Exponential Lower Bound for 2-Query LDCs via a Quantum Argument

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STOC 03, joint with Iordanis Kerenidis (LRI Paris)

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- Disadvantage: if you only want one bit x_i, you still need to decode the whole $C(x)$

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  2. $d(y, C(x)) \leq \delta m \Rightarrow \Pr[A^y(i) = x_i] \geq 1/2 + \varepsilon$

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$$\Pr[A^y(i) = x_i] \geq 1 - 2\delta$$

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- Step 2:

  $(1, \delta, \varepsilon)$-LQDC needs length $m \geq 2^{\Omega(n)}$, because it implies a $\log m$-qubit random access code for x

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- Hence Hadamard code is a 1-query LQDC!

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$$\frac{11}{14}p + \frac{3}{14}(1-p) = \frac{1}{2} + \frac{4\varepsilon}{7}$$

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- Given $|A_i(x)\rangle = \sum_{j\in A_i}(-1)^{C(x)_j}\alpha_{ij}|j\rangle$,

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- By Birthday Paradox: $r \approx m^{1-\lfloor 2/q \rfloor}$ suffices
- RAC: $m^{1-\lfloor 2/q \rfloor} \cdot \log m = \Omega(n) \Rightarrow n \geq \left(\frac{n}{\log n}\right)^{1+\frac{1}{\lceil q/2 \rceil - 1}}$
- Slightly improved by Woodruff

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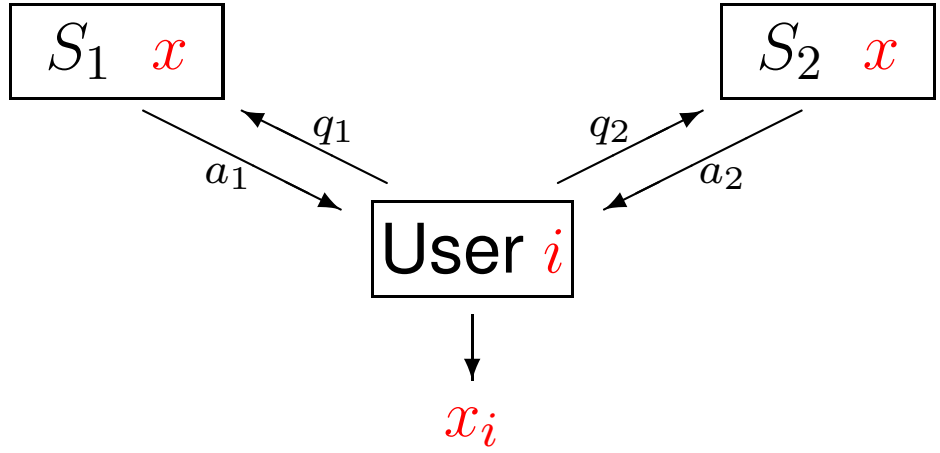

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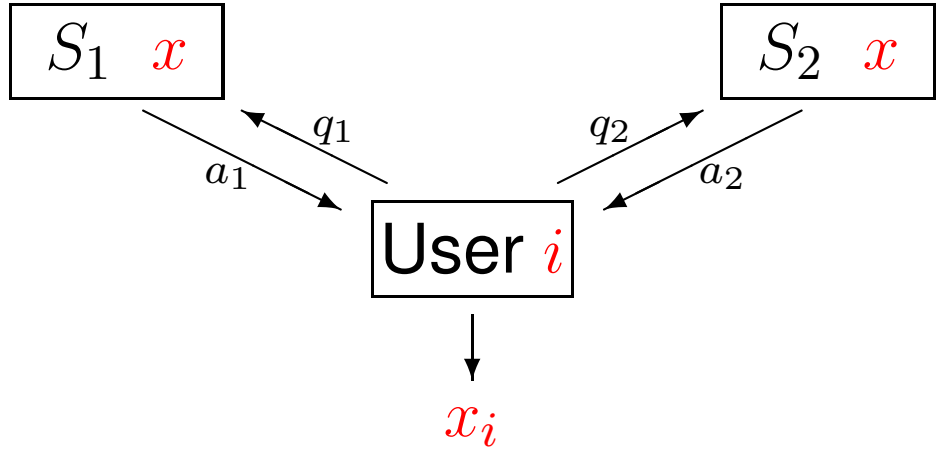

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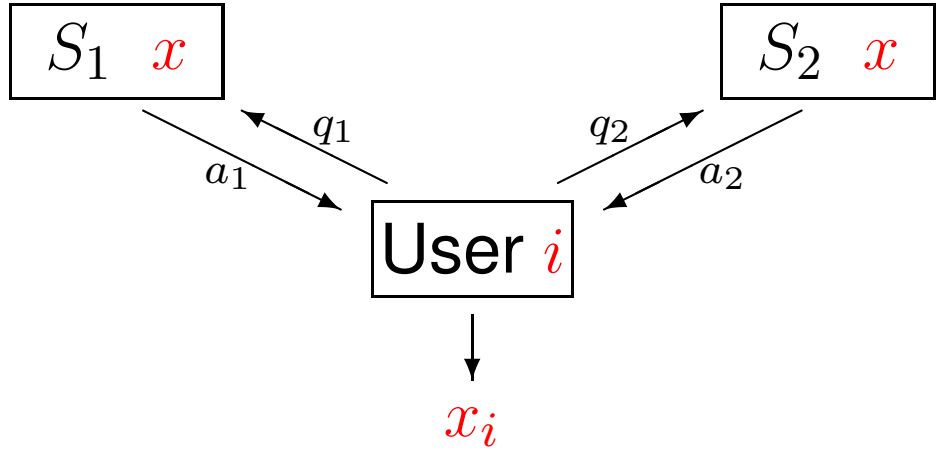

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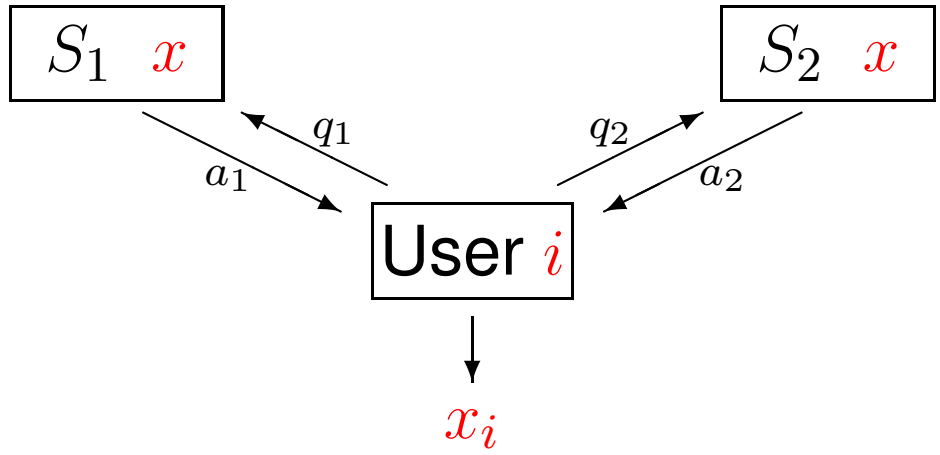

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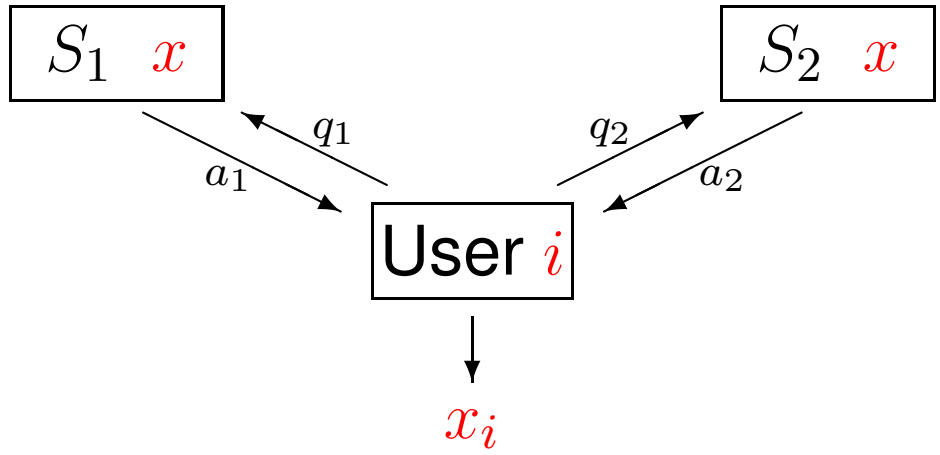

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- From Yekhanin's construction: $O(n^{\varepsilon})$ 2-server QPIRs

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- Future Work: optimal LDC bounds for $q > 2$?