L^1TV computes the Flat Norm

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Credit and Acknowledgements

- This is joint work with Simon Morgan and Wotao Yin.
- Beneficial conversations with Bill Allard and Bob Hardt.
- Benefit of parallel work with Selim Esedoḡlu on the L^1TV functional.
- Inspired by the work of Glaunès, Joshi and Vaillant.

Shapes as Currents

Shapes Subsets in $\mathbb{R}^m$ for some m: mathematical structure? Measures? Points? Manifolds?

Currents n-Forms can be integrated against oriented n-manifolds. n-Manifolds are therefore dual to n-forms. The full dual space is much bigger than oriented n-manifolds. This dual space is the space of *Currents*.

Rectifiable Sets Countable union of subsets of C^1 n-submanifolds of $\mathbb{R}^{n+k}$ plus $\mathcal{H}^n$ negligible set.

Simple n-vector The wedge product of n vectors. Think of it as an oriented n-plane.

Shapes as Currents

Representation We can represent many currents by integration. We describe T by its value on ϕ: T is the n-current, ϕ is an n-form, ξ is the n-vector field, θ is the real or integral density, and $\mathcal{H}^n \llcorner N$ is Hausdorff measure restricted to the support of the current, N.

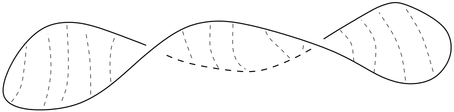
$$T(\phi) = \int_{\mathbb{R}^{n+k}} < \phi, \xi > \theta(x) d\mathcal{H}^n \llcorner N$$

Boundaries We define the boundary very simply:

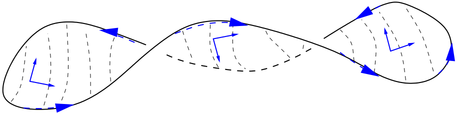
$$\partial T(\phi) = T(d\phi)$$

Notice this agrees with Stokes Theorem: we get what we expect when T is regular enough.

Shapes as Currents: Pictures

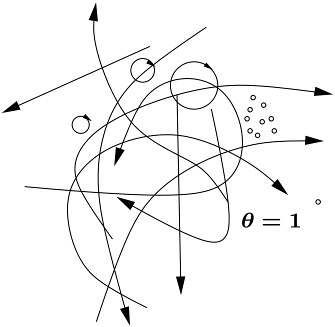


Manifold $\boldsymbol{M}$, with boundary $\boldsymbol{\partial M}$

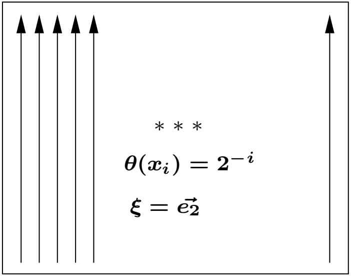


The Current $\boldsymbol{T_M}$ showing orientation of $\boldsymbol{T_M}$ and $\boldsymbol{\partial T_M}$

Shapes as Currents: Pictures



A nice rectifiable **1**-current



Union of vertical unit segments with rational x-coordinate.

A nice **1**-current carried by a rectifiable set and having non-integral density

L1TV: Definition and Specialization to Sets

$$F_\lambda(u) \equiv \int |\nabla u| dx + \lambda \int |u - d| dx$$

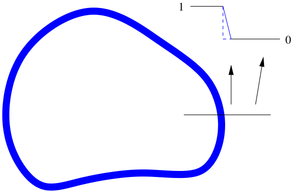
Chan and Esedoglu show that:

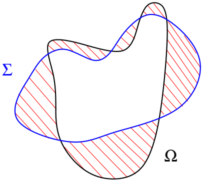
- $d = \chi_\Omega \Rightarrow$ for some Σ, $u = \chi_\Sigma$ is a minimizer.
- More Precisely: If u is any minimizer of $F_\lambda(u)$ then for almost all $\mu \in [0, 1]$, $\chi_{\{x:u>\mu\}}$ is also a minimizer of $F_\lambda(u)$.

L1TV: Definition and Specialization to Sets

$u = \chi_\Sigma$ **and** $d = \chi_\Omega \Rightarrow F_\lambda(\Sigma) \equiv F_\lambda(\chi_\Sigma) = \textbf{Per}(\Sigma) + \lambda|\Sigma\Delta\Omega|$

- $u = \chi_\Sigma \rightarrow$
  $\int |\nabla u| dx$ = perimeter of Σ
- $u = \chi_\Sigma, d = \chi_\Omega \rightarrow$
  $\lambda \int |u - d| dx = \lambda \int |\chi_\Sigma - \chi_\Omega| dx = \lambda \text{Area}(\Sigma \triangle \Omega)$

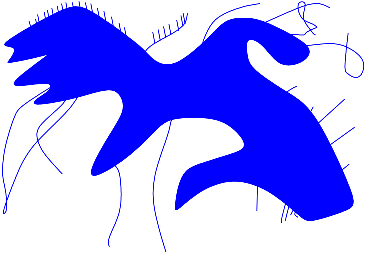


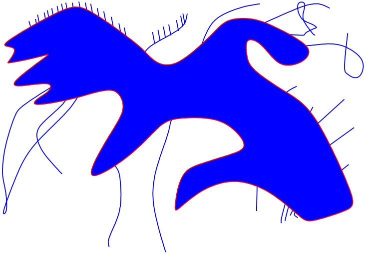


L1TV: Definition and Specialization to Sets

Suppose Ω is really wild? The "argument" above depended on Ω being nice. What can we conclude about TV(χ_Ω) in this case?

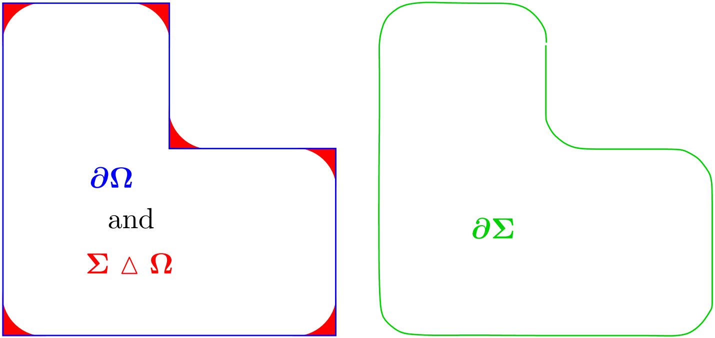
- There is a set $\partial^*\Omega$ called *reduced boundary* of Ω that coincides with the boundary that test functions can see
- TV(χ_Ω) picks up the boundary that integration against smooth test functions "sees".





L1TV: Definition and Specialization to Sets

If we input Ω we get a minimizer $\Sigma(\Omega, \lambda)$ that looks like this:



The Flat Norm

The flat norm can be defined in two equivalent ways:

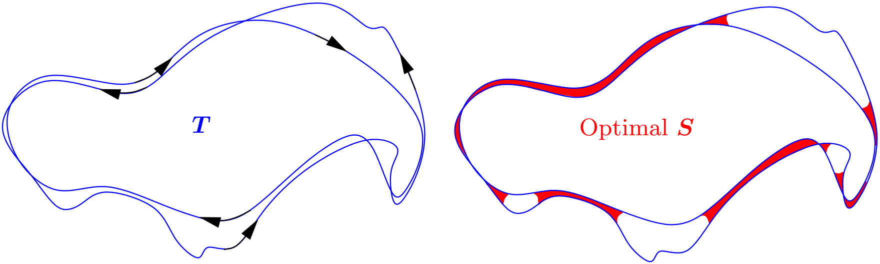
Forms:

$$\mathbb{F}(T) = \sup\{T(\phi) | \phi \leq 1, d\phi \leq 1\}$$

Decomposition:

$$\mathbb{F}(T) = \min_{S \in \mathcal{D}_n} \{M(S) + M(T - \partial S\}$$

The Flat Norm: A picture



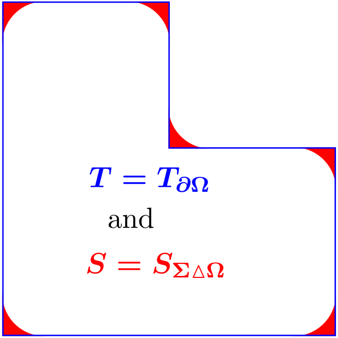
$$\mathbb{F}(T) = M(S) + M(T - \partial S)$$

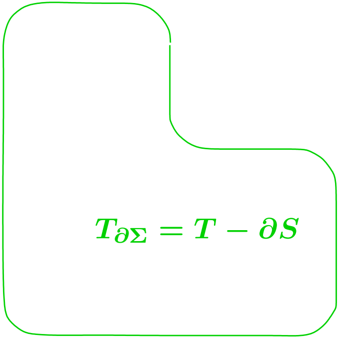
Result: L1TV computes the Flat Norm

Now for the key result:

$$F_1(\Sigma(\Omega, 1)) = \mathbb{F}(\partial\Omega)$$

A Picture:





Flat Norm with Scale

Inspired by this result, we introduce the *flat norm with scale*:

Forms:

$$\mathbb{F}_{\lambda}(T) = \sup\{T(\phi) | \phi \leq 1, d\phi \leq \lambda\}$$

Decomposition:

$$\mathbb{F}_{\lambda}(T) = \min_{S \in \mathcal{D}_n} \{\lambda M(S) + M(T - \partial S\}$$

and get ...

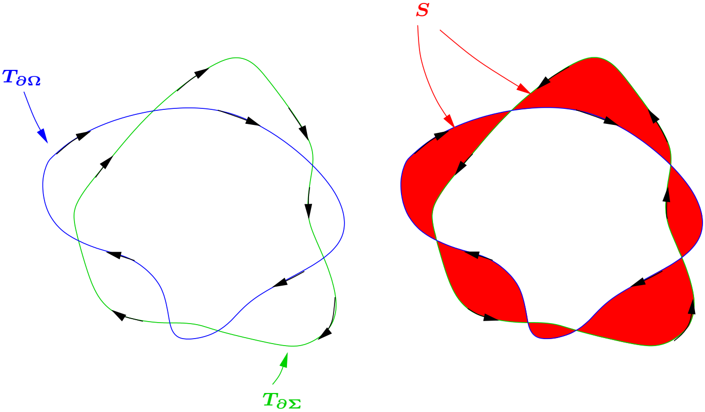
L1TV computes the Flat Norm with Scale

We obtain the natural generalization of the first result:

$$\boldsymbol{F_\lambda}(\boldsymbol{\Sigma}(\boldsymbol{\Omega}, \boldsymbol{\lambda})) = \mathbb{F}_{\boldsymbol{\lambda}}(\partial\boldsymbol{\Omega})$$

Now we sketch the proof with a picture.

Pictorial Proof that L1TV computes the Flat Norm



$T_{\partial\Sigma} = T_{\partial\Omega} - \partial S_{\Omega \triangle \Sigma}$

$$
\begin{aligned}
F(\Sigma) &= \lambda|\Sigma \triangle \Omega| + \mathrm{Per}(\Sigma) \\
&= \lambda M(S_{\Sigma\triangle\Omega}) + M(T_{\partial\Sigma}) \\
&= \lambda M(S_{\Sigma\triangle\Omega}) + M(T_{\partial\Omega} - \partial S_{\Sigma\triangle\Omega})
\end{aligned}
$$

... Now minimize both sides to get result

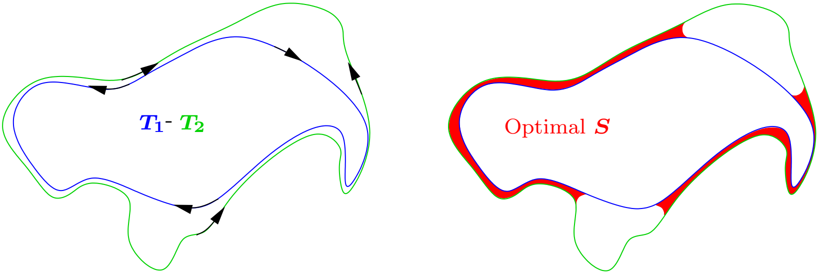
Applications

What does this result get us?

- Computational path to the flat norm and flat norm decomposition.
- Generalization of L1TV to non-boundaries and higher codimension.
- A new nonlinear, multiscale decomposition of general shapes.
- A practical distance in shape space: Statistics on shapes, as suggested by Glaunès et al.

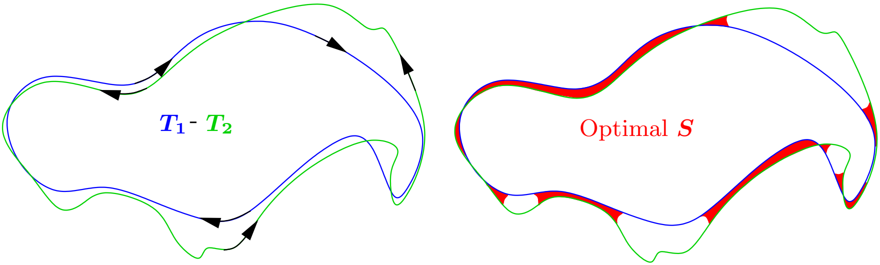
Applications: Computation of Flat Norm

Using L^1TV:



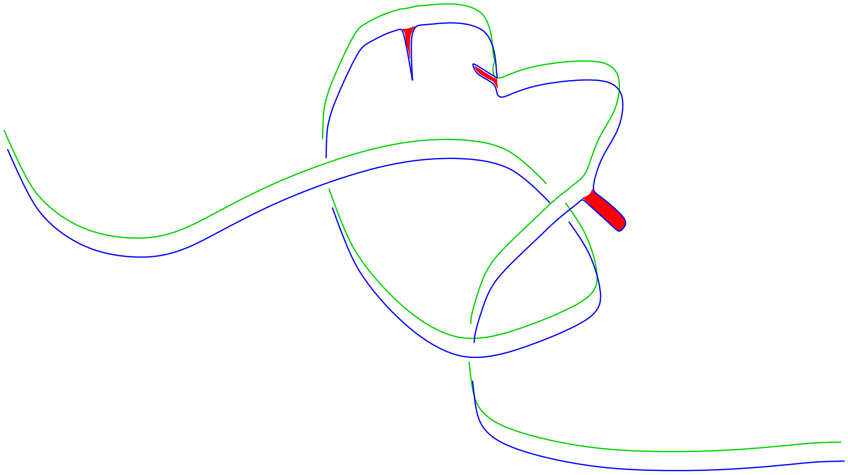
Applications: Computation of Flat Norm

Slight detail: $T_1 - T_2$ below is not the boundary of a set.

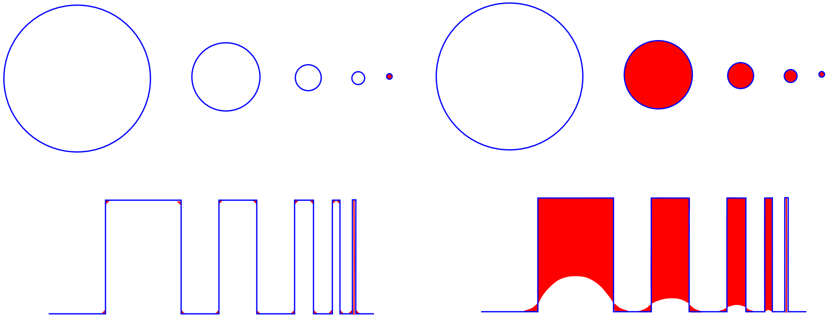


Reasonable solution: consider the difference between $\partial\Omega_1$ and $\partial\Omega_2$ to be $\partial(\Omega_1 \triangle \Omega_2)$. (Or simply use direct methods for calculating $\mathbb{F}_\lambda$.)

Applications: Non-boundaries, codimension > 1



Applications: Multiscale decomposition



$\lambda \gg 1$ $\lambda \ll 1$

Compare: L^1TV work of Chan and Esedoglu, work of Vese and collaborators (UCLA/CAM preprint depository)

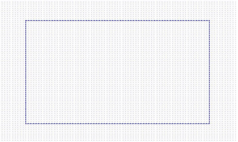
Applications: Shape Statistics

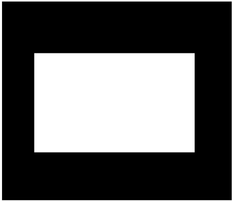
The distance promises to be useful for statistics in shape spaces. For example, one can define a scale dependent mean of the shapes $\{T_i\}_{i=1}^{i=N}$ as a shape $\hat{T}_\lambda$ such that:

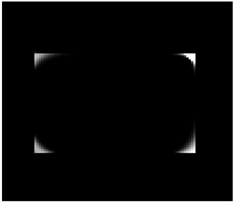
$$\hat{T}_\lambda = \mathbf{argmin}_{T \in \mathcal{D}_n} \sum_{i=1}^{i=N} \mathbb{F}_\lambda(T_i - T)$$

(See Glaunès and Joshi's work and Joshi's talk on Friday)

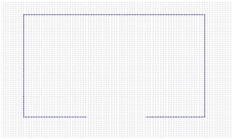
Numerical Examples

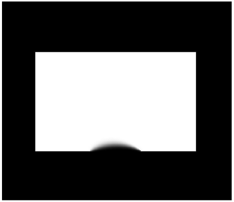


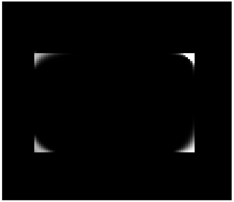




Numerical Examples







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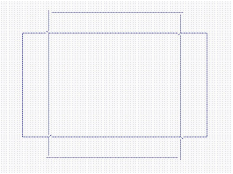

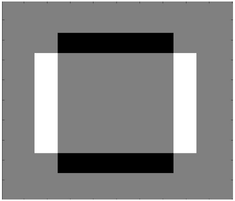

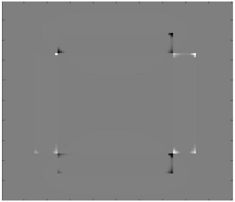

Summary and Next Steps

Current work includes:

- A conjecture relating $\mathcal{F}$ and $\mathbb{F}$ when T is an integral current. This will rigorously justify the dual calculations to get integral minimizers. (see S. Morgan and KRV)
- Numerical methods for higher codimension implementing the generalization of L^1TV to higher codimension.
- Applications of the multiscale distances and decompositions to various shape and image analysis problems.
- Development of statistics in shape spaces using the flat norm.

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