The Bulk-Edge Correspondence via FREDHOLM THEORY

JACOB SHAPIRO

JAN 17 2020, IPAM UCLA
Theory & Computation for 2D Materials

Based on joint work with:
- Graf.
- Fonseca, Sheta, Wang & Yamakawa

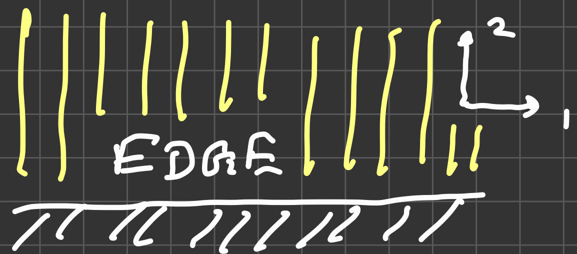
MAIN MESSAGE

H Bulk (∞ space) Hamiltonian

$\mathrm{sgn}(H)$ flat Hamiltonian

Bulk top. invariant

$\Lambda(H)$ truncated, edge, half-∞ Hamiltonian



Edge top. invar.

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Bulk-edge correspondence (in the spectral gap regime) amounts to the statement that

$$\boxed{\underbrace{\mathrm{sgn} \circ \Lambda}_{\text{EDGE}} \sim \underbrace{\Lambda \circ \mathrm{sgn}}_{\text{BULK}}}$$

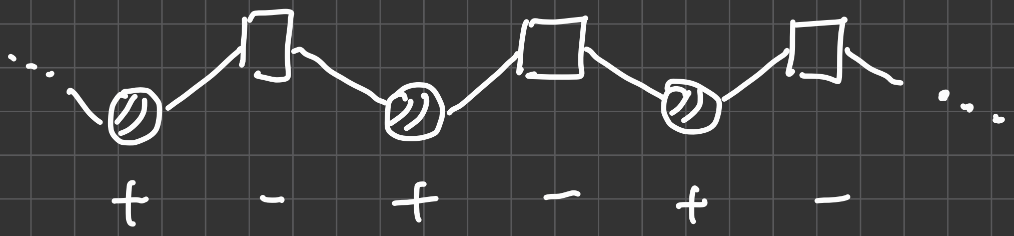
where $\sim$ is equivalence when computing indices.

TODAY

1. 1D CHIRAL EXAMPLE.
2. FREDHOLM BASICS.
3. 2D IQHE and $\mathbb{Z}_2$ EXAMPLES.

	Symmetry			Dimension							
AZ	T	C	S	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

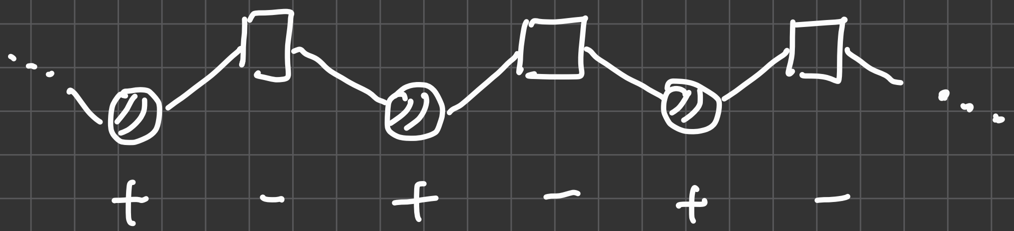
SIMPLEST EXAMPLE: CHIRAL 1D



1D chain w/ two species of sites

Hamiltonian is chiral iff it only has off-diagonal terms in the $+-$ (chirality) basis.

SIMPLEST EXAMPLE: CHIRAL 1D (SSH e.g.)



1D chain w/ two species of sites

Hamiltonian is chiral iff it only has off-diagonal terms in the +− (chirality) basis.

$$\Longrightarrow \quad H = \begin{bmatrix} 0 & S^* \\ S & 0 \end{bmatrix} \quad \exists \quad S : \underset{\substack{\uparrow \\ +\text{ sites}}}{\mathcal{H}_+} \longrightarrow \underset{\substack{\uparrow \\ -\text{ sites}}}{\mathcal{H}_-} .$$

S need not be self-adjoint. E.g.: S = shift operator

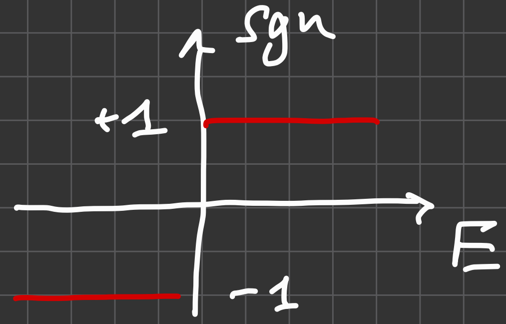
$$\Pi = \begin{bmatrix} \mathbb{1}_{\mathcal{H}_+} & 0 \\ 0 & -\mathbb{1}_{\mathcal{H}_-} \end{bmatrix} \qquad \{H, \Pi\} = 0 \,, \quad \Pi^2 = +\mathbb{1}.$$

$$\Downarrow$$

$\sigma(H)$ symmetric about 0.

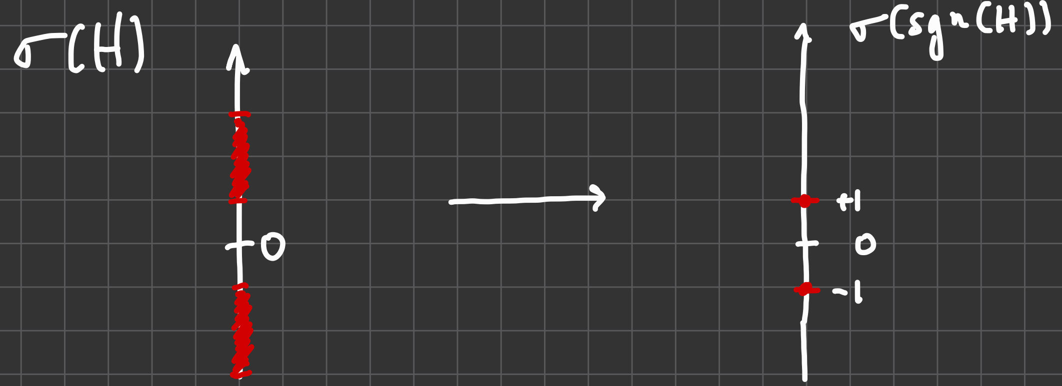
SIMPLEST EXAMPLE: CHIRAL 1D (cont.)

$\sigma(H)$ symmetric about 0. $\Longrightarrow$ $E_F := 0$.



Spectral gap for $H \iff 0 \notin \sigma(H)$.

$\Longrightarrow$ $\mathrm{sgn}(H) \equiv H|H|^{-1}$ makes sense



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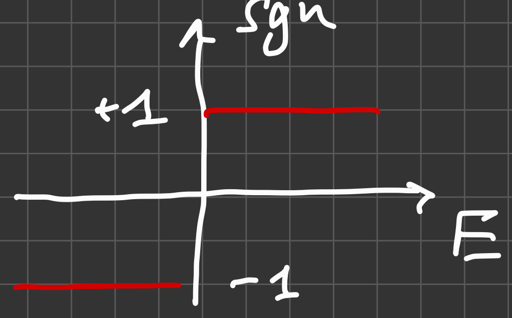

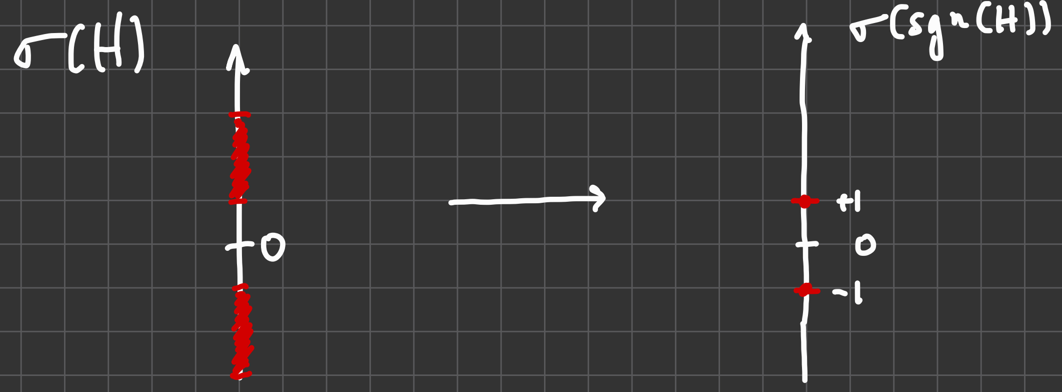

chiral polarization

Bulk index $\mathcal{N} := \frac{1}{2}\mathrm{tr}\left(\Pi \, \mathrm{sgn}(H)\, [\Lambda, \mathrm{sgn}(H)]\right)$

where Λ projects onto RHS of space:

$$(\dots, \psi_{-1}, \psi_0, \psi_1, \psi_2, \dots) \xmapsto{\Lambda} (\dots, 0, 0, \psi_1, \psi_2, \dots)$$

After some basic calculation,

$$\mathcal{N} := \frac{1}{2}\operatorname{tr}(\Pi \operatorname{sgn}(H)[\Lambda, \operatorname{sgn}(H)]) = \dots =$$

$$= \operatorname{tr}(U^*[\Lambda, U]) \qquad \text{where} \quad \operatorname{sgn}(H) = \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix}$$

i.e., U is the polar part in the polar decomp. of S: $\quad U \equiv S|S|^{-1}$.

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S invertible $\Rightarrow$ U is unitary.

$\Lambda = \Lambda^* = \Lambda^2$ is a proj.

$$\Rightarrow \mathcal{N} = \operatorname{tr}(U^*[\Lambda, U]) = \operatorname{tr}(U^*\Lambda U - \Lambda)$$

AVRON SEILER SIMON index of pair of proj.

$$= \dots = \operatorname{index}(\Lambda U \Lambda + \Lambda^\perp) \in \mathbb{Z} \qquad \boxed{\Lambda^\perp \equiv \mathbb{1} - \Lambda}$$

FEDOSOV

FREDHOLM INDEX

$$=: \operatorname{index}(\mathbb{\Lambda}(U)) \qquad \mathbb{\Lambda}(U) \equiv \Lambda U \Lambda + \Lambda^\perp.$$

Fredholm index ?

FREDHOLM THEORY

Operator A is FREDHOLM iff:

$\ker(A^*) \cong \mathrm{coker}(A)$
if $\mathrm{range}(A)$ closed.

1. $\dim \ker A < \infty$
2. $\dim \ker A^* < \infty$
3. $\mathrm{range}(A)$ is closed $\Leftrightarrow \exists\, \varepsilon > 0 : \|A\varphi\| \geq \varepsilon \|\varphi\|$ $\forall\, \varphi \in (\ker A)^{\perp}$.

(outside the kernel inverse bdd.)

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$$\in \mathbb{Z}$$

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Then we def.

$$\operatorname{index} A := \dim \ker A - \dim \ker A^* \in \mathbb{Z}$$

(degree of non-invertibility)

<u>Facts</u>:

① index stable under cpt. perturbations.

② index stable under norm-small perturbations.

FREDHOLM THEORY EXAMPLES

① $\mathbb{1}$ is fredholm w/ index 0.

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1. $\mathbb{1}$ is fredholm w/ index 0.
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3. Right shift is fredholm on:

$$\ell^2(\mathbb{N}) \quad \text{w/} \quad \text{index} = -1,$$
$$\ell^2(\mathbb{Z}) \quad \text{w/} \quad \text{index} = 0.$$

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$\ell^2(\mathbb{N})$ w/ index $= -1$,

$\ell^2(\mathbb{Z})$ w/ index $= 0$.

④ X^{-1} is <u>not</u> fredholm on $\ell^2(\mathbb{N})$ though it has finite kernel & and is self-adjoint.

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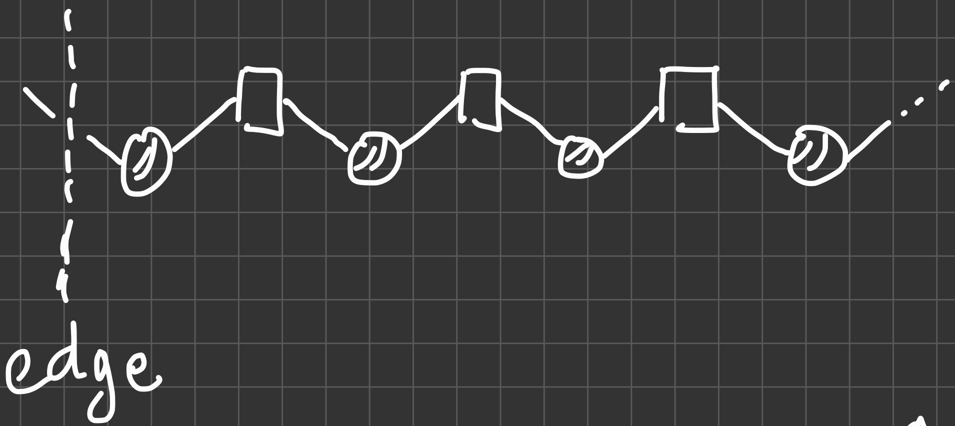
5. $\Lambda U \Lambda + \Lambda^\perp$ for chiral 1D bulk system <u>is</u> fredholm because: (a) H is local: $\|H_{xy}\| \leq C e^{-\mu\|x-y\|}$
   (b) H has gap.

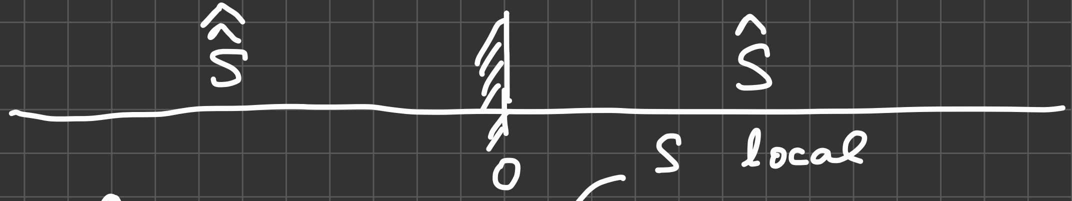
Back to CHIRAL 1D

For half-∞ geometry, we truncate $H \mapsto \hat{H}$.



(any B.C.)

Since $[\Pi, X] = 0$, $\hat{H} =: \begin{bmatrix} 0 & \hat{S}^* \\ \hat{S} & 0 \end{bmatrix}$ and $\hat{S}$ is the truncation of S. By locality of H, S is also local (and invertible, so Fredholm).



$$S = \hat{\hat{S}} \oplus \hat{S} + \text{cpt.} \implies \hat{S} \text{ also Fredholm.}$$

CHIRAL 1D EDGE INVARIANT

$\hat{S}$ is fredholm $\Rightarrow$ Def. $\hat{\mathcal{N}} = \mathrm{index}(\hat{S}) \in \mathbb{Z}$.

CHIRAL 1D BEC

$$\mathcal{N} \equiv \mathrm{index}(\Lambda U \Lambda + \Lambda^\perp) \overset{?}{=} \mathrm{index}(\hat{S}) \equiv \hat{\mathcal{N}}$$

$U \equiv S|S|^{-1} =: \mathrm{polar}(S)$

$\hat{S} \cong \Lambda S \Lambda + \Lambda^\perp =: \Lambda(S)$

<u>Fact</u>: $\mathrm{index}(A) = \mathrm{index}(VV^* - V^*V)$ where $V := \mathrm{polar}(A)$.

$\Longrightarrow \mathrm{index}(A) = \mathrm{index}(\mathrm{polar}(A))$.

So BEC reduces to the question

$$\boxed{\mathrm{index}(\Lambda(\mathrm{polar}(S))) = \mathrm{index}(\mathrm{polar}(\Lambda(S)))}\ ?$$

Do truncation and flattening commute on the index?

Answer: Yes! By homotopy: $S_t := tS + (1-t)\,\mathrm{polar}(S)$

Spectral gap and locality of S guarantee

$\Lambda(S_t)$ is fredholm $\forall\, t \in [0,1]$

$t \mapsto \mathrm{index}(\Lambda(S_t))$ is const. by cont. of index.

$\mathcal{N} = \mathrm{index}(\Lambda(S_0))$

$\hat{\mathcal{N}} = \mathrm{index}(\Lambda(S_1))$

$\Longrightarrow \boxed{\mathcal{N} = \hat{\mathcal{N}}}$. ✓

2D BULK SYSTEMS w/ or w/o TRI

H is gapped @ 0 energy (WLOG), $E_F := 0$,

$$P := \chi_{(-\infty,0)}(H) \qquad \text{Fermi projection}$$

Fermionic ground state.

Note: $P = \frac{1}{2}(\mathbb{1} - \mathrm{sgn}(H)) \longrightarrow P$ related to flattening.

Let $U := \exp(i \arg(X_1 + iX_2))$ (Laughlin's flux insertion).

Then $PUP + P^{\perp}$ is fredholm (by gap of H

$$\downarrow$$

$\|P_{xy}\| \leq C e^{-\mu\|x-y\|}$).

IQHE BULK

For the IQHE, the top. invar. is the Hall conductivity:

Kubo

Berry curvature integral

$$\mathcal{N} = 2\pi i \,\mathrm{tr}(P [[\Lambda_1, P], [\Lambda_2, P]]),$$

$(\Lambda_1, \Lambda_2$ proj. onto right / upper half space$)$;

trace class bcs. P is local.

Bellissard et al

After some work, $\mathcal{N} = \dots = \mathrm{index}(PUP + P^\perp)$

$\Rightarrow \mathcal{N} \in \mathbb{Z}.$

TIME REVERSAL

Operator $\Theta : \mathcal{H} \to \mathcal{H}$ s.t. : (1) anti-unitary, anti-linear:

$$\langle \Theta \psi, \Theta \varphi \rangle = \langle \varphi, \psi \rangle ,$$

(2) $\Theta^2 = -\mathbb{1}$,

(3) $[\Theta, X] = 0$.

TIME REVERSAL

Operator $\Theta : \mathcal{H} \to \mathcal{H}$ s.t.: ① anti-unitary, anti-linear:

$$\langle \Theta\psi, \Theta\varphi\rangle = \langle \varphi, \psi\rangle ,$$

② $\Theta^2 = -\mathbb{1}$, Fermions

③ $[\Theta, X] = 0$.

H TRI iff $[\Theta, H] = 0$.

$$\Downarrow$$

$[\Theta, P] = 0$ by functional calculus.

By anti-$\mathbb{C}$-linearity, $U = -\Theta U^* \Theta$, as $U \equiv e^{i\arg(X_1 + iX_2)}$.

$P\Theta = \Theta P \to P = -\Theta P \Theta$

So $\boxed{PUP + P^\perp = -\Theta (PUP + P^\perp)^* \Theta}$ for TRI H.

Fact: ① $\mathrm{index}(AB) = \mathrm{index}(A) + \mathrm{index}(B)$. (log rule)

② $\mathrm{index}(A^*) = -\mathrm{index}(A)$.

So for TRI H,

$$\mathcal{N} = \mathrm{index}(PUP+P^\perp) = \mathrm{index}(-\Theta(PU^*P+P^\perp)\Theta)$$

$$= \underbrace{\mathrm{index}(-\Theta)}_{\substack{=0 \\ \text{as } \Theta \text{ bijective}}} + \mathrm{index}((PUP+P^\perp)^*) + \underbrace{\mathrm{index}(\Theta)}_{=0}$$

$$= -\mathrm{index}(PUP+P^\perp) = -\mathcal{N} \Rightarrow \boxed{\mathcal{N}=0}.$$

For Fredholm operators A which obey

$$\boxed{A = -\Theta A^* \Theta} \qquad (\Theta\text{-ODD FREDHOLM OP.})$$

$\mathrm{index}(A) = 0$ so define instead:

$$\mathrm{index}_2(A) := \dim\ker A \mod 2 \in \mathbb{Z}_2.$$

H - Odd Fredholm theory ?

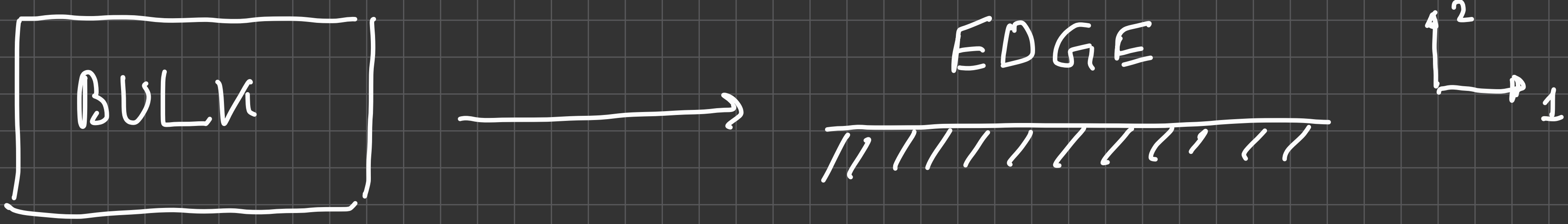
$TRI^2 = -\mathbb{1}$ 2D BULK SYSTEMS

Since $\mathcal{N} = \text{index}(PUP + P^\perp) = 0$ always,

define $\mathcal{N}_2 := \text{index}_2\,(PUP + P^\perp) \in \mathbb{Z}_2$.

This is the FU-KANE-MELE Pfaffian index for TRI vector bundles when $\exists$ translation invariance.

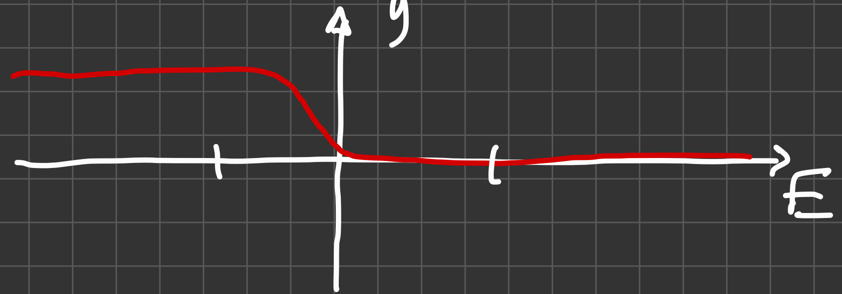
2D EDGE PICTURE



In the IQHE, invariant is edge Hall conductivity.

$$\hat{\sigma} = i\, \mathrm{tr}\big(g'(\hat{H}) \underbrace{[\hat{H}, \Lambda_1]}_{\text{velocity in 1-direction}} \big)$$

where g is a smooth version of $\chi_{(-\infty,0)}$:



$\mathrm{supp}(g')$ within bulk gap $\Rightarrow$ $g'(H)$ "projects" onto edge states.

Thm. (Kellendonk-Richter-Schulz-Baldes)

$$\hat{N} = \operatorname{index}\left(\Lambda_1 \exp(-2\pi i\, g(\hat{H}))\Lambda_1 + \Lambda_1^{\perp}\right) \in \mathbb{Z}.$$

As before, $[\hat{H}, \Theta] = 0$ for TRI, so $[g(\hat{H}), \Theta] = 0$.

$$\Rightarrow \hat{N} = \operatorname{index}\left(\Lambda_1 \exp(-2\pi i\, g(\hat{H}))\Lambda_1 + \Lambda_1^{\perp}\right) = 0.$$

Instead, for TRI edge systems take

$$\hat{N}_2 := \operatorname{index}_2\left(\Lambda_1 \exp(-2\pi i\, g(\hat{H}))\Lambda_1 + \Lambda_1^{\perp}\right).$$

2D BEC (TRI or not)

(Can add subscript 2 everywhere]

$$\mathcal{N} \equiv \underbrace{\text{index}(PUP + P^{\perp})}_{\text{index}(\Lambda_1 e^{-2\pi i P\Lambda_2 P}\Lambda_1 + \Lambda_1^{\perp})} \overset{?}{=} \text{index}(\Lambda_1 e^{-2\pi i g(\hat{H})}\Lambda_1 + \Lambda_1^{\perp}) \equiv \hat{\mathcal{N}}$$

(Kitaev)

- By homotopy, $P\Lambda_2 P \longrightarrow \Lambda_2 P \Lambda_2$, truncation of P.
- $P \longrightarrow g(H)$ in spec. gap regime.
- $\hat{H} \approx \Lambda_2 H \Lambda_2$

So,

$$\text{index}(\Lambda_1 e^{-2\pi i \Lambda_2 g(H)\Lambda_2}\Lambda_1 + \Lambda_1^{\perp}) \overset{?}{=} \text{index}(\Lambda_1 e^{-2\pi i \Lambda_2 g(\Lambda_2 H \Lambda_2)\Lambda_2}\Lambda_1 + \Lambda_1^{\perp})$$

Yes, by $A_t := t\Lambda_2 g(H)\Lambda_2 + (1-t)\Lambda_2 g(\Lambda_2 H\Lambda_2)\Lambda_2 \quad t \in [0,1]$.

MAIN INGREDIENT FOR THE 2D HOMOTOPIES

Fact: $\exp(2\pi i Q) = \mathbb{1}$ for Q projection

Lemma: If $A^2 - A$ decays into the bulk,
$\exp(2\pi i A) - \mathbb{1}$ decays into the bulk.

(decay into the bulk means:
$\| B_{xy} \|$ decays in x_2, y_2.)