

Particle and Game Theoretic Methods for Multi-scale and High Dimensional Problems

Jack Xin

Department of Mathematics
University of California, Irvine.

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Outline

- Fisher-Kolmogorov-Petrovsky-Piskunov **Front Speeds in Flows**.
- **Feynman-Kac Representation** and **Particle Approximation**.
- Learning **Invariant Measure w. Physical Parameter Dependence**.
- Entropy production in time reversal of diffusion processes.
- Computing principal eigenvalue by interacting particle method.
- Challenges: concentrations, large gradients and high dimensions.
- **Curvature G-equation and Two-Player Game Representation**.
- **One-Way Reachability, Minimum Principle**, Cell Problems.
- Bounded Oscillation Estimate, Front Speeds in Cell Flows.
- Conclusions and Future Work.

FKPP Variational Formula in Stationary Ergodic Media

$$u_t = \kappa \Delta_x u + \mathbf{v}(t, x) \cdot \nabla_x u + u(1 - u), \quad x \in \mathbb{R}^d,$$

\mathbf{v} is space-time stationary ergodic, mean zero, volume preserving. To calculate front speed c^* along direction \mathbf{e} , let w solve linear equation parameterized by $\lambda > 0$ ($u = \exp\{\lambda \mathbf{e} \cdot x\} w$):

$$w_t = \mathcal{L}w := \kappa \Delta_x w + (2\kappa \lambda \mathbf{e} + \mathbf{v}) \cdot \nabla_x w + (1 + \kappa \lambda^2 + \lambda \mathbf{e} \cdot \mathbf{v})w,$$

with $w(0, x) = 1$. Almost surely,

$$\mu(\lambda) = \lim_{t \rightarrow \infty} t^{-1} \ln w$$

exists as [principal Lyapunov exponent](#), convex and superlinear in large λ .

$$c^*(\mathbf{e}) = \inf_{\lambda > 0} \frac{\mu(\lambda)}{\lambda}.$$

Freidlin, Gärtner, 1979: spatially periodic (1D random) media; Nolen, X, 2009: space-time stationary ergodic flow.

Lagrangian Approximation in Space-Time Periodic Flow

- Write $\mathcal{L} = L + M = \text{Markovian} + \text{Potential}$,

$$M \cdot := c(t, \mathbf{x}) \cdot = (1 + \kappa \lambda^2 + \lambda \mathbf{e} \cdot \mathbf{v}) \cdot$$

Feynman-Kac formula gives:

$$\mu = \lim_{t \rightarrow \infty} t^{-1} \ln \left(\mathbb{E} \exp \left\{ \int_0^t c(t-s, \mathbf{X}_s^{t, \mathbf{x}}) ds \right\} \right),$$

$$d \mathbf{X}_s^{t, \mathbf{x}} = \mathbf{v}(t-s, \mathbf{X}_s^{t, \mathbf{x}}) ds + \sigma d \mathbf{W}_s, \quad \mathbf{X}_0^{t, \mathbf{x}} = \mathbf{x}.$$

- Direct approximation of this formula is challenging, as the main contribution to \mathbb{E} comes from sample paths that visit maximal points of time-dependent potential c .

Lagrangian Approximation in Space-Time Periodic Flow

- Consider a “normalized version”, the Feynman-Kac semi-group:

$$\Phi_t^c(\nu_0)(\phi) := \frac{\mathbb{E}[\phi(\mathbf{X}_t^{t,x}) \exp\{\int_0^t c(t-s, \mathbf{X}_s^{t,x}) ds\}]}{\mathbb{E}[\exp\{\int_0^t c(t-s, \mathbf{X}_s^{t,x}) ds\}]} := \frac{P_t^c(\nu_0)(\phi)}{P_t^c(\nu_0)(1)}$$

acting on initial probability measure ν_0 ($\mathbf{x} \sim \nu_0$), converges weakly to an **invariant measure** ν_c as $t \uparrow \infty$, for any test function ϕ . Moreover,

$$P_t^c(\nu_c) = \exp\{\mu t\} \nu_c \quad \text{or} \quad \mu = t^{-1} \ln \mathbb{E}_{\nu_c}[P_t^c(\nu_c)].$$

- Discretize $\mathbf{X}_s^{t,x}$ as $\mathbf{X}_i^{\Delta t}$ by explicit Euler, $i = 1, \dots, n \times m$, $m = \frac{T}{\Delta t}$, approximate evolution of probability measure $\Phi_t^c(\nu)$ by a particle system, with a resampling technique to reduce variance.

Lagrangian Approximation in Space-Time Periodic Flow

- Let

$$P_n^{c,\Delta t}(\nu_0)(\phi) := \mathbb{E} \left[\phi(\mathbf{X}_{nm}^{\Delta t}) \exp \left\{ \Delta t \sum_{i=1}^m c((m-i)\Delta t, \mathbf{X}_{i+(n-1)m}^{\Delta t}) \right\} \right]$$

- As $n \rightarrow \infty$, the sampled FK semi-group actions on ν :

$$\mathbb{E}_{\nu_0} \frac{P_n^{c,\Delta t}(\nu_0)(\phi)}{P_n^{c,\Delta t}(\nu_0)(1)} \rightarrow \int_D \phi d\nu_{c,\Delta t}, \quad \forall \text{ smooth } \phi,$$

D : space periodic cell, $\nu_{c,\Delta t}$: approximate invariant measure.

Theorem (Lyu, Wang, X, Zhang, SINUM 2022)

There exists $q \in (0, 1)$ so that:

$$\mu_{\Delta t}^n := (nT)^{-1} \ln[P_n^{c,\Delta t}(\nu_0)(1)] \rightarrow T^{-1} \ln[P_1^{c,\Delta t}(\nu_{c,\Delta t})(1)] = \mu + o((\Delta t)^q)$$

Genetic Interacting Particle Algorithm

- Initialize 1st generation of particles $\xi_1^0 = (\xi_1^{0,1}, \dots, \xi_1^{0,N_0}) \in (\mathbb{T}^d)^{N_0}$, unif. distributed over \mathbb{T}^d ($d \geq 2$). Let g be the generation no. in approximating $\nu_{c,\Delta t}$. Each generation moves and mutates m -times, with a life span T (time period), time step $\Delta t = T/m$.

for $g = 1 : G - 1$

for $i = 0 : m - 1$

$\zeta_g^i \leftarrow$ one-step-advection-diffusion update on ξ_g^i

with fitness $F \leftarrow \exp\{c(T - i\Delta t, \zeta_g^i) \Delta t\}$.

$E_{g,i} := \frac{1}{\Delta t} \ln$ (mean population fitness), growth rate.

Normalize fitness to weight $\mathbf{p} := F / \text{SUM}(F)$.

$\xi_g^{i+1} \leftarrow$ resample ζ_g^i via multinomial distribution with weight \mathbf{p} .

end for

$\xi_{g+1}^0 \leftarrow \xi_g^m$, $E_g \leftarrow \text{mean}(E_{g,i})$ over i .

end for

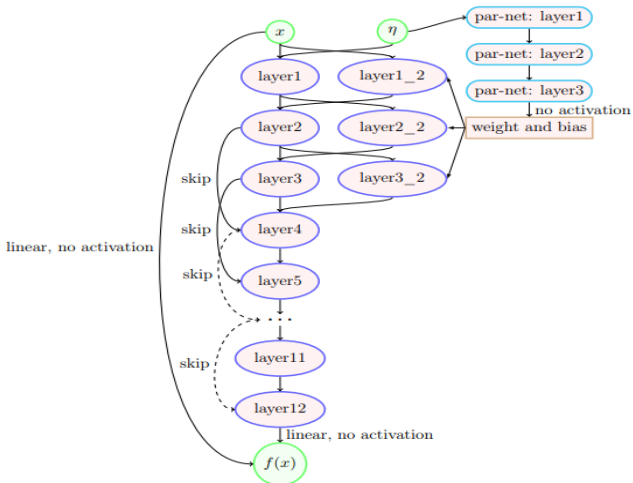
- Output: approximate $\mu_{\Delta t} \leftarrow \text{mean}(E_g)$, and ξ_G^0 .

Genetic Algorithm, Bottleneck and Learning

- Feynman-Kac (F-K) semigroup, particle methods of invariant measure and principal eigenvalue, are well-known in physics, large deviation, Monte Carlo.
- Ferré & Stoltz, 2019: error estimates of discrete F-K and particle approximation in *spatially periodic media*.
- Advantages of interacting particle method for generating training data: (1) meshfree, (2) self-adaptive, (3) not sensitive to high dimensions (costs about the same from 2D to 3D).
- **Computational bottleneck**: long time evolution of large number of particles in the small κ (large Péclet number $\sim \kappa^{-1}$) regime.
- **Invariant measure learning**: map uniform distribution to invariant measure at κ based on a few less expensive simulations at $\kappa_j > \kappa$.

Nonlocal Resnet for optimal transport (OT) map

η : physical parameter input. $X = Y = \mathbb{R}^d$.



Deep Particle Network Training

- Training data: interacting particle algorithm generates samples of invariant measure at $\kappa_i = 2^{-2-0.25(i-1)}$, $i = 1, \dots, 8 (= n_\eta)$, and $N_0 = 40000$ particle evolution for $G = 2048$, $\Delta t = 2^{-8}$, $T = 1$.
- At each κ_i , randomly sample $N = 2000$ points \mathcal{Y}_i without replacement.
- $\{\mathcal{Y}_1, \dots, \mathcal{Y}_8\}$: one mini-batch of training data.
- Total 5 mini-batches for 50000 gradient descent steps of training.
- On a quad-core CPU desktop with an RTX2080 8GB GPU.

Arnold-Beltrami-Childress Flow

- (Arnold '65; Hénon '66; Dombre, Frisch, Greene, Hénon, Mehr & Soward '86):

$$x' = A \sin z + C \cos y$$

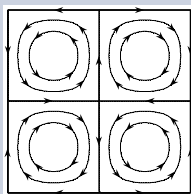
$$y' = B \sin x + A \cos z$$

$$z' = B \cos x + C \sin y$$

weakly chaotic at $A = B = C = 1$.

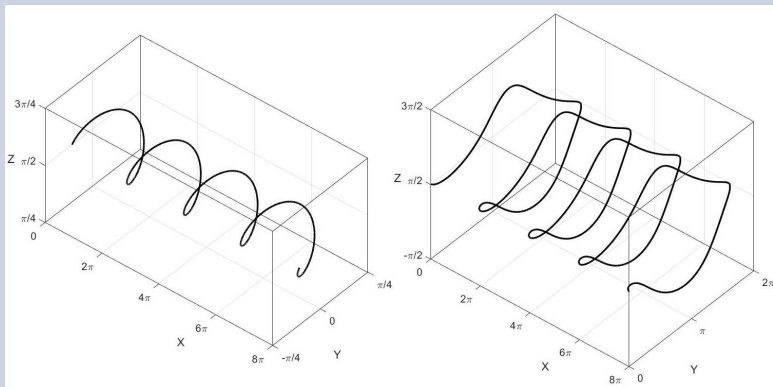
- Cellular (Hamiltonian) flow or BC flow ($A = 0$) with $\pi/4$ rotation:

$$\mathbf{v}(x, y) = (\partial_y \mathcal{H}, -\partial_x \mathcal{H}), \quad \mathcal{H} = \sin(x) \sin(y)$$



Ballistic Orbits in ABC (L), Komogorov(R) Flow

- Kolmogorov (K) flow: ABC ($A=B=C=1$) with only sine terms.
- Construction of orbits by symmetry: X, Yu, Zlatoš (ABC, SIAM Math Anal. 2016), Kao, Liu, X (K flow, MMS 2022).



A=B=C=1, plus random shear perturbation

- $\delta (\sin(z) + \cos(y), \sin(x) + \cos(z) + \epsilon \cdot \xi(x, \omega), \sin(y) + \cos(x))$,
 $\kappa = 1, T = 512, \Delta t = 2^{-11}, T = 512, N = 100,000$ particles.

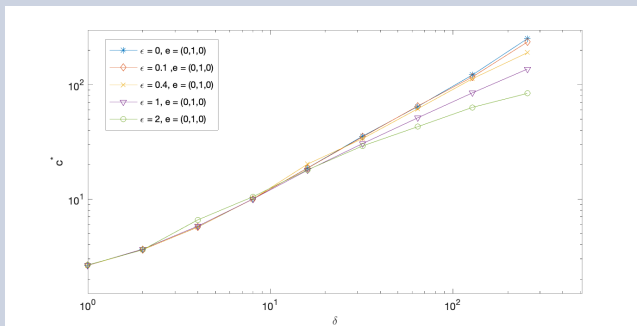
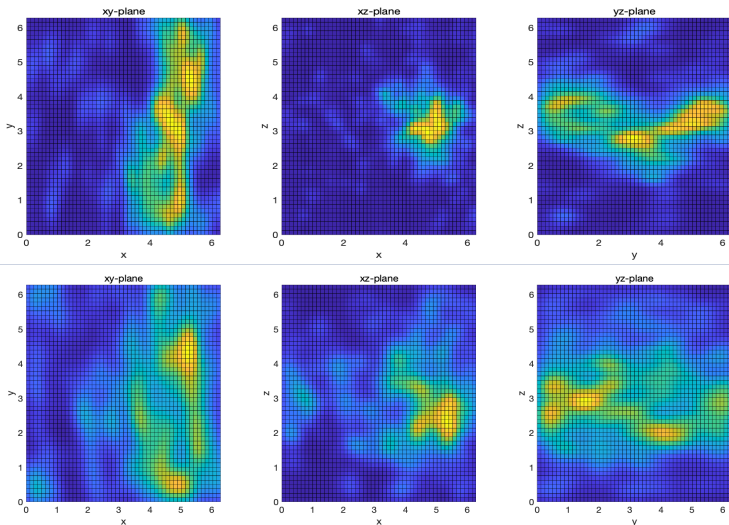


Figure: $\xi \approx$ OU process, generated by random Fourier series.

- $c^* = O(\delta^{0.9842})$ at $\delta \approx 100, \epsilon = 0$. As $\epsilon \uparrow$, vortex tube structure in ABC flow facilitating transport is destroyed by $(0, \xi(x, \omega), 0)$, $c^* \downarrow$.

Defocusing of invariant measure: $\epsilon = 0$ (top), 2 (bottom).



Kolmogorov Flows

- Galloway & Proctor 1992, Childress & Gilbert 1995:

$$x' = \sin z$$

$$y' = \sin x$$

$$z' = \sin y$$

Disorder (mixing) dominates phase space, integrable regions shrink to small islands.

- **Time-Periodic Kolmogorov flow** field (strongly mixing):

$$\mathbf{v} = (\sin(z + \sin(2\pi t)), \sin(x + \sin(2\pi t)), \sin(y + \sin(2\pi t))).$$

DP Generated Invariant Measure Projected on yz plane

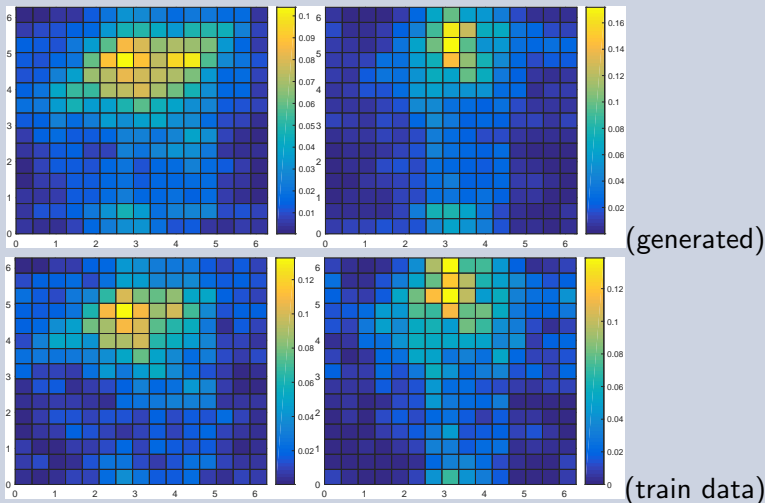


Figure: 1st/2nd col: $\kappa = 2^{-2.5}/2^{-3}$ in time-periodic K flow.

DP Predicted Inv. Meas. vs. Ground Truth on yz plane

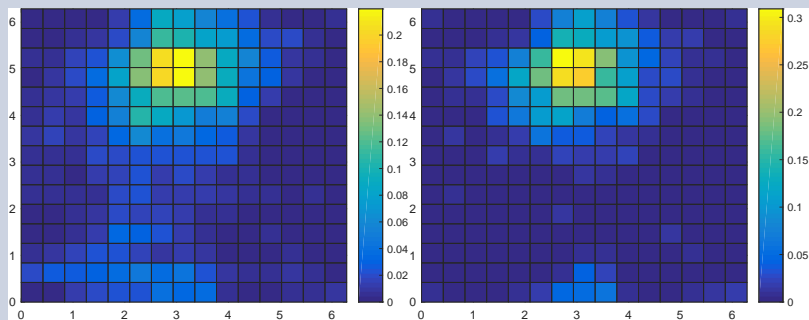


Figure: Invariant measures projected to yz plane in time-periodic K flow at $\kappa = 2^{-4}$: prediction (left), ground truth (right), at test value $\kappa = 2^{-4}$.

Time Reversal of Diffusion Processes

- Stochastic differential equation (SDE) in \mathbb{R}^d :

$$dX_s = h(X_s, s) ds + g(s) dW_s := -\nabla V(X_s) ds + b(X_s) ds + \sqrt{2\epsilon} dW_s \quad (1)$$

where V is a smooth potential function with at least quadratic growth at infinity, b is a bounded divergence free and smooth vector field, $\epsilon \in (0, 1)$, and W is standard Wiener process.

- Kolmogorov (1937): The initial distribution $\sim \exp\{-V/\epsilon\}$ is realized by solving (1) backward in time from terminal distribution iff $b = 0$.
- Let terminal (prior) distribution p_T at time T be normal.
MCMC sampler: $p_{data} = Z^{-1} \exp\{-V\}$, V is a closed form log-likelihood function.
- Mapping p_{prior} to p_{data} (since 2010's): variational autoencoder, generative adversarial network, flow-based models (e.g. neural ODEs, finding h from data), deep particle (2022).

Entropy Production

- Denoising diffusion probabilistic (score-based generative) modeling.
- Map p_{data} to p_{prior} by (1) or **adding noise** in time to increase entropy.
- Invert the map (**denoising**) by solving SDE (in reverse time):

$$d\tilde{X}_s = [h(\tilde{X}_s, s) - g^2(s)\nabla_{\tilde{X}_s} \log p_s(\tilde{X}_s)] ds + g(s) d\tilde{W}_s, \quad (2)$$

Anderson (1982), where p_s is distribution of forward time SDE (1).

- Entropy integral to quantify reversability:

$$S_t = \epsilon^{-1} \int_0^t \langle b(X_s), \circ dX_s \rangle$$

work done by **non-gradient** part of the drift h .

Entropy Production

- Let $P_t^{\lambda, \epsilon}$ be the probability measure of (1) from initial measure λ . Its moment generating function is ($\alpha \in \mathbb{R}$):

$$\chi_t^\epsilon(\alpha) = \int_{C([0, t]; \mathbb{R}^d)} \exp\{-\alpha S_t^\epsilon\} dP_t^{\lambda, \epsilon}$$

with representation:

$$\chi_t^\epsilon(\alpha) = \int_{\mathbb{R}^d} d\xi (\exp\{t A^{\epsilon, \alpha}\} 1)(\xi)$$

where $A^{\epsilon, \alpha}$ acts on smooth and compactly supported functions f as:

$$\begin{aligned} A^{\epsilon, \alpha} f &= \epsilon \Delta f + \langle -\nabla V + (1 - 2\alpha)b, \nabla f \rangle - \frac{\alpha(1 - \alpha)}{\epsilon} |b|^2 f \\ &\quad + \frac{\alpha}{\epsilon} \langle b, \nabla V \rangle f + \alpha(\nabla \cdot b) f \end{aligned}$$

over $\alpha \in [-\delta, 1 + \delta]$ for some small $\delta > 0$.

Entropy Production and Leading Eigenvalue

- Let $\lambda^{\varepsilon, \alpha}$ be the **leading eigenvalue** (the one with the largest real part) of $A^{\varepsilon, \alpha}$. Then:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \chi_t^\varepsilon(\alpha) = \lambda^{\varepsilon, \alpha} \quad (3)$$

convex in α and symmetric about $\alpha = 1/2$.

- Legendre transform of $\lambda^{\varepsilon, \alpha}$ in α** is the large deviation rate function of $t^{-1} S_t^\varepsilon$, so characterizes the stochastic **growth rate of entropy**.
- $A^{\varepsilon, \alpha}$ shares $\lambda^{\varepsilon, \alpha}$ with operator:

$$\begin{aligned} B^{\varepsilon, \alpha} f &:= \exp((-2\varepsilon)^{-1} V) A^{\varepsilon, \alpha} (\exp((2\varepsilon)^{-1} V)) \\ &= \varepsilon \Delta f + \langle (1 - 2\alpha)b, \nabla f \rangle - \frac{1}{4\varepsilon} |\nabla V|^2 f + \frac{1}{2\varepsilon} \langle b, \nabla V \rangle f \\ &\quad - \frac{\alpha(1 - \alpha)}{\varepsilon} |b|^2 f + \frac{1}{2} (\Delta V) f - \alpha (\nabla \cdot b) f := (L + U) f. \end{aligned}$$

Probabilistic Representation

- Markov generator:

$$L f := \varepsilon \Delta f + \langle (1 - 2\alpha)b, \nabla f \rangle$$

with corresponding SDE:

$$dX_t = (1 - 2\alpha)b(X_t) dt + \sqrt{2\varepsilon} dB_t.$$

- Assume: at least quadratic growth of potential $|\nabla V|^2 \geq a|x|^2$ for $|x| \geq R$, const. $(a, R) > 0$, $\lim_{|x| \rightarrow \infty} \Delta V / |\nabla V| = 0$; and bounded drift

$$\|b\|_{C^1(\mathbb{R}^d)} \leq C.$$

- Evolution operator P_t^U :

$$P_t^U \varphi(x) = \mathbb{E} \left[\varphi(X_t) \exp \left(\int_0^t U(X_s) ds \right) \mid X_0 = x \right], \quad (4)$$

$\mathbb{E}[\cdot]$ on Brownian motion and φ measurable.

- Let \hat{P}_t^U be the time discretized P_t^U , $t = n\Delta t$.

Discrete FK semigroup

- \exists invariant measure $\hat{\nu}_*$ under discrete FK semigroup so that:

$$\hat{P}_t^U(\hat{\nu}_*) = \exp\{\hat{\lambda}_{\Delta t}^{\varepsilon, \alpha} t\} \hat{\nu}_* \quad \text{or} \quad \hat{\lambda}_{\Delta t}^{\varepsilon, \alpha} = t^{-1} \ln \mathbb{E}_{\hat{\nu}_*}[\hat{P}_t^U(\hat{\nu}_*)]$$

and

$$\hat{\lambda}_{\Delta t}^{\varepsilon, \alpha} = \lim_{k \rightarrow +\infty} \frac{1}{k\Delta t} \log \mathbb{E} \left[\exp \left(\Delta t \sum_{j=1}^{k-1} U(X_{j\Delta t}) \right) \middle| X_0 \sim \nu \right].$$

- Interacting Particle method + HKU cluster: 2 Intel Xeon Gold 6226R (16 Core) CPU's and 96GB RAM.
- Experiments: $\#(\text{particles})=500,000$, $\Delta t = 2^{-8}$; $\varepsilon = 0.1, 0.01, 0.001$.
At each ε , $\alpha \in \left[-\frac{1}{10}, \frac{11}{10}\right]$, compute $\lambda^{\varepsilon, \alpha}$ for $\alpha = -\frac{1}{10} + \frac{j}{31} \frac{12}{10}$ with $j = 0, 1, \dots, 31$. The computation of $\lambda^{\varepsilon, \alpha}$ for each ε with 32 different values of α is performed in parallel on 32 cores of the CPU's.

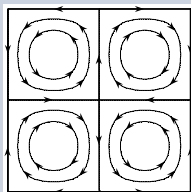
2D example

- Double-well potential

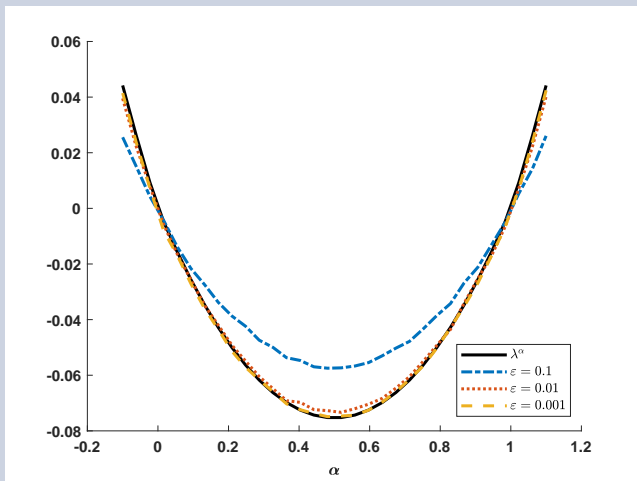
$$V(x_1, x_2) = x_1^4 - 2x_1^2 + (1 + a(x_1 - 1)^2)x_2^2 + x_2^4$$

with $a = 0.4$, and cell (Hamiltonian) flow

$$b(x_1, x_2) = \pi^{-1}(\cos(\pi x_1) \sin(\pi x_2), -\sin(\pi x_1) \cos(\pi x_2))$$

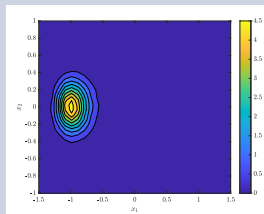


- $\lambda^\alpha = \lim_{\varepsilon \rightarrow 0} \lambda^{\varepsilon, \alpha}$ known in closed-form.

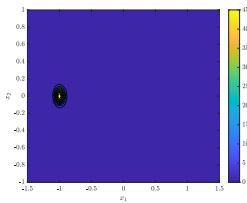
2D example: computed $\lambda^{\varepsilon, \alpha}$ and exact λ^α as $\varepsilon \downarrow 0$ 

2D example: approximate invariant measures

- $\varepsilon = 0.1$



0.01



0.001

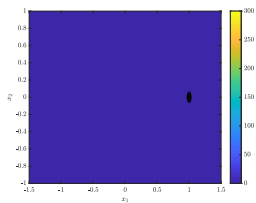
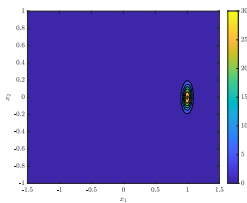
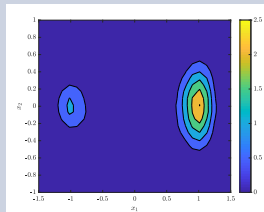
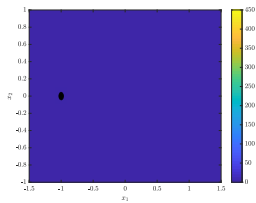
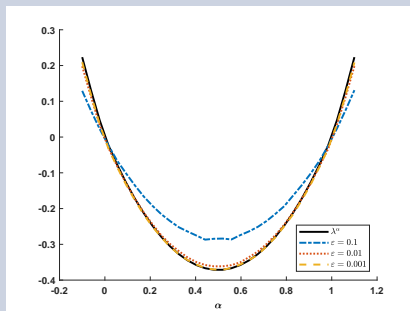


Figure: Distribution of particles at $T = 2048$, and $\alpha \approx 0.5968$ (top)/1.0613.

4D example: $\lambda^{\varepsilon, \alpha}$ in doublewell potential ($a = 0.4$)

$$V(x_1, x_2, x_3, x_4) = x_1^4 - 2x_1^2 + (1 + a(x_1 - 1)^2)x_2^2 + x_2^4 + x_3^4 - 2x_3^2 + (1 + a(x_3 - 1)^2)x_4^2 + x_4^4$$

$$b(x_1, x_2, x_3, x_4) = \pi^{-1}(\cos(\pi x_1) \sin(\pi x_2), -\sin(\pi x_1) \cos(\pi x_2), 2 \cos(\pi x_3) \sin(\pi x_4), -2 \sin(\pi x_3) \cos(\pi x_4)),$$



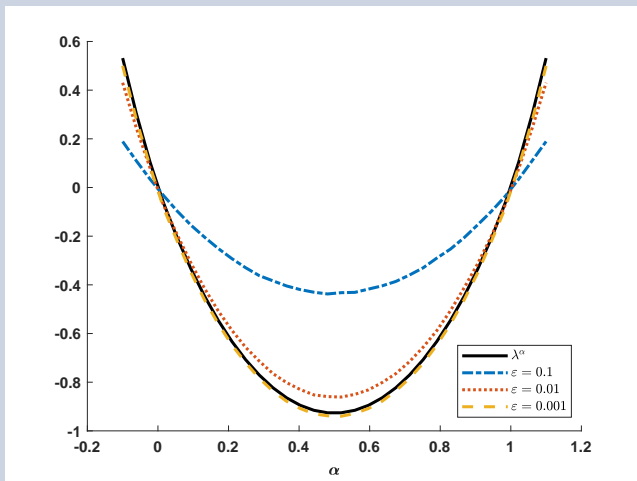
8D example: $\lambda^{\varepsilon, \alpha}$ in potential and Hamiltonian flow.

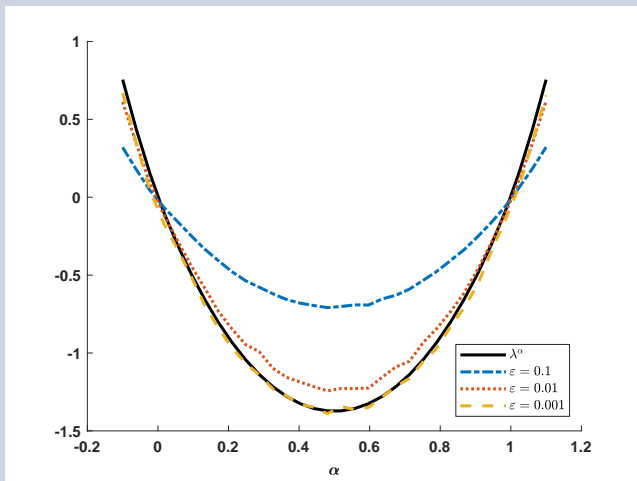
$$\begin{aligned}
 V(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = & \frac{x_1^2 + x_2^2}{2} + \frac{x_1^4 + x_2^4}{8} + \frac{x_3^2 + x_4^2}{2} + \frac{x_3^4 + x_4^4}{8} \\
 & + x_5^4 - 2x_5^2 + (1 + a(x_5 - 1)^2)x_6^2 + x_6^4 \\
 & + x_7^4 - 2x_7^2 + (1 + a(x_7 - 1)^2)x_8^2 + x_8^4
 \end{aligned}$$

with $a = 0.3$.

$$\begin{aligned}
 b(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = & \pi^{-1}(\cos(\pi x_1) \sin(\pi x_2), -\sin(\pi x_1) \cos(\pi x_2), \\
 & \tilde{b}_2 \cos(\pi x_3) \sin(\pi x_4), -\tilde{b}_2 \sin(\pi x_3) \cos(\pi x_4), \\
 & \cos(\pi x_5) \sin(\pi x_6), -\sin(\pi x_5) \cos(\pi x_6), \\
 & 2 \cos(\pi x_7) \sin(\pi x_8), -2 \sin(\pi x_7) \cos(\pi x_8)),
 \end{aligned}$$

with $\tilde{b}_2 = 0.5$.

8D example: computed $\lambda^{\varepsilon, \alpha}$ and exact λ^α as $\varepsilon \downarrow 0$ 

16D example: computed $\lambda^{\varepsilon, \alpha}$ and exact λ^α as $\varepsilon \downarrow 0$ 

16D example: projected invariant measures

- $\varepsilon = 0.1$

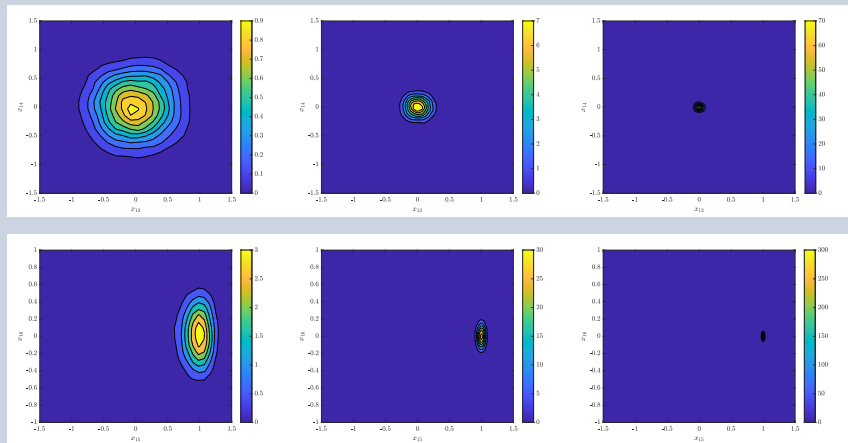
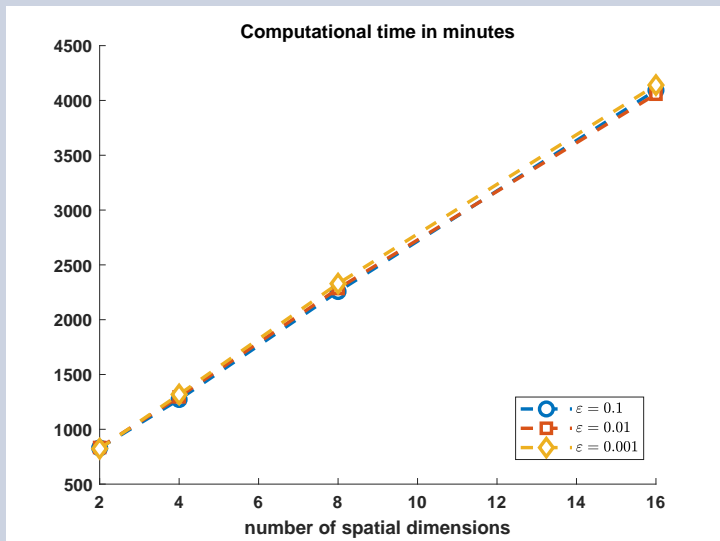


Figure: $T = 2048$, $\alpha \approx 0.2097$, projected on (x_{12}, x_{14}) (top)/ (x_{15}, x_{16}) .

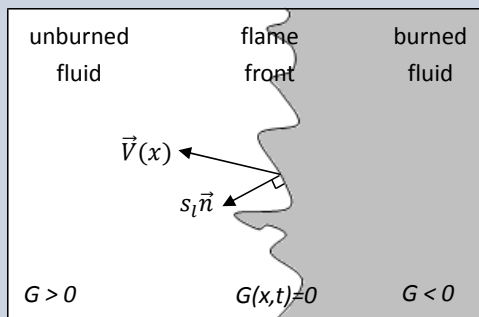
Linear Growth: comput. time (min) vs. dimension.



G-equation: Hamilton-Jacobi Level Set Equation

- Markstein (1950's), Williams (1980's): convex yet non-coercive Hamiltonian.

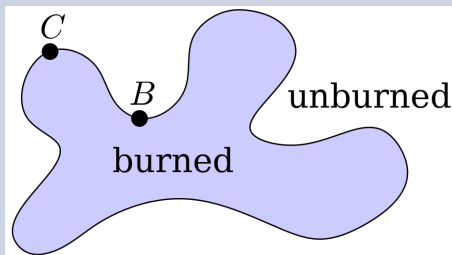
$$G_t + s_l |DG| + V(x) \cdot DG = 0.$$



- Motion law:** normal velocity $v_{\vec{n}} = \vec{V} \cdot \vec{n} + s_l$,
 s_l laminar speed, $\vec{n} = \frac{DG}{|DG|}$.

Curvature Effect on Laminar Speed

- **Curvature effect**, Markstein (1951): as flame front bends toward cold region (unburned area, point C), propagation slows down. As flame bends toward hot spot (burned area, point B), it burns faster.



- Empirical linear relation proposed by Markstein:

$$s_l = s_l^0 (1 - \tilde{d} \kappa)_+$$

κ : mean curvature along the flame front. \tilde{d} : Markstein length.

$(\cdot)_+$ prevents “unburn”. **Folklore: curvature slows flame on average.**

Curvature G-equation in Cell Flow

- After normalizing consts, a **non-convex & non-coercive** Hamiltonian:

$$G_t + \left(1 - d \operatorname{div} \left(\frac{DG}{|DG|} \right)\right)_+ |DG| + V(x) \cdot DG = 0$$

- Let $V(x) = A D^\perp \sin(x_1) \sin(x_2)$, intensity $A > 0$.

Theorem (Gao, Long, X, Yu. J. Geometric Analysis, 2024)

For any unit vector $p \in \mathbb{R}^2$ and initial data $G(x, 0) = p \cdot x$, there exists a positive number $\bar{H}_{A,d}(p)$ such that

$$|G(x, t) - p \cdot x + \bar{H}_{A,d}(p) t| \leq C, \quad \forall (t, x) \in \mathbb{R}^2 \times [0, \infty),$$

for a constant C depending only on A and d .

Corrector

- Look for a solution of the form $p \cdot x - \bar{H}(p)t + v(x)$, v is the so called corrector satisfying (a.k.a. cell problem):

$$\left(1 - d \operatorname{div} \left(\frac{p + Dv}{|p + Dv|} \right)\right)_+ |p + Dv| + V(y) \cdot (p + Dv) = \bar{H}(p), \quad (5)$$

subject to 2π -periodic boundary condition in y .

- Theorem follows by comparing G with $v \pm \text{const}$.
- To construct a solution, consider a modified cell (discount) problem as Lions-Varadhan-Papanicolaou (1980's, *coercive* periodic Hamiltonian):

$$\lambda v + \left(1 - d \operatorname{div} \left(\frac{p + Dv}{|p + Dv|} \right)\right)_+ |p + Dv| + V(y) \cdot (p + Dv) = 0, \quad (6)$$

for a parameter $\lambda > 0$.

- Existence and uniqueness of $v = v_\lambda$ to (6) known by Perron's method (Crandall-Ishii-Lions 1992). Comparison principle implies:

$$\max_{x \in \mathbb{R}^2} |\lambda v_\lambda(x)| \leq 1 + \max_{\mathbb{R}^2} |V|.$$

- Aim to show

$$\lim_{\lambda \rightarrow 0} \lambda v_\lambda(x) = -\bar{H}(p) \quad \text{uniformly on } \mathbb{R}^2$$

then (6) \rightarrow (5) with standard sub/super-solution technique on v_λ .

- Key is the *bounded oscillation* estimate:

$$\max_{x, y \in [-\pi, \pi]^2} |v_\lambda(x) - v_\lambda(y)| \leq C = C(d, A). \quad (7)$$

- In the absence of curvature ($d = 0$), (7) follows from [mutual reachability](#) of x and y by a controlled trajectory (X-Yu, 2010). If y is reachable from x by ξ for $|\alpha(t)| \leq 1$,

$$\dot{\xi}(t) = \alpha(t) - V(\xi(t))$$

subject to $\xi(0) = x$ within time T , then

$$v_\lambda(x) \leq v_\lambda(y) + C T,$$

for incompressible V in any dimension.

Deterministic Two-Person Game in 2D

- A deterministic game (Kohn-Serfaty, 2006): consider the discrete dynamical system $\{x_n\}_{n=1}^N \subset \mathbb{R}^2$ associated with the game starting from $x_0 = x$: for $n = 0, 1, 2, \dots, N - 1$,

$$\begin{cases} x_{n+1} = x_n + \tau\sqrt{2d}b_n\vec{\eta}_n + \tau^2\vec{\eta}_n^\perp - \tau^2 V(x_n) \\ x_0 = x, \end{cases}$$

where $|\vec{\eta}_n| \leq 1$ and $b_n \in \{-1, 1\}$.

- Player I controls direction via $\vec{\eta}_n$ and player II controls sign via b_n .
- Let $g = g(x)$ be a final payoff function. Player I (II) aims to minimize (maximize) $g(x_N)$. If both players proceed optimally, the value function

$$u(x, N\tau^2) := g(x_N),$$

converges to the solution of curvature G-eq with initial data $g(x)$:

$$\lim_{N\tau^2 \rightarrow t, \tau \rightarrow 0} u(x, N\tau^2) = G(x, t).$$

Inequalities from One-Way Reachability via Game

- A typical scenario to get an **upper bound of the game value** is for player I to devise a strategy so that the game trajectory, starting at a point P , ends at a point Q in a desired region U in N moves despite any strategy of player II; then

$$u(P, N\tau^2) \leq g(Q) \leq \max_{q \in U} g(q).$$

- For stationary equation, suppose: (1) w satisfies

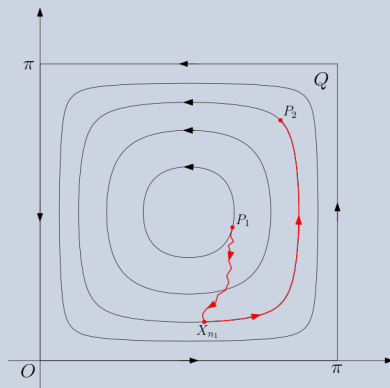
$$\left(1 - d \operatorname{div} \left(\frac{Dw}{|Dw|} \right)\right)_+ |Dw| + V(x) \cdot Dw \leq C$$

implying $w - Ct$ is a sub-solution to G-eq; (2) player I has a strategy to steer the game trajectory from x to a region D (invariant under $-V$ flow) within time T regardless how player II plays; then

$$w(x) - CT \leq G(x, N\tau^2) \leq \max_{y \in D} w(y).$$

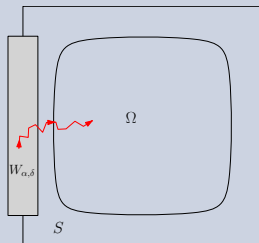
- Unlike the $d = 0$ case, **two-way reachability does NOT hold**.

Interior One-way Reachability



Player I's strategy: at each step, choose $\eta = \frac{\nu}{|V|}$ until it reaches the target lower level set of \mathcal{H} : $\mathcal{H}(X_{n+1}) - \mathcal{H}(X_n) \leq -|DH(X_n)|\tau^2 + O(\tau^3)$.
Then switch to $\eta = 0$ to reach the target point P_2 .

Quarter Cell Boundary Crossing: PDE Method



Evolve G-equation with initial data $(\Omega \subset S = (0, 1)^2)$:

$g_S(x) = -\frac{2}{\pi} \arctan(\text{dist}(x, \partial S))$, for $x \in S$; else, $\frac{2}{\pi} \arctan(\text{dist}(x, \partial S))$.

The edge $\{0\} \times (0, 1)$ is part of the zero level set of G where:

$$v_{\vec{n}} = (1 - d\kappa)_+ + V(x) \cdot \vec{n} \approx 1,$$

so $\exists t_\delta > 0$ s.t. $G((0, \theta), t) < 0$ for $(\theta, t) \in [\delta, 1 - \delta] \times (0, t_\delta]$. An optimal game trajectory starting from $W_{\alpha, \delta}$ must pass through the edge $\{0\} \times [\delta, 1 - \delta]$ during the time interval $(0, t_\delta]$.

PDE estimates from: 1-Way Reachability

Lemma

Let $G(x, t)$ be the unique solution with $G(x, 0) = p \cdot x$. There exist positive constants β and C depending only on d and V s.t. for all $(x, t) \in \mathbb{R}^2 \times [0, \infty)$,

$$G(x, t) - p \cdot x \leq -\beta t + C \quad (8)$$

and

$$\max_{x \in \mathbb{R}^2} \lambda v_\lambda(x) < -\frac{\beta}{2} + \lambda C. \quad (9)$$

Inequality (8) follows from [estimating travel times of game trajectories across cells and 1-way reachability inequalities](#).

Inequality (9) from constructing a super-solution to a time-dependent variant of the discount cell problem via the inequality (8).

Minimum Principle from: 1-Way Reachability

- Applying inequality (9) at $\lambda < \beta/(4C)$, we have from the discount cell problem:

$$\left(1 - d \operatorname{div} \left(\frac{p + Dv_\lambda}{|p + Dv_\lambda|} \right)\right)_+ |p + Dv_\lambda| + V(y) \cdot (p + Dv_\lambda) \geq \beta/4$$

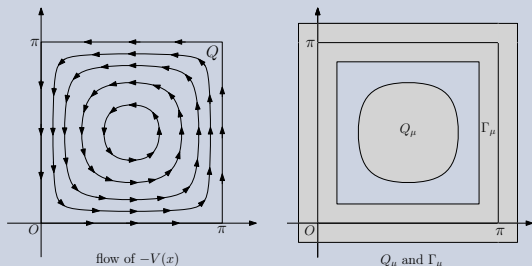
- **minimum principle**: The minimum value of $u_\lambda := p \cdot x + v_\lambda$ in a domain can only be attained on its boundary.
- $u = u_\lambda$ is a viscosity sub-solution of the stationary G-equation:

$$\left(1 - d \operatorname{div} \left(\frac{Du_\lambda}{|Du_\lambda|} \right)\right)_+ |Du_\lambda| + V(y) \cdot (Du_\lambda) = 1 + \max_{[-\pi, \pi]^2} |V|(y) := \alpha$$

$$\text{satisfying } u_\lambda(x_0) \leq \max_{y \in \bar{S}} u_\lambda(y) + \alpha T_0$$

if a bounded set S is reachable from x_0 within T_0 , and S is invariant under $-V$ flow.

Quarter Cell Interior Oscillation Estimate



- For each point $x \in \partial Q_\mu$ and each point $y \in \overline{Q}_\mu$, $u_\lambda(x) \geq u_\lambda(y) - C_\mu$ for some constant $C_\mu > 0$. Accordingly:

$$\min_{x \in \partial Q_\mu} u_\lambda(x) \geq \max_{x \in \overline{Q}_\mu} u_\lambda(x) - C_\mu.$$

- By **minimum principle**: $\min_{x \in \partial Q_\mu} u_\lambda(x) = \min_{x \in \overline{Q}_\mu} u_\lambda(x)$, and so:

$$\max_{x \in \overline{Q}_\mu} u_\lambda(x) - \min_{x \in \overline{Q}_\mu} u_\lambda(x) = \max_{x, y \in \overline{Q}_\mu} |u_\lambda(x) - u_\lambda(y)| \leq C_\mu.$$

Conclusions and Future Work

- Deep Particle (DP) framework to compute and learn multi-scale PDEs with physical parameter dependence based on data from stochastic interacting particle (SIP) systems.
- Learned and generated complex phenomena of FKPP in 3D flows.
- Computed 1st eigenvalue of advection dominated reaction-diffusion operators on \mathbb{R}^d ($d = 16$) by SIP with linear complexity.
- Integrated two-person game and PDE method for existence of average curvature dependent front speeds in cell flows.
- Ongoing/future: (1) DP for reaction-diffusion models with interacting particle description arising in cancer cell spreading. (2) curvature G-equation in 3D: non-existence of \bar{H} in shear flow $V = (0, 0, Af(x_1, x_2))$ if A exceeds a finite value, i.e. averaging fails in some direction (Mitake, Mooney, Tran, X, Yu, '23).

Happy Birthday Russ !