Synthesis and analysis type methods for signal reconstruction from random observations

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Structure and Randomness in System Identification and Learning

IPAM, UCLA

Outline

- Introduction
  - Applications
  - Mathematical Formulation & Methods
- Extensions to other dictionaries
  - Analysis methods
  - Signal space methods
  - Total variation methods

The mathematical problem (notation)

1. Signal of interest $f \in \mathbb{C}^d (= \mathbb{C}^{N \times N})$

2. Measurement operator $\mathscr{A} : \mathbb{C}^d \to \mathbb{C}^m$.

3. Measurements $y = \mathscr{A} f + \xi$.

$$\begin{bmatrix} \\ y \\ \\ \end{bmatrix} = \begin{bmatrix} & & \\ & \mathscr{A} & \\ & & \end{bmatrix} \begin{bmatrix} \\ f \\ \\ \\ \\ \end{bmatrix} + \begin{bmatrix} \\ \xi \\ \\ \end{bmatrix}$$

4. *Problem:* Reconstruct signal f from measurements y

Sparsity

Measurements $y = \mathscr{A} f + \xi$.

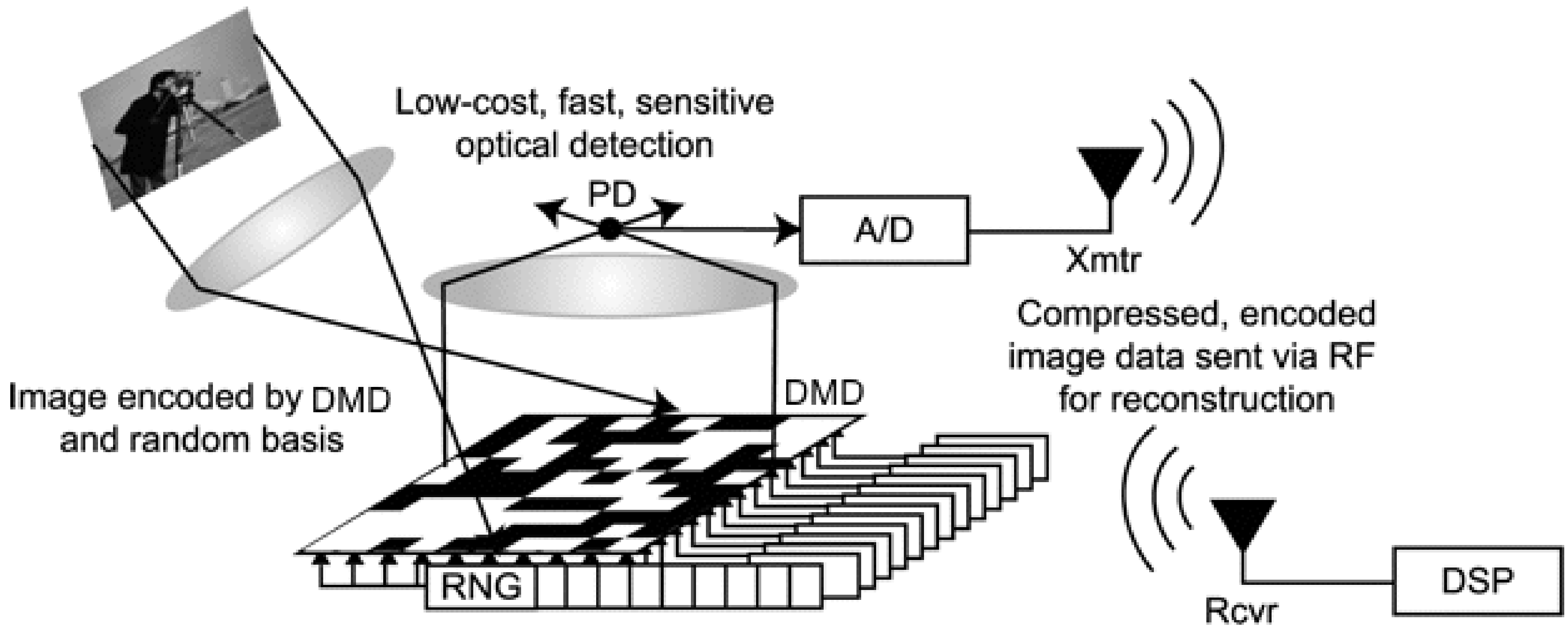
$$\begin{bmatrix} \\ y \\ \\ \end{bmatrix} = \begin{bmatrix} & & \\ & \mathscr{A} & \\ & & \end{bmatrix} \begin{bmatrix} f \\ \\ \\ \\ \end{bmatrix} + \begin{bmatrix} \\ \xi \\ \\ \end{bmatrix}$$

Assume f is *sparse*:

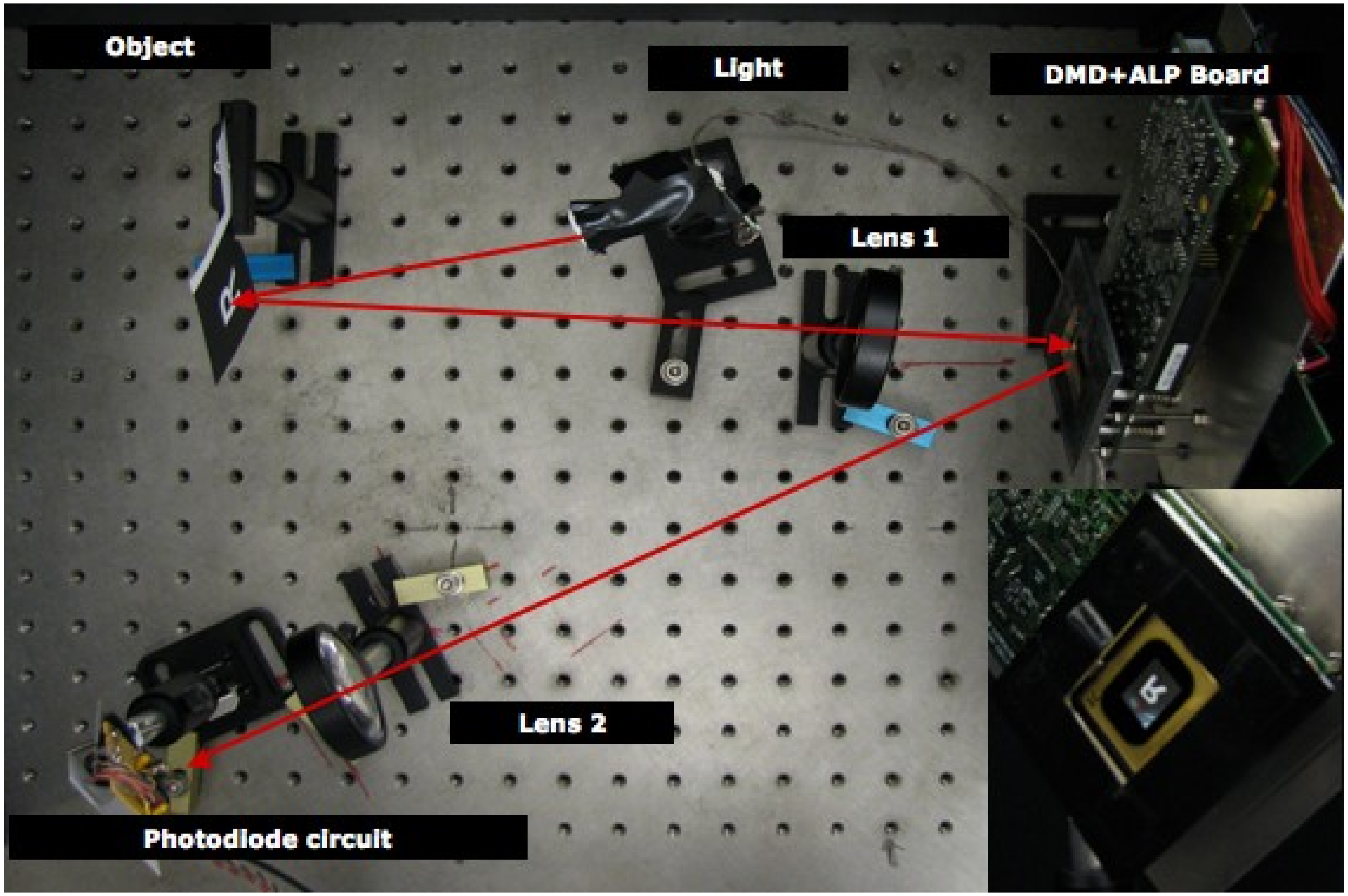
- In the coordinate basis: $\|f\|_0 \stackrel{\text{def}}{=} |\operatorname{supp}(f)| \leq s \ll d$
- In orthonormal basis: $f = Bx$ where $\|x\|_0 \leq s \ll d$
- In other dictionary: $f = Dx$ where $\|x\|_0 \leq s \ll d$

In practice, we encounter *compressible* signals.

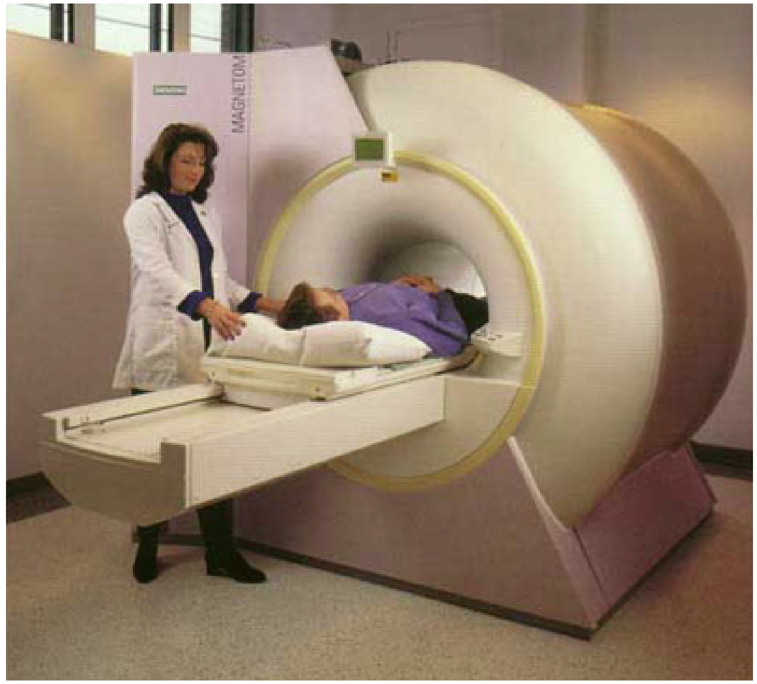
Digital Cameras



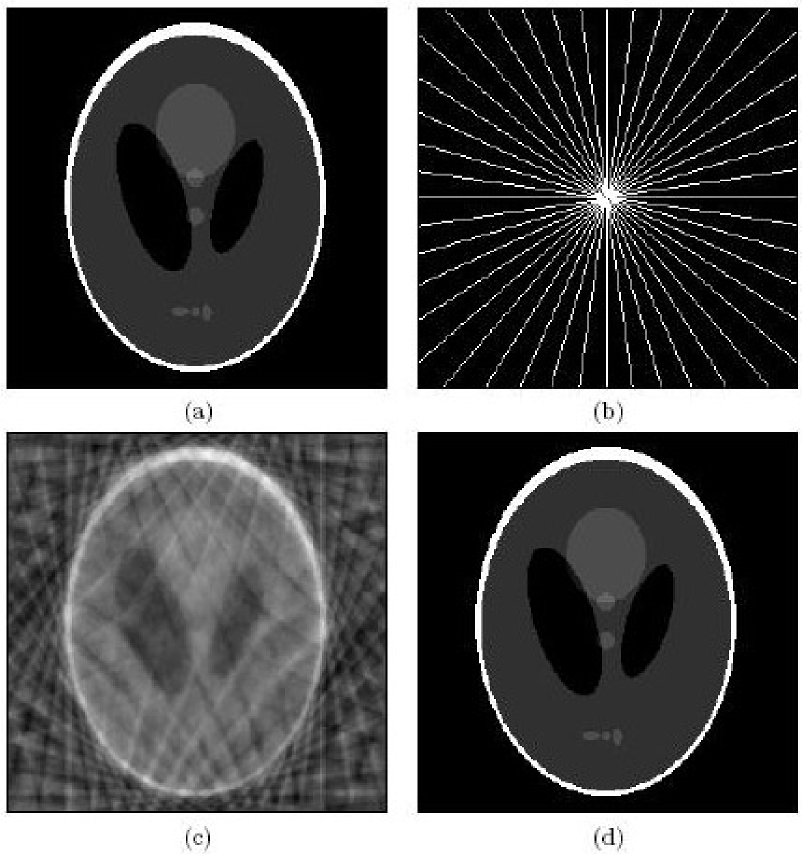
Digital Cameras



MRI

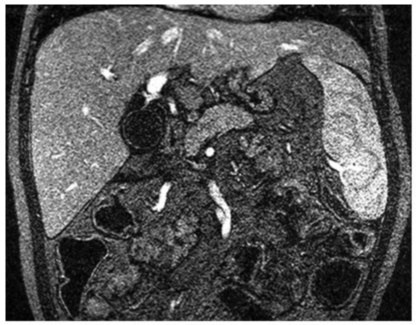


MRI

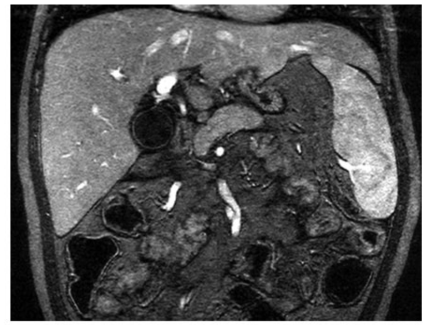


(Candès et.al.)

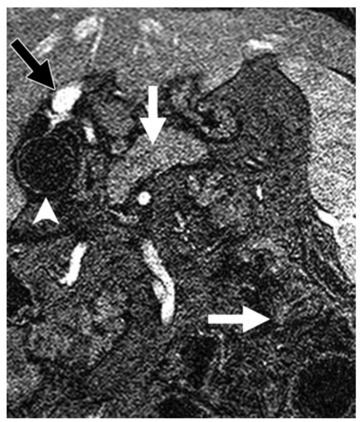
Pediatric MRI



(a)



(b)



(c)

(d)

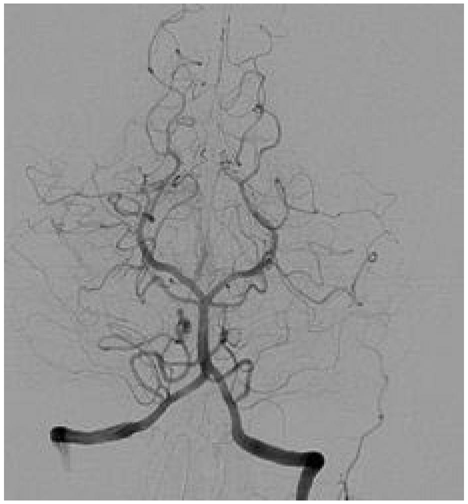
(Vasanawala et.al.)

Many more...

- Radar, Error Correction
- Computational Biology, Geophysical Data Analysis
- Data Mining, classification
- Neuroscience
- Imaging
- Sparse channel estimation, sparse initial state estimation
- Topology identification of interconnected systems
- ...

Sparsity...

Sparsity in coordinate basis: f=x



Reconstructing the signal f from measurements y

- ℓ_1-minimization [Candès-Romberg-Tao]

Let A satisfy the *Restricted Isometry Property* and set:

$$\hat{f} = \underset{g}{\operatorname{argmin}} \|g\|_1 \quad \text{such that} \quad \|\mathscr{A} f - y\|_2 \leq \varepsilon,$$

where $\|\xi\|_2 \leq \varepsilon$. Then we can stably recover the signal f:

$$\|f - \hat{f}\|_2 \lesssim \varepsilon + \frac{\|x - x_s\|_1}{\sqrt{s}}.$$

This error bound is optimal.

Restricted Isometry Property

- $\mathscr{A}$ satisfies the Restricted Isometry Property (RIP) when there is $\delta < c$ such that

$$(1-\delta)\|f\|_2 \leq \|\mathscr{A} f\|_2 \leq (1+\delta)\|f\|_2 \quad \text{whenever } \|f\|_0 \leq s.$$

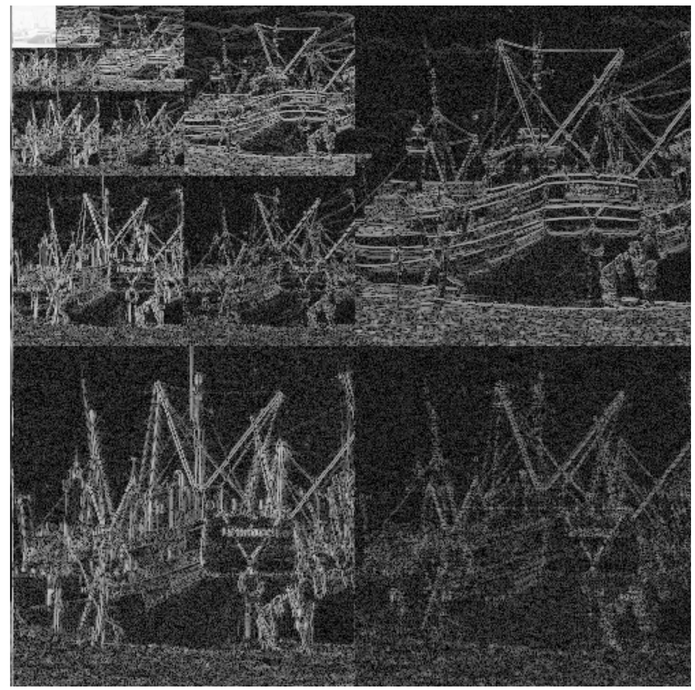
- Gaussian or Bernoulli measurement matrices satisfy the RIP with high probability when

$$m \gtrsim s \log d.$$

- Random Fourier and others with fast multiply have similar property: $m \gtrsim s \log^4 d$.

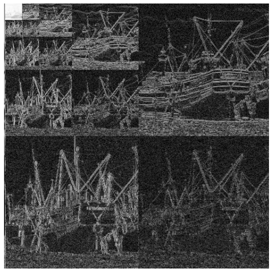
Sparsity...

In orthonormal basis: $f = Bx$



Natural Images

Images are compressible in *Wavelet bases*.



$$f = \sum_{j,k=1}^{N} x_{j,k} H_{j,k}, \qquad x_{j,k} = \langle X, H_{j,k} \rangle, \qquad \|f\|_2 = \|x\|_2,$$

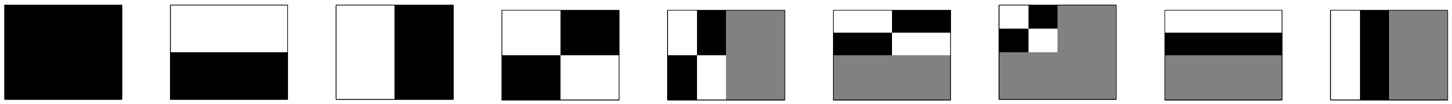


Figure 1: Haar basis functions

Wavelet transform is *orthonormal* and multi-scale. Sparsity level of image is higher on detail coefficients.

Natural images

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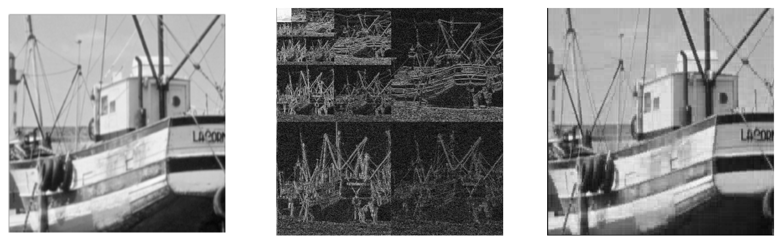


Figure 2: Boats image, 2D Haar transform, and compression from 10% of Haar coefficients

$$f = H^{-1}H(f) = \sum_{j,k=1}^{N} x_{j,k} H_{j,k}$$

f is s-sparse (in Haar basis) if $\|x\|_0 \leq s$

f_s^w is the best s-term approximation to f in Haar basis

Image compression: $f \to f_s^w$

Sparsity in orthonormal basis B

- L1-minimization Method

For orthonormal basis B, $f = Bx$ with x sparse, one may solve the ℓ_1-minimization program:

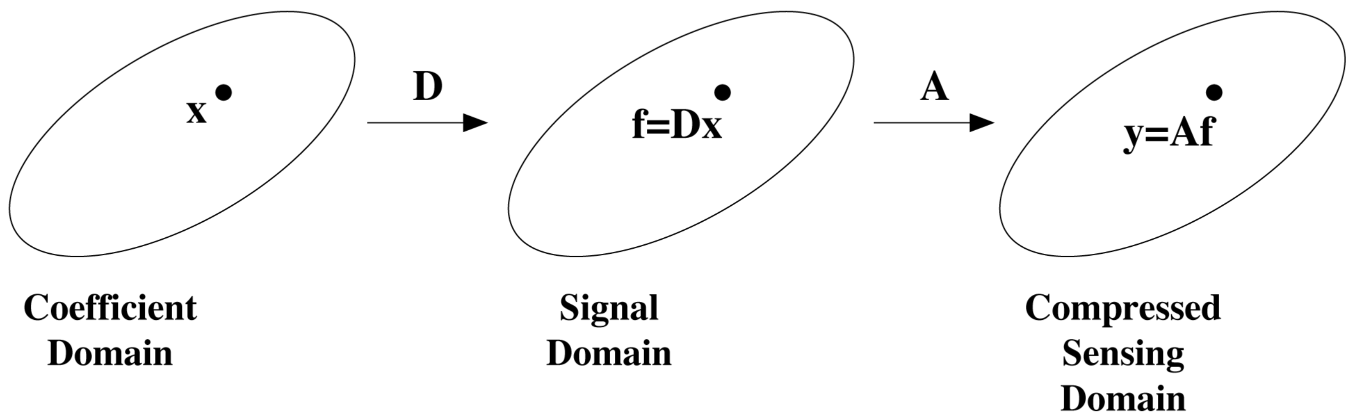
$$\hat{f} = \underset{\tilde{f} \in \mathbb{C}^n}{\operatorname{argmin}} \|B^{-1}\tilde{f}\|_1 \quad \text{subject to} \quad \|\mathscr{A}\tilde{f} - y\|_2 \leq \varepsilon.$$

Same results hold.

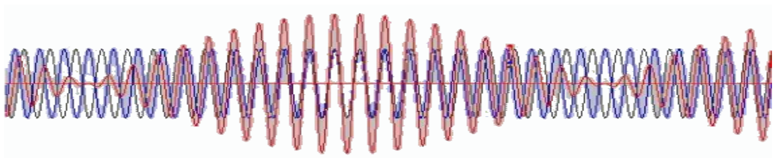
Sparsity...

In arbitrary dictionary: $f = Dx$

The CS Process



Example: Oversampled DFT



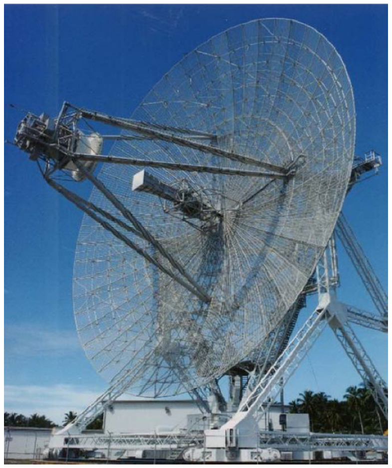
- $n \times n$ DFT: $d_k(t) = \frac{1}{\sqrt{n}} e^{-2\pi i k t / n}$

- Sparse in the DFT → superpositions of sinusoids with frequencies in the lattice.

- Instead, use the *oversampled DFT*:

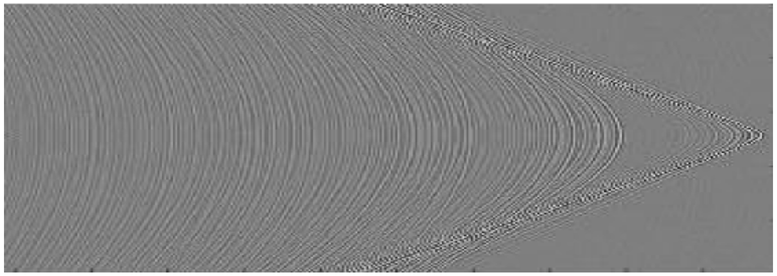
- Then D is an overcomplete frame with highly coherent columns → *conventional CS does not apply*.

Example: Gabor frames



- Gabor frame: $G_k(t) = g(t - k_2 a) e^{2\pi i k_1 b t}$
- Radar, sonar, and imaging system applications use Gabor frames and wish to recover signals in this basis.
- Then D is an overcomplete frame with possibly highly coherent columns $\rightarrow$ *conventional CS does not apply*.

Example: Curvelet frames



- A Curvelet frame has some properties of an ONB but is overcomplete.
- Curvelets approximate well the curved singularities in images and are thus used widely in image processing.
- Again, this means D is an overcomplete dictionary $\rightarrow$ *conventional CS does not apply*.

Example: UWT

- The undecimated wavelet transform has a translation invariance property that is missing in the DWT.
- The UWT is overcomplete and this redundancy has been found to be helpful in image processing.
- Again, this means D is a redundant dictionary → *conventional CS does not apply*.

ℓ_1-Synthesis Method

- For arbitrary tight frame D, one may solve the ℓ_1-synthesis program:

$$\hat{f} = D\left(\underset{\tilde{x}\in\mathbb{C}^n}{\operatorname{argmin}} \|\tilde{x}\|_1 \quad \text{subject to} \quad \|\mathscr{A} D\tilde{x} - y\|_2 \le \varepsilon\right).$$

Some work on this method [Candès et.al., Rauhut et.al., Elad et.al.,...]

ℓ_1-Analysis Method

- For arbitrary tight frame D, one may solve the ℓ_1-analysis program:

$$\hat{f} = \underset{\tilde{f} \in \mathbb{C}^n}{\operatorname{argmin}} \|D^* \tilde{f}\|_1 \quad \text{subject to} \quad \|\mathscr{A} \tilde{f} - y\|_2 \leq \varepsilon.$$

Condition on A?

✦ D-RIP

We say that the measurement matrix $\mathscr{A}$ obeys the *restricted isometry property adapted to* D (D-RIP) if there is $\delta < c$ such that

$$(1-\delta)\|Dx\|_2^2 \leq \|\mathscr{A}Dx\|_2^2 \leq (1+\delta)\|Dx\|_2^2$$

holds for all s-sparse x.

✦ Similarly to the RIP, many classes of random matrices satisfy the D-RIP with $m \approx s\log(d/s)$.

CS with tight frame dictionaries

✦ Theorem [Candès-Eldar-N-Randall]

Let D be an arbitrary tight frame and let $\mathscr{A}$ be a measurement matrix satisfying D-RIP. Then the solution $\hat{f}$ to ℓ_1-analysis satisfies

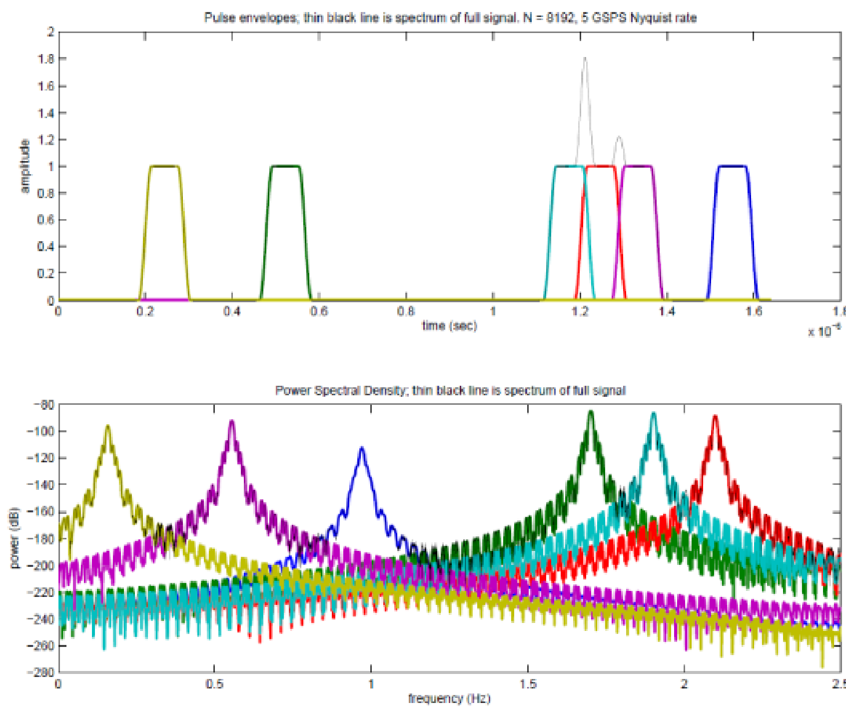
$$\|\hat{f} - f\|_2 \lesssim \varepsilon + \frac{\|D^* f - (D^* f)_s\|_1}{\sqrt{s}}.$$

✦ In other words, This result says that ℓ_1-analysis is very accurate when $D^* f$ has rapidly decaying coefficients and D is a tight frame.

ℓ_1-analysis: Experimental Setup

$n = 8192, m = 400, d = 491,520$

A: $m \times n$ Gaussian, D: $n \times d$ Gabor



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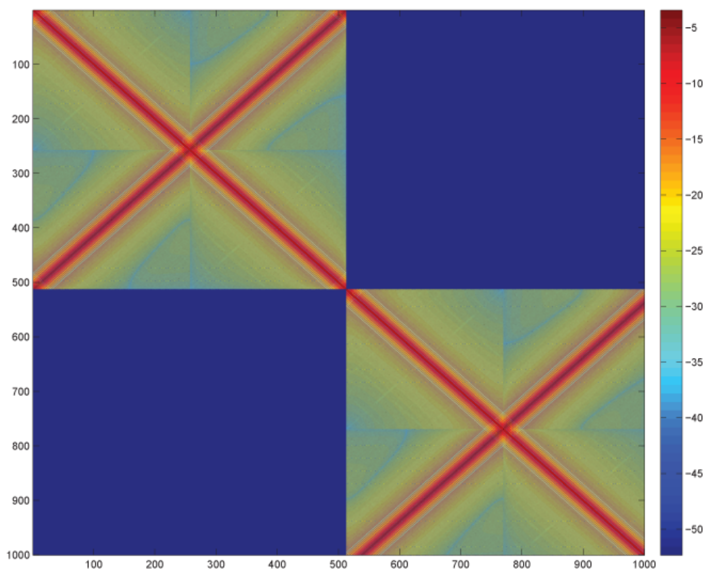
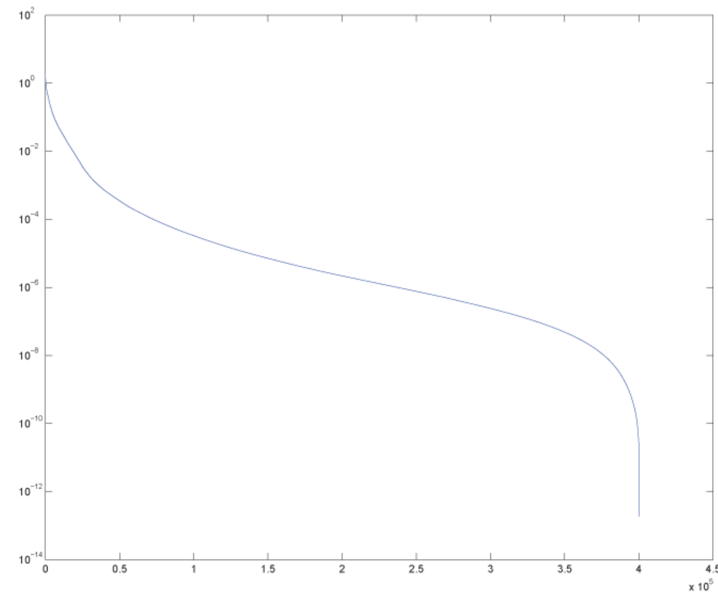
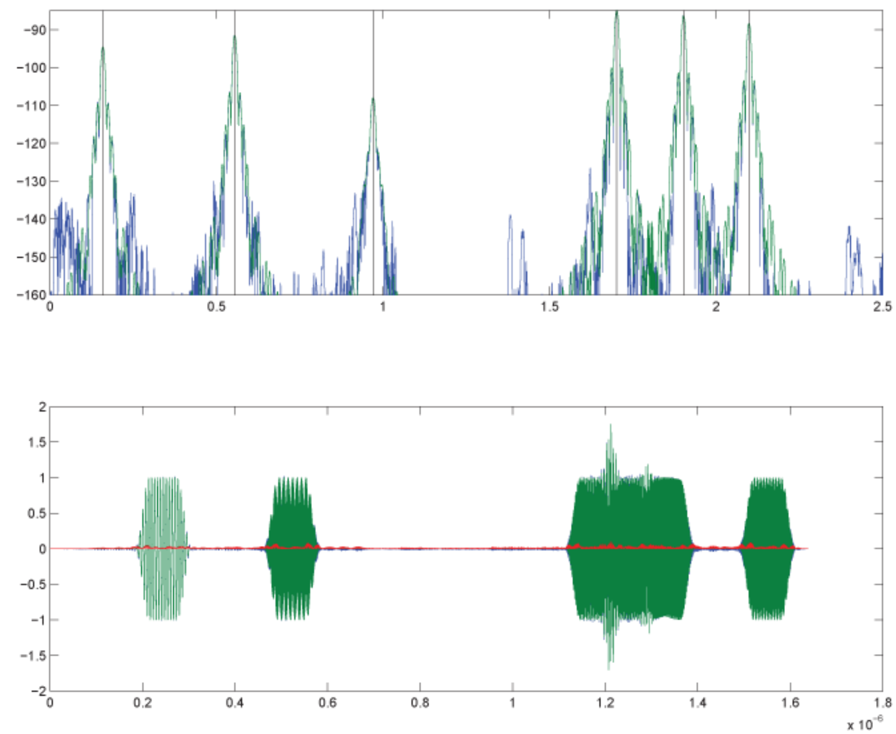




ℓ_1-analysis: Experimental Results



Other algorithms

- ℓ_1-analysis is very accurate when $D^* f$ has rapidly decaying coefficients and D is a tight frame. This is precisely because this method operates in "analysis" space.
- What about operating in signal or coefficient space?

Is it really a pipe?

(Thanks to M. Davenport for this clever analogy.)

CoSaMP

COSAMP (N-Tropp)

input: Sampling operator A, measurements y, sparsity level s
initialize: Set $x^0 = 0$, $i = 0$.
repeat
signal proxy: Set $p = A^*(y - Ax^i)$, $\Omega = \mathrm{supp}(p_{2s})$, $T = \Omega \cup \mathrm{supp}(x^i)$.
signal estimation: Using least-squares, set $b|_T = A_T^\dagger y$ and $b|_{T^c} = 0$.
prune and update: Increment i and to obtain the next approximation, set $x^i = b_s$.
output: s-sparse reconstructed vector $\widehat{x} = x^i$

Signal Space CoSaMP

SIGNAL SPACE COSAMP (Davenport-N-Wakin)

input: $\boldsymbol{A}$, $\boldsymbol{D}$, $\boldsymbol{y}$, s, stopping criterion
initialize: $\boldsymbol{r} = \boldsymbol{y}$, $\boldsymbol{x}^0 = 0$, $\ell = 0$, $\Gamma = \emptyset$
repeat

proxy: $\boldsymbol{h} = \boldsymbol{A}^* \boldsymbol{r}$
identify: $\Omega = \mathscr{S}_{\boldsymbol{D}}(\boldsymbol{h}, 2s)$
merge: $T = \Omega \cup \Gamma$
update: $\widetilde{\boldsymbol{x}} = \operatorname{argmin}_{\boldsymbol{z}} \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{z}\|_2 \quad \text{s.t.} \quad \boldsymbol{z} \in \mathscr{R}(\boldsymbol{D}_T)$
$\Gamma = \mathscr{S}_{\boldsymbol{D}}(\widetilde{\boldsymbol{x}}, s)$
$\boldsymbol{x}^{\ell+1} = \mathscr{P}_{\Gamma} \widetilde{\boldsymbol{x}}$
$\boldsymbol{r} = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^{\ell+1}$
$\ell = \ell + 1$
output: $\widehat{\boldsymbol{x}} = \boldsymbol{x}^{\ell}$

Signal Space CoSaMP

- Here we must contend with

$$\Lambda_{\text{opt}}(\boldsymbol{z}, s) := \underset{\Lambda:|\Lambda|=s}{\operatorname{argmin}} \|\boldsymbol{z} - \mathscr{P}_\Lambda \boldsymbol{z}\|_2, \quad \mathscr{P}_\Lambda : \mathbb{C}^n \to \mathscr{R}(\boldsymbol{D}_\Lambda).$$

- Estimate by $\mathscr{S}_{\boldsymbol{D}}(\boldsymbol{z}, s)$ with $|\mathscr{S}_{\boldsymbol{D}}(\boldsymbol{z}, s)| = s$, that satisfies

$$\left\|\mathscr{P}_{\Lambda_{\text{opt}}(\boldsymbol{z},s)}\boldsymbol{z} - \mathscr{P}_{\mathscr{S}_{\boldsymbol{D}}(\boldsymbol{z},s)}\boldsymbol{z}\right\|_2 \le \min\left(\epsilon_1 \left\|\mathscr{P}_{\Lambda_{\text{opt}}(\boldsymbol{z},s)}\boldsymbol{z}\right\|_2, \epsilon_2 \left\|\boldsymbol{z} - \mathscr{P}_{\Lambda_{\text{opt}}(\boldsymbol{z},s)}\boldsymbol{z}\right\|_2\right)$$

for some constants $\epsilon_1, \epsilon_2 \ge 0$.

Approximate Projection

- Practical choices for $\mathscr{S}_{\boldsymbol{D}}(\boldsymbol{z}, s)$:
- Any sparse recovery algorithm!
- OMP
- CoSaMP
- ℓ_1-minimization followed by hard thresholding

Signal Space CoSaMP

✦ Theorem [Davenport-N-Wakin] Let D be an arbitrary tight frame, A be a measurement matrix satisfying D-RIP, and f a sparse signal with respect to D. Then the solution $\hat{f}$ from *Signal Space CoSaMP* satisfies

$$\|\hat{f} - f\|_2 \lesssim \varepsilon.$$

(And similar results for approximate sparsity.)

Signal Space CoSaMP: Experimental Results

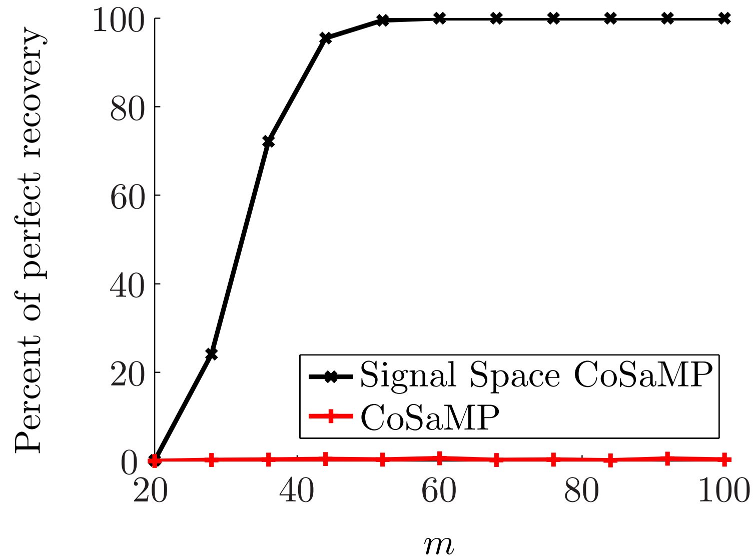


Figure 3: *Performance in recovering signals having a $s = 8$ sparse representation in a dictionary $\boldsymbol{D}$ with orthogonal, but not normalized, columns.*

Signal Space CoSaMP: Experimental Results

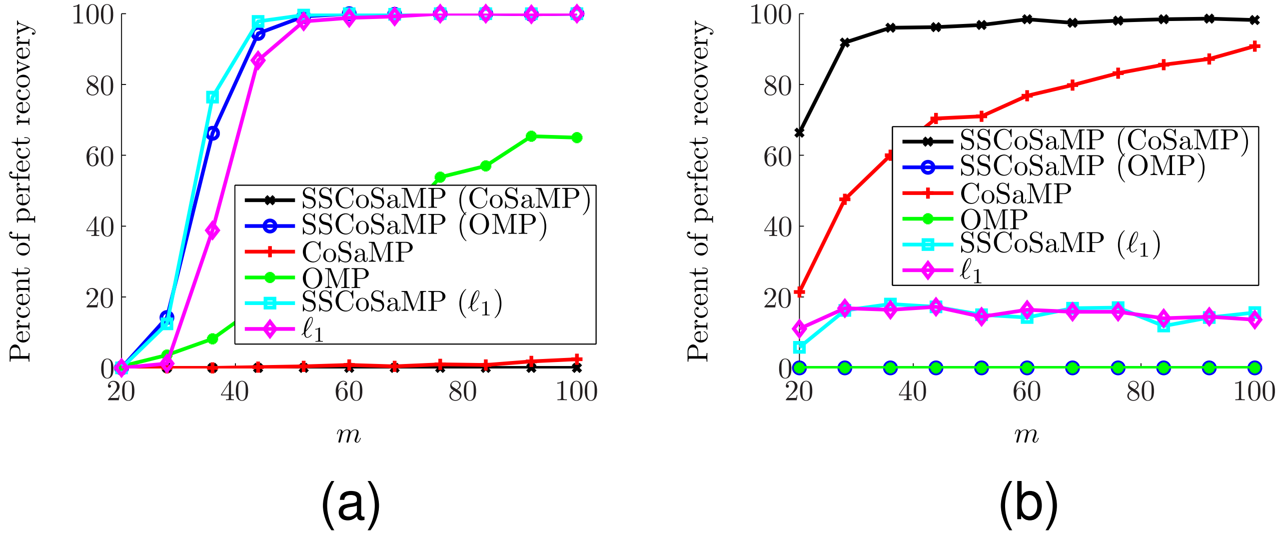


Figure 4: *Results with* $s = 8$ *sparse representation in a 4× overcomplete DFT dictionary: (a) well-separated coefficients, (b) clustered coefficients.*

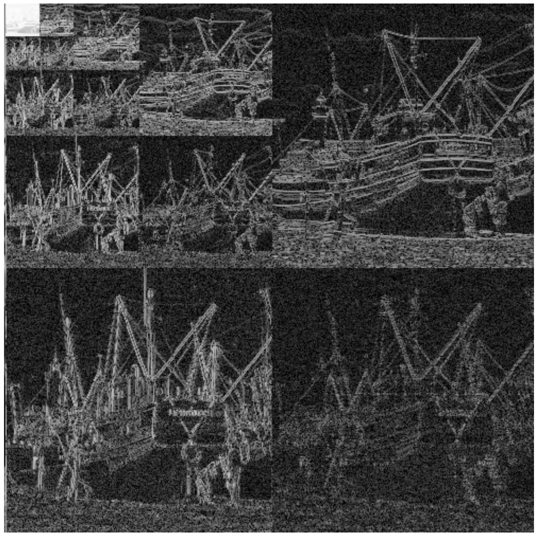
Natural images

Sparse...

256×256 “Boats" image

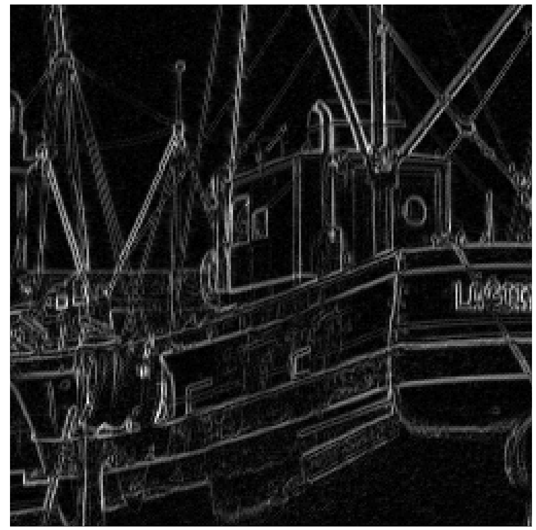
Natural images

Sparse wavelet representation...



Natural images

Images are compressible in *discrete gradient*.



Natural images

Images are compressible in *discrete gradient*.

The discrete directional derivatives of an image $f \in \mathbb{C}^{N \times N}$ are

$$f_x : \mathbb{C}^{N \times N} \to \mathbb{C}^{(N-1) \times N}, \qquad (f_x)_{j,k} = f_{j,k} - f_{j-1,k},$$

$$f_y : \mathbb{C}^{N \times N} \to \mathbb{C}^{N \times (N-1)}, \qquad (f_y)_{j,k} = f_{j,k} - f_{j,k-1},$$

the discrete gradient operator is

$$\nabla[f] = (f_x, f_y)$$

Natural Notation

Images are compressible in *discrete gradient*.

- $\|f\|_p := \left(\sum_{j=1}^{N}\sum_{k=1}^{N}|f_{j,k}|^p\right)^{1/p}$
- f is s-sparse if $\|f\|_0 := |\{(j,k) : f_{j,k} \neq 0\}| \leq s$
- f_s is the best s-sparse approximation to f
- “Phantom": $\|\nabla[f]\|_0 = .03N^2$
- “Boats": $\|\nabla[f] - \nabla[f]_s\|_2$ decays quickly in s

Sparsity in gradient

- CS Theory

The gradient operator ∇ *is not an orthonormal basis or a tight frame.*

Comparison of two compressed sensing reconstruction algorithms

- Haar-minimization (L_1-Haar)

$$\hat{f}_{Haar} = \operatorname{argmin} \| H(Z) \|_1 \quad \text{subject to} \quad \| \mathscr{A} Z - y \|_2 \leq \varepsilon$$

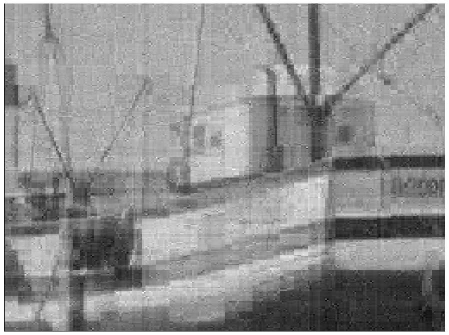
- Total Variation minimization (TV)

$\hat{f}_{TV} = \operatorname{argmin} \| \nabla [Z] \|_1 \quad \text{subject to} \quad \| \mathscr{A} Z - y \|_2 \leq \varepsilon$, where $\| Z \|_{TV} = \| \nabla [Z] \|_1$ is the *total-variation norm*.

Imaging via compressed sensing

(a) Original

(b) TV



(c) L_1-Haar

Figure 5: Reconstruction using $m = .2N^2$

Imaging via compressed sensing

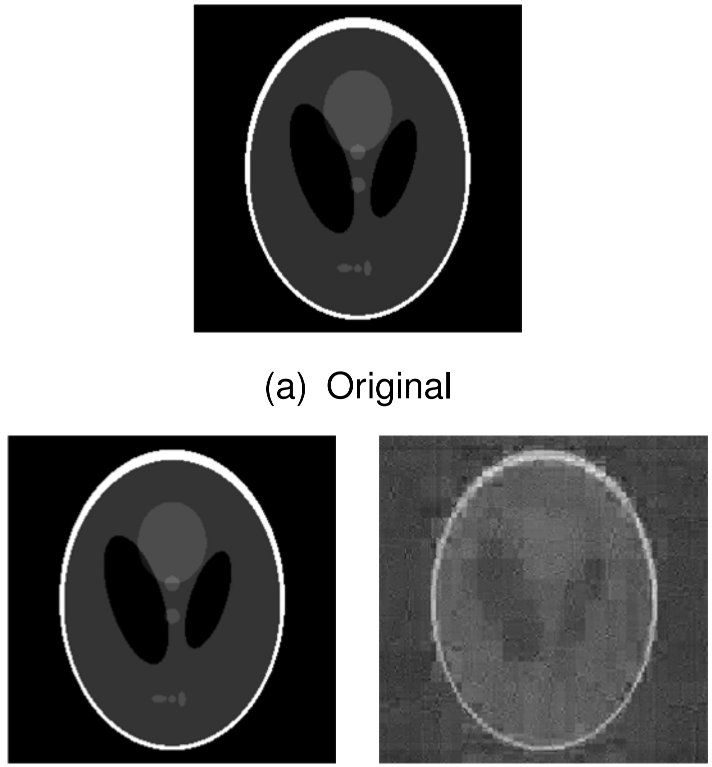
(a) Original

(b) TV

(c) L_1-Haar

Figure 6: Reconstruction using $m = .2N^2$ measurements

Imaging via compressed sensing



(a) Original

(b) TV

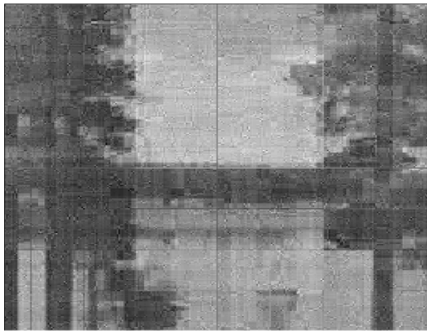
(c) L_1-Haar

Figure 7: Reconstruction using $m = .2N^2$ measurements.

Imaging via compressed sensing

(a) (Quantization)

(b) TV



(c) L_1-Haar

Figure 8: Reconstruction using $m = .2N^2$ measurements

Imaging via compressed sensing

InView (Austin TX)

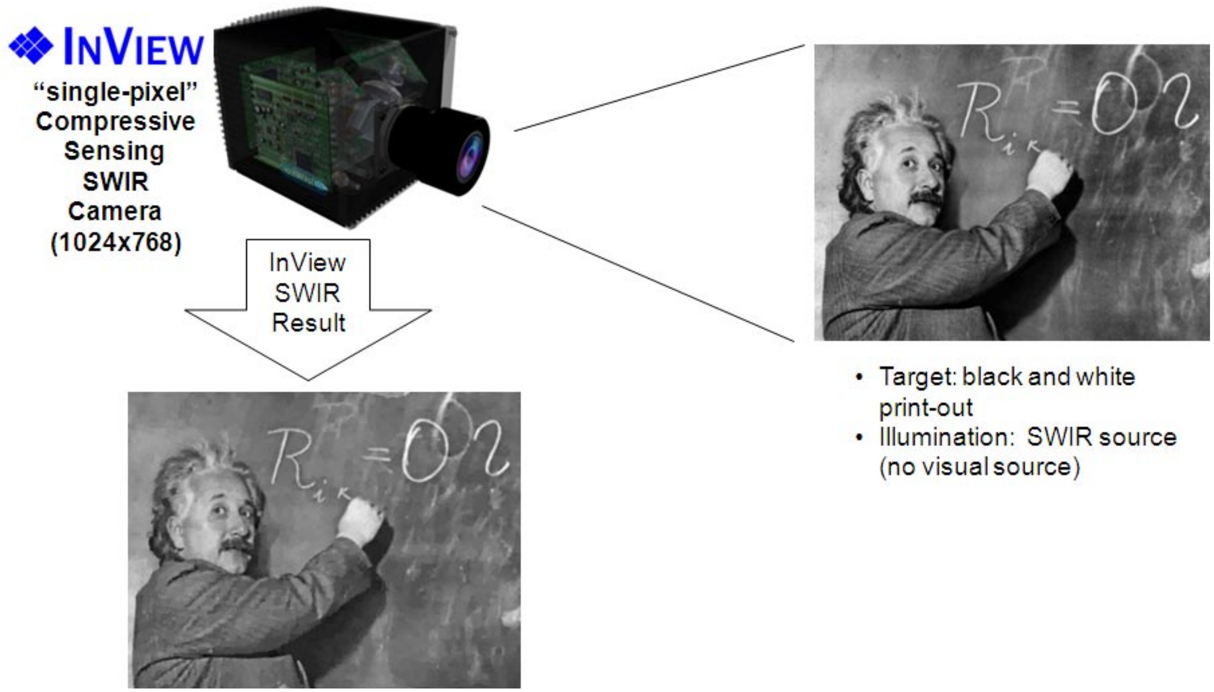


Figure 9: SWIR Reconstruction using $m = .5N^2$ measurements

Imaging via compressed sensing

InView (Austin TX)

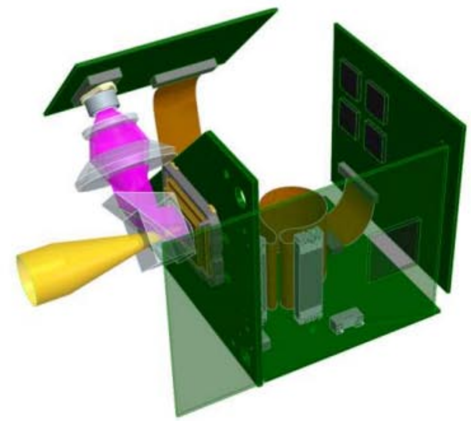



Figure 10: InView SWIR camera

Empirical → Theoretical?

- TV Works

Empirically, it has been well known that

$$\hat{f}_{TV} = \operatorname{argmin} \|Z\|_{TV} \quad \text{subject to} \quad \|\mathscr{A}Z - y\|_2 \leq \varepsilon, \qquad (TV)$$

provides quality, stable image recovery.

- No provable stability guarantees.

Stable signal recovery using total-variation minimization

Theorem 1. [N-Ward] *From $m \gtrsim s\log(N)$ linear RIP measurements, for any $f \in \mathbb{C}^{N\times N}$,*

$$\hat{f} = \operatorname{argmin} \|Z\|_{TV} \quad \textit{such that} \quad \|\mathscr{A}(Z) - y\|_2 \leq \varepsilon,$$

satisfies

$$\|f - \hat{f}\|_{TV} \lesssim \|\nabla[f] - \nabla[f]_s\|_1 + \sqrt{s}\varepsilon \qquad \text{(gradient error)}$$

and

$$\|f - \hat{f}\|_2 \lesssim \log(N) \cdot \left[\frac{\|\nabla[f] - \nabla[f]_s\|_1}{\sqrt{s}} + \varepsilon\right] \qquad \text{(signal error)}$$

This error guarantee is optimal up to the $\log(N)$ factor

Higher dimensional objects

Movies, higher dimensional objects?

Theorem 2. [N-Ward] *From $m \gtrsim s\log(N^d)$ linear RIP measurements, for any $f \in \mathbb{C}^{N^d}$,*

$$\hat{f} = \operatorname{argmin} \|Z\|_{TV} \quad \textit{such that} \quad \|\mathscr{A}(Z) - y\|_2 \le \varepsilon,$$

satisfies

$$\|f - \hat{f}\|_{TV} \lesssim \|\nabla[f] - \nabla[f]_s\|_1 + \sqrt{s}\varepsilon \qquad \text{(gradient error)}$$

and

$$\|f - \hat{f}\|_2 \lesssim \log(N^d/s) \cdot \left[\frac{\|\nabla[f] - \nabla[f]_s\|_1}{\sqrt{s}} + \varepsilon\right] \qquad \text{(signal error)}$$

This error guarantee is optimal up to the $\log(N^d/s)$ factor

Proof Sketch

✦ Strengthened Sobolev inequalities for random subspaces

Proposition 3. [Sobolev inequality for discrete images] *Let $X \in \mathbb{R}^{N \times N}$ be mean-zero. Then*

$$\|X\|_2 \leq \|X\|_{TV}$$

Proposition 4. [New: Strengthed Sobolev inequality] *With probability $\geq 1 - e^{-cm}$, the following holds for all images $X \in \mathbb{R}^{N \times N}$ in the null space of an $m \times N^2$ random Gaussian matrix*

$$\|X\|_2 \lesssim \frac{[\log(N)]^{3/2}}{\sqrt{m}} \|X\|_{TV}$$

Strengthened Sobolev inequalities

Proof ingredients:

- [CDPX 99:] Denote the bivariate Haar wavelet coefficients of $X \in \mathbb{R}^{N \times N}$ by $c_{(1)} \geq c_{(2)} \geq \cdots \geq c_{(N^2)}$. Then

$$|c_{(k)}| \lesssim \frac{\|X\|_{TV}}{k}$$

  That is, the sequence is in weak-ℓ_1.

- If $\Phi : \mathbb{R}^d \to \mathbb{R}^m$ has (properly normalized) i.i.d. Gaussian entries then with probability exceeding $1 - e^{-cm}$, Φ has the RIP of order $s \sim \frac{m}{\log d}$:

$$\frac{3}{4}\|f\|_2 \leq \|\Phi f\|_2 \leq \frac{5}{4}\|f\|_2 \quad \text{for all } s\text{-sparse } f.$$

Strengthened Sobolev inequalities: proof

- Let $\Phi : \mathbb{R}^d \to \mathbb{R}^m$ be a Gaussian matrix and $\Phi X = 0$.
- Suppose that $\Psi = \Phi \mathcal{H}^* : \mathbb{R}^d \to \mathbb{R}^m$ has the RIP of order $2s$.
- Decompose $c = \mathcal{H} X$ into s-sparse blocks $c = c_{S_0} + c_{S_1} + c_{S_2} + \dots$
- Then $\Psi c = \Phi \mathcal{H}^* \mathcal{H} X = \Phi X = 0$ and

$$
\begin{aligned}
0 &\geq \|\Psi(c_{S_0} + c_{S_1})\|_2 - \sum_{j \geq 2} \|\Psi c_{S_j}\|_2 \\
\text{(RIP of } \Psi\text{)} \quad &\geq \frac{3}{4}\|c_{S_0} + c_{S_1}\|_2 - \frac{5}{4}\sum_{j \geq 2} \|c_{S_j}\|_2 \\
\text{(block trick)} &\geq \frac{3}{4}\|c_{S_0}\|_2 - \frac{5/4}{\sqrt{s}}\sum_{j \geq 1} \|c_{S_j}\|_1 \\
(c \text{ in weak } \ell_1) &\geq \frac{3}{4}\|c_{S_0}\|_2 - \frac{5/4}{\sqrt{s}}\|X\|_{TV} \log(d/s)
\end{aligned}
$$

Strengthened Sobolev inequalities: proof

So

- $\|c_{S_0}\|_2 \leq \frac{1}{\sqrt{s}}\|X\|_{TV}\log(d/s)$,
- $\|c - c_{S_0}\|_2 \leq \frac{1}{\sqrt{s}}\|X\|_{TV}$ (c is in weak ℓ_1)

Then

$$\begin{aligned} \|X\|_2 &= \|c\|_2 \leq \|c_{S_0}\|_2 + \|c - c_{S_0}\|_2 \\ &\leq \frac{\log(d/s)}{\sqrt{s}}\|X\|_{TV} \end{aligned}$$

Proof is complete, because with probability $1 - \varepsilon^{-cm}$, RIP of $\Phi\mathcal{H}^*$ holds with $s \sim m/\log(d)$.

Stable signal recovery using total-variation minimization

Method of proof:

- First prove stable *gradient* recovery
- Translate stable *gradient* recovery to stable *signal* recovery using the strengthened Sobolev inequality.

Thank you!

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