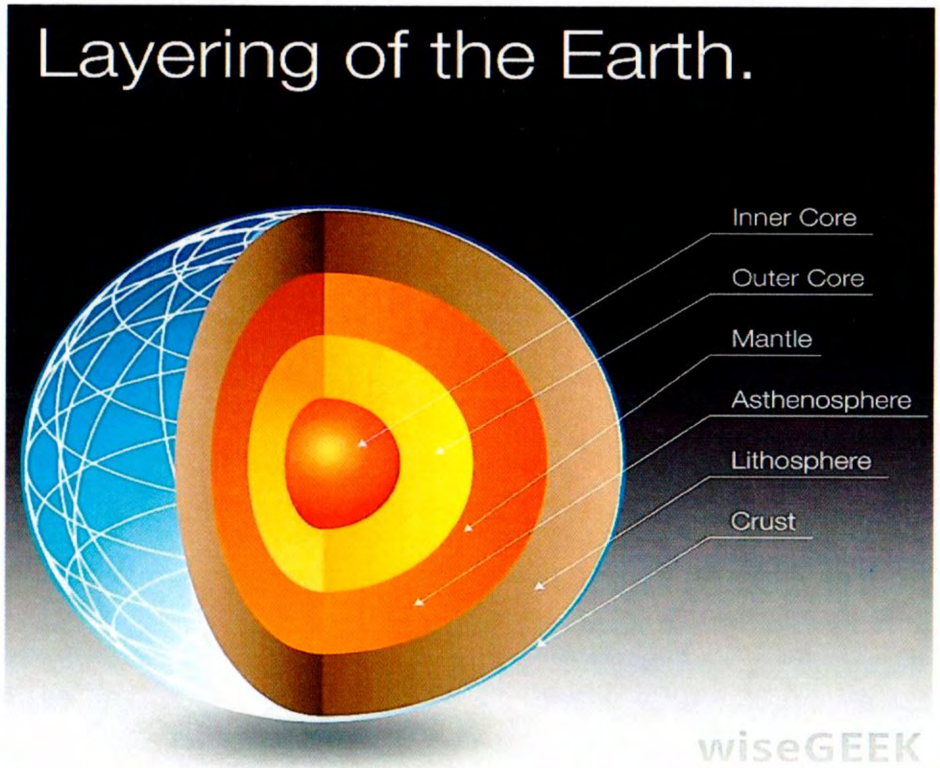
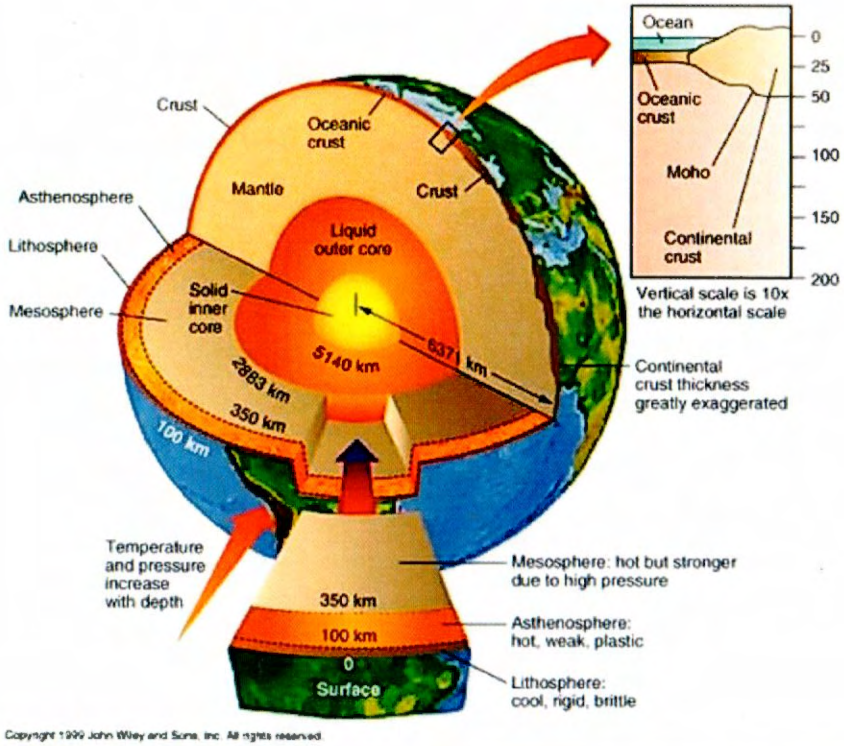
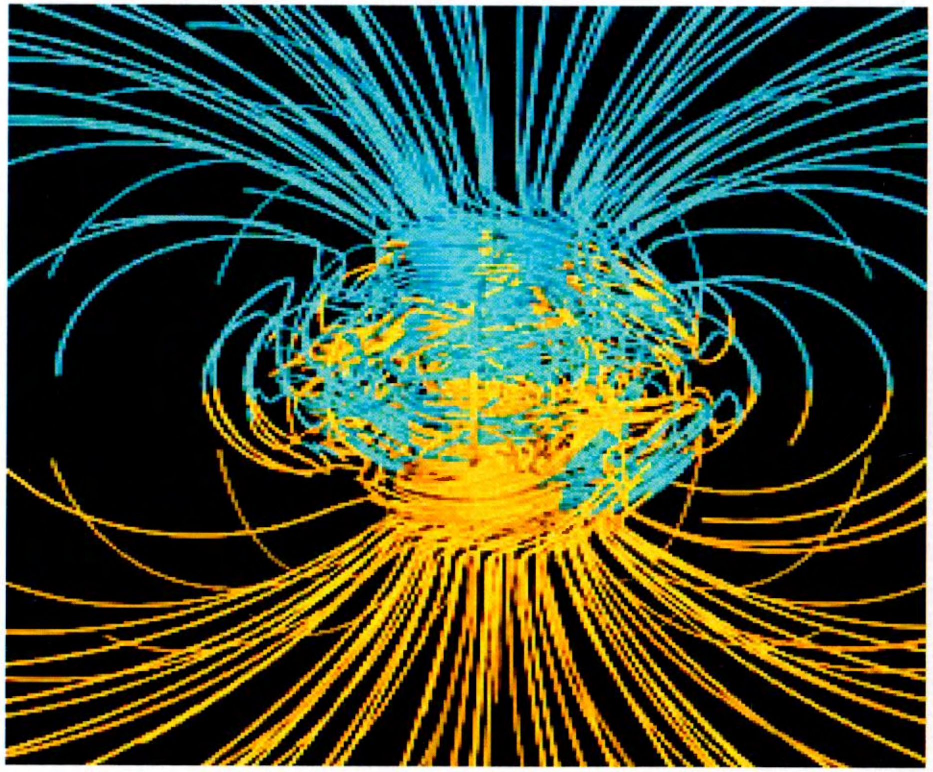
Asymptotics for Magnetostrophic turbulence in the Earth's fluid core

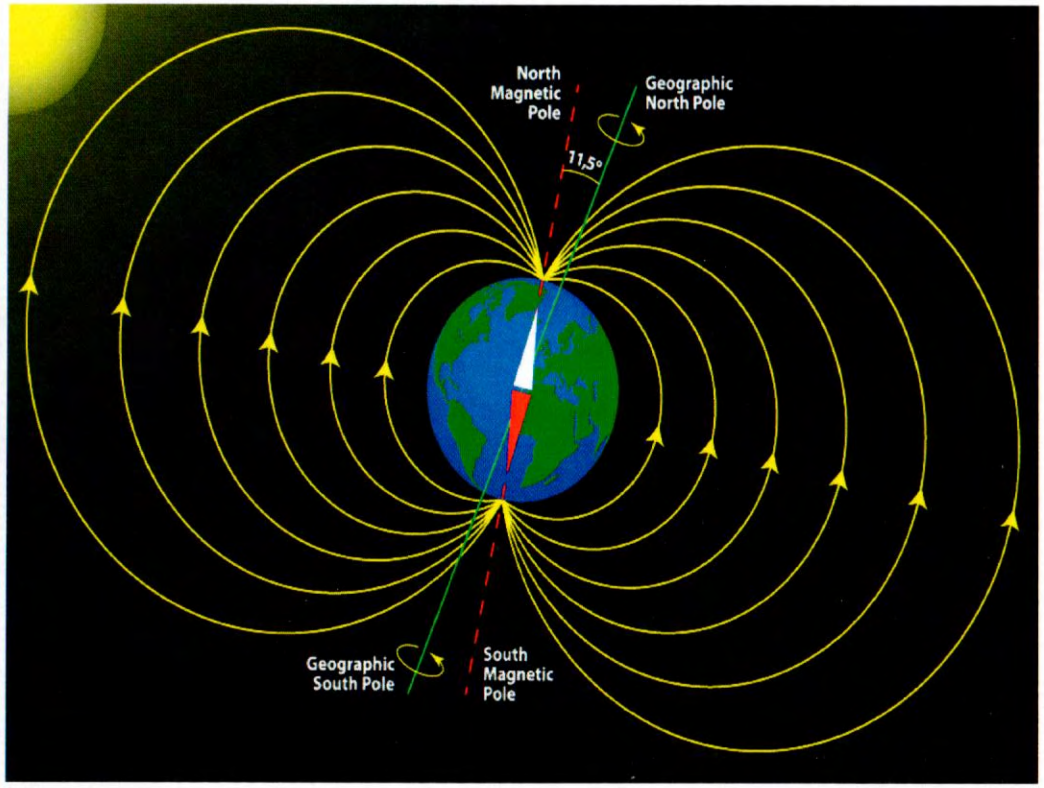
Susan Friedlander, USC

Vlad Vicol, Walter Rusin, Anthony Suen
Nathan Glatt-Holtz, Juraj Foldes, Geordie Richards


Layering of the Earth.
Inner Core
Outer Core
Mantle
Asthenosphere
Lithosphere
Crust
wiseGEEK


Ocean
Oceanic crust
Moho
Continental crust
0
25
50
100
150
200
Kilometers
Vertical scale is 10x the horizontal scale
Crust
Oceanic crust
Mantle
Crust
Asthenosphere
Lithosphere
Mesosphere
Liquid outer core
Solid inner core
5140 km
6371 km
2883 km
350 km
100 km
Continental crust thickness greatly exaggerated
Temperature and pressure increase with depth
Mesosphere: hot but stronger due to high pressure
350 km
100 km
0
Surface
Asthenosphere: hot, weak, plastic
Lithosphere: cool, rigid, brittle
Copyright 1999 John Wiley and Sons, Inc. All rights reserved.




North Magnetic Pole
Geographic North Pole
11,5°
Geographic South Pole
South Magnetic Pole

MHD system: Coriolis, Lorentz, Gravity

$$N^2[R_0(\partial_t u + u\cdot\nabla u) + \hat{e}_3 \times u]$$
$$= -\nabla P + \hat{e}_2\cdot\nabla b + R_m\, b\cdot\nabla b + N^2\theta\hat{e}_3 + \nu\Delta u$$
$$R_m[\partial_t b + u\cdot\nabla b - b\cdot\nabla u] = \hat{e}_2\cdot\nabla u + \Delta b$$
$$\partial_t\theta + u\cdot\nabla\theta = \kappa\Delta\theta + S$$
$$\nabla\cdot u = 0 \qquad \nabla\cdot b = 0$$

u velocity, b magnetic field, θ temperature
P magnetic + fluid pressure

dimensionless parameters

N^2,	R_0,	R_m,	ν	κ
$O(1)$	$O(10^{-3})$	$O(10^{-2})$	very small	very small

Moffatt – Loper Model: leading order.

$$N^2 \hat{e}_3 \times u = -\nabla p + \hat{e}_2 \cdot \nabla b + N^2 \theta \hat{e}_3 + \nu \Delta u$$

$$0 = \hat{e}_2 \cdot \nabla u + \Delta b$$

$$\partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta + S$$

$$\nabla \cdot u = 0 , \quad \nabla \cdot b = 0$$

$$\left\{ \left[\nu \Delta^2 - (\hat{e}_2 \cdot \nabla)^2 \right]^2 + N^4 (\hat{e}_3 \cdot \nabla)^2 \Delta \right\} u$$

$$= -N^2 \left[\nu \Delta^2 - (\hat{e}_2 \cdot \nabla)^2 \right] \nabla \times (\hat{e}_3 \times \nabla \theta)$$

$$+ N^4 (\hat{e}_3 \cdot \nabla) \Delta (e_3 \times \nabla \theta)$$

Domain: $\mathbb{T}^3 \times (0, \infty)$; θ has zero ~~vertical~~ mean

Magnetogeostrophic Equation (MG)

$$\partial_t \theta + (u\cdot\nabla)\theta = \kappa\Delta\theta + S$$

$$u = M[\theta]$$

Components of Fourier multiplier symbol $\hat{M}(k)$

$$\hat{M}_1 = \left[N^4 k_2 k_3 |k|^2 - N^2 k_1 k_3 (k_2^2 + \nu |k|^4)\right] / D$$

$$\hat{M}_2 = \left[-N^4 k_1 k_3 |k|^2 - N^2 k_2 k_3 (k_2^2 + \nu |k|^4)\right] / D$$

$$\hat{M}_3 = \left[N^2 (k_1^2 + k_2^2)(k_2^2 + \nu |k|^4)\right] / D$$

where $D = N^4 |k|^2 k_3^2 + (\nu |k|^4 + k_2^2)^2$

recall $k_3 \neq 0$

Observations about the symbol

1) anisotropic, even in wave number k

2) $\underline{\nu > 0}$; $\hat{M}(k) \sim 1/k^2$ as $k \to \infty$

3) $\underline{\nu = 0}$:

On the curved regions in Fourier space $k_3 = O(1)$, $k_1 \sim k_2^2$ and $|k_1| \to \infty$

$$\hat{M}(k) \sim C k_1$$

i.e. when $\nu > 0$ M is smoothing of degree 2

when $\nu = 0$ M is singular of degree -1.

Hierarchy of active scalar equations

$$\partial_t \theta + u \cdot \nabla \theta = (\kappa \Delta^\alpha \theta)$$

1. Inviscid ($\nu = 0$) MG : singular order 1

   $\kappa = 0$, Hadamard ill-posed

   $\kappa > 0$, critical case, globally well posed

2. SQG : singular order zero

   $\kappa = 0$, open

   $\kappa > 0$, critical case, globally well posed

3. 2D Euler in vorticity form: smoothing degree 1

   well posed.

4. viscous MG: smoothing degree 2

   "better" than 2D Euler

Inviscid, nondiffusive MG_0 ($\nu = 0, \kappa = 0$)

Friedlander - Vicol (2011)

1) Instability of MG_0 linearised about a particular steady state
Use continued fractions to construct an eigenvalue with arbitrarily large real part

2. Ill-posedness in Sobolev spaces for the full nonlinear problem follows by showing the solution map is not Lipschitz with respect to initial data.

Result requires the derivative loss and the fact that $\hat{M}$ is even.

* Inviscid critical MG. ($\nu=0$, $\kappa>0$, $\alpha=1$)

Friedlander & Vicol (2011)

1) Linear parabolic PDE

$$\partial_t \theta + v \cdot \nabla \theta = \kappa \Delta \theta, \quad \nabla \cdot v = 0$$

$$v \in L^2_t L^2_x \cap L^\infty_t BMO^{-1}_x$$

then weak solutions are Hölder continuous

Proof uses De Giorgi iteration.

2) Use this result to prove that Leray-Hopf weak solutions to an active scalar nonlinear PDE of the type * are classical solutions.

Viscous, nondiffusive MG_r ($r>0$, $\kappa=0$)

Friedlander & Suen (2015)

Th 1. Let $\theta_0 \in L^3$. There exists a unique global weak solution to
$$\theta_t + u\cdot\nabla\theta = 0 \;, \quad u = M_r[\theta]$$
such that $\theta \in BC((0,\infty); L^3)$, $u \in C((0,\infty); W^{3,3})$.
In particular, $\theta(\cdot,t) \to \theta_0$ weakly as $t \to 0^+$.

Th 2. Let $\theta_0 \in W^{s,3}$, $s>0$. There exists a unique solution $\theta(\cdot,t) \in W^{s,3}$ for all $t \geq 0$.
In particular, for $s=1$
$$\|\nabla\theta\|_{L^3} \leq C_1 \|\nabla\theta_0\|_{L^3} \exp\left(t\, C_2 \|\theta_0\|_{W^{1,3}}\right)$$

Note: L^3 is the critical Lebesgue space with respect to the natural scaling of the MG system in the sense that if $\theta(x,t)$ is a solution, then $\theta_\lambda(x,t) = \lambda^3 \theta(\lambda x, \lambda^2 t)$ is also a solution with corresponding drift velocity given by
$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t) = M[\theta_\lambda]$ for $\lambda > 0$.

The theorems require no smallness condition

Viscous, diffusive MG ($\nu > 0$, $\kappa > 0$)

Th 3. There exists a unique global in time mild solution to

$$\theta_t + u \cdot \nabla\theta = \kappa\Delta\theta, \quad u = M_\nu[\theta], \quad \theta_0 \in L^3$$

such that $\theta \in BC((0,\infty); L^3)$

$$t^{s/2 + 1/2 - 3/2p}\, \theta \in C((0,\infty); \dot{W}^{s,p}), \quad s \in [0,1), \ p \in (3,\infty).$$

In particular, $\theta(\cdot,t) \to \theta_0$ in L^3 as $t \to 0^+$ and $\|\theta(\cdot,t)\|_{\dot{W}^{s,p}} \to 0$ as $t \to \infty$

The solution is instantaneously C^∞ smoothed out and in $\dot{W}^{s,p}$ for all $t > 0$.

Limit as $\kappa \to 0$

Th 4. Let θ_κ be the solution with $\theta_0 \in L^3$. Then there exisits a sequence $\{\kappa_n\}$ with $\lim_{n\to\infty} \kappa_n = 0$ such that

$$\theta_{\kappa}(\cdot, t) \to \theta(\cdot, t) \text{ weakly in } L^3 \text{ as } n\to\infty$$

for all $t \geq 0$ where θ is the solution to the non diffusive MG_r equation.

If also $\nabla\theta_0 \in L^2$, then for any $T > 0$

$$\lim_{\kappa\to 0} \kappa \int_0^T \int |\nabla \theta_\kappa|^2 \, dx\, ds = 0$$

Stochastically Forced MHD System

$\nu > 0, \quad \kappa > 0, \quad \lim R_o \to 0, \quad \lim R_m \to 0$

$$R_o(\partial_t u + u\cdot\nabla u) + \hat{e}_3 \times u$$
$$= -\nabla p + \hat{e}_2\cdot\nabla B + R_m B\cdot\nabla B + \theta\hat{e}_3 + \nu\Delta u$$

$$R_m(\partial_t B + u\cdot\nabla B - B\cdot\nabla u] = \hat{e}_2\cdot\nabla u + \Delta B$$

$$d\theta + u\cdot\nabla\theta = \kappa\Delta\theta + \sigma\, dW$$

White in time, spatially correlated, Gaussian noise $\sigma\, dW$:

$$\sigma\, dW = \sum_{k\in\mathbb{Z}^3} \alpha_k \sigma_k\, dW^k$$

σ_k are $\sin k\cdot x$ and $\cos k\cdot x$

W^k independent 1-D Brownian motions

$\alpha_k \in \mathbb{R}$ are the amplitudes

Fundamental postulates of turbulence

Consider energy cascading from large to small spatial scales.

Our system is driven by "spectrally degenerate" stochastic forcing, i.e. noise acts only through a narrow range of low frequencies

Hypo-elliptic situation substantially more difficult than forcing on all spatial scales.

Martingale Solutions for the Full System

(i) Given any initial probability distribution μ there exists a stochastic process (u, B, θ) which is weakly continuous and solves the full evolution system.

(ii) For every $R_0, R_m > 0$ there exists a stationary Martingale solution that satisfies the uniform moment bound

$$\sup_{R_0, R_m \in (0,N]} \mathbb{E} \exp\left(\eta\left(R_0\|u\|^2 + R_m\|B\|^2 + \|\theta\|^2\right)\right) \le C_N < \infty$$

Proved using a Galerkin regularization similar to that used for 3D Navier Stokes

The MG$_r$ equation is the limit system obtained with $\nu > 0$, $\kappa > 0$, $R_o = 0$, $R_m = 0$

Very singular limit

Full system supports initial conditions on all variables u, B and θ

The limit system (MG$_r$ active scalar) allows initial conditions only on θ.

Multi-time scale analysis with three time scales:

$O(1)$, $O(R_o^{-1})$, $O(R_m^{-1})$

Limit System:
MG_r active scalar PDE, $r>0$, $\kappa>0$.

$$\partial_t \theta + (u\cdot\nabla)\theta = \kappa\Delta\theta + \sigma\, dW, \quad \theta|_{t=0} = \theta_0$$

$$u = M_u[\theta] \qquad b = M_b[\theta]$$

Components of Fourier multiplier symbol $\hat{M}_u$

$$\hat{M}_{u_1} = [k_2 k_3 |k|^2 - k_1 k_3 (k_2^2 + r|k|^4)]/D$$

$$\hat{M}_{u_2} = [-k_1 k_3 |k|^2 - k_2 k_3 (k_2^2 + r|k|^4)]/D$$

$$\hat{M}_{u_3} = [(k_1^2 + k_2^2)(k_2^2 + r|k|^4)]/D$$

where $D = |k|^2 k_3^2 + (k_2^2 + r|k|^4)^2$

$$\hat{M}_b = ik_2 \hat{M}_u / |k|^2$$

Results for the limit system.

Stochastically forced MG_r equation, $\kappa > 0$, $r > 0$.

Well-posedness:

The PDE possesses unique, pathwise solutions which satisfy exponential moment bounds.

Extension of known results in the deterministic case

Markovian Dynamics of the limit system.

Use the framework of Hairer-Mattingly

- Show that a form of the Hormander bracket condition is satisfied
- verify a form of asymptotic strong Feller
- an irreducibility condition
- certain exponential moment bounds

We infer that the contractivity property of H-M is satisfied in a suitably chosen Wasserstein metric $\mathcal{W}$.

<u>Th</u>: Let $\{P_t\}_{t\geq 0}$ be the Markov semigroup associated to the MG_r equation

Then $\{P_t\}_{t\geq 0}$ is contractive in $\mathcal{W}$:

$$\mathcal{W}(\mu_1 P_t, \mu_2 P_t) \leq C e^{-\gamma t} \mathcal{W}(\mu_1, \mu_2), \quad t \geq 0$$

It then follows that $\{P_t\}_{t>0}$ possesses a unique ergodic invariant Measure μ.

Furthermore μ satisfies attraction properties:

- it is exponentially mixing
- obeys a strong law of large numbers
- obeys a central limit system

Finite time convergence as $R_o, R_m \to 0$.

We use the powerful observation of Hairer-Mattingly [08] that if one can establish a contraction property for the limit system the convergence of statistically steady states can be reduced to convergence of solutions of the full system on *finite* time scales

Key observation for our problem is to show that a difference in initial conditions on ~~θ~~, namely u and B has negligible ~~effect~~ on algebraic order in $R_o + R_m$.

Note: the convergence for the SSS do not require uniform convergence in u and B up to the initial time $t=0$.

Asymptotics for the full evolution system

Denote variables Θ, U, B for the full system
θ, u, b for the limit system

Results as $R_o \to 0$, $R_m \to 0$

(i) Assume $\|\Theta(0) - \theta(0)\| \to 0$

Then for any $t > 0$ $\exists\, \gamma > 0$ and $p > \gamma$ so that

- $\mathbb{E} \sup_{s \in [0,t]} \|\Theta(s) - \theta(s)\|^p \le C\left(\|\Theta(0) - \theta(0)\| + R_o + R_m\right)^\gamma \to 0$

  and furthermore

- $\mathbb{E} \int_0^t \|U(s), B(s) - M_{u,b}[\theta]\|^2 ds \le C\left(\|\Theta(0) - \theta(0)\| + R_o + R_m\right)^\gamma \to 0$

(ii) Convergence of statistically steady states (invariant measures)

- For every $R_o, R_m > 0$ the full system possesses at least one SSS μ_{R_o,R_m} which satisfies certain exponential moment bounds independent of R_o, R_m.
- Any collection μ_{R_o,R_m} converges to μ at an algebraic rate in a suitable metric $\mathcal{W}$.
- In particular for any sufficiently regular observable ψ

$$\left| \int \psi(u, B, \cancel{\phi}) \, d\mu_{R_o,R_m} - \int \psi(M_{u,b}(\theta), \theta) \, d\mu \right| \to 0$$

as $R_o, R_m \to 0$

Conclusion

Our analysis demonstrates that the unique invariant statistics for the limit equation (i.e. stochastically forced MG_r) approximate any reasonable invariant statistics of the full MHD equations describing dynamo action in the Earth's fluid core. Thus we give a rigorous foundation for the Moffatt and Loper model of Magnetostrophic turbulence.

Thank You!

F.-Vicol, Ann I.H.P. Non lineaire, 2011

F.-Vicol, Nonlinearity, 2011

F.-Suen, Nonlinearity, 2015

F.-Foldes, Glatt-Holtz, Richards

to appear SIMA, 2017