Understanding Search Trees via Statistical Physics

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March 12, 2007

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Sorting and Search

The Goal: Store data efficiently so that the search time is minimum

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Linear Sorting: Store the data sequentially onto a linear table

$$\{6, 4, 5, 8, 9, 1, 2, 10, 3, 7\}$$

Search for 7: Search proceeds sequentially by comparison

$$t_{\rm search} = 10 \sim O(N) \rightarrow \text{BAD}$$

Tree Sorting: of $\{6, 4, 5, 8, 9, 1, 2, 10, 3, 7\}$

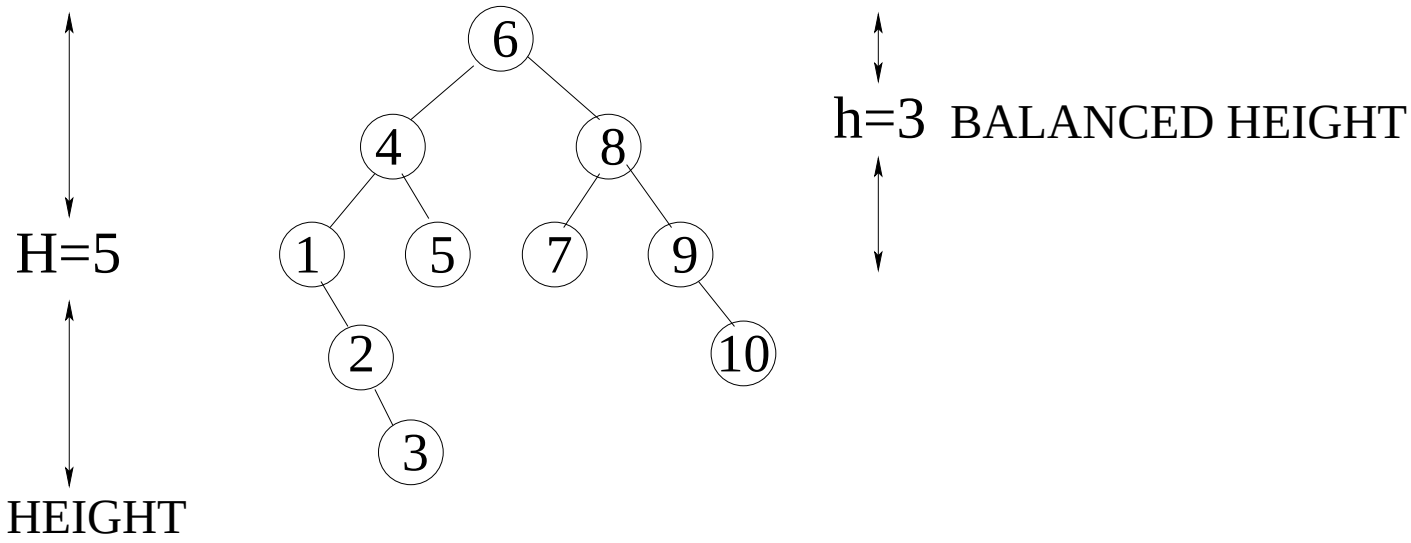

Figure: Binary Search Tree with $N = 10$ Elements.

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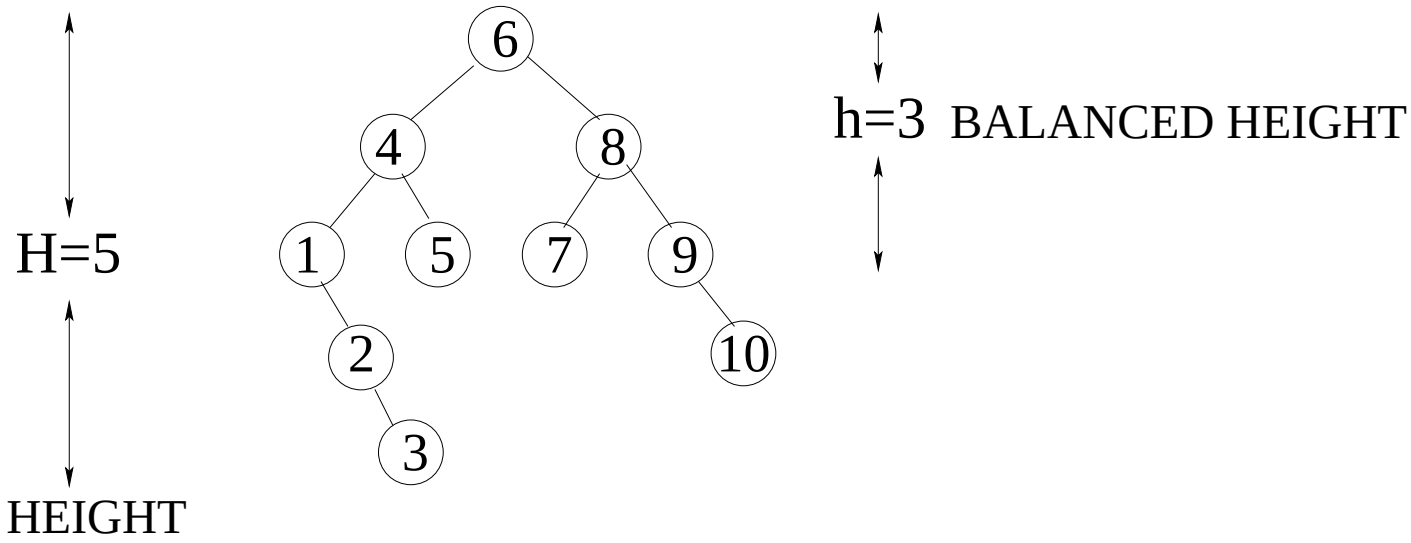

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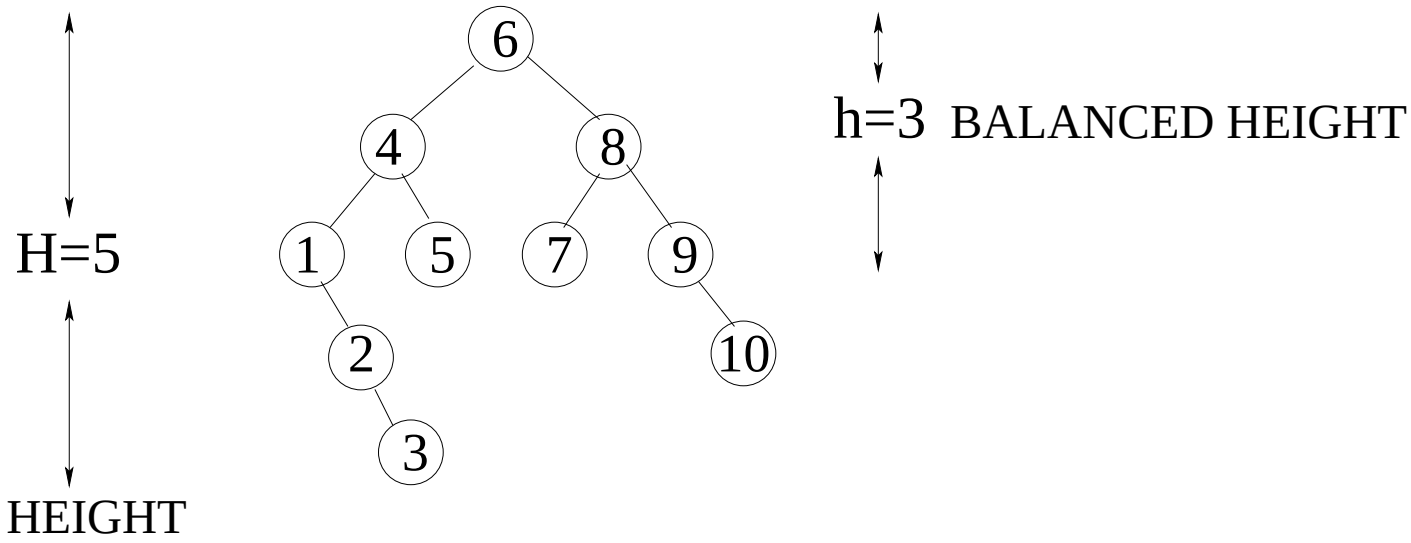

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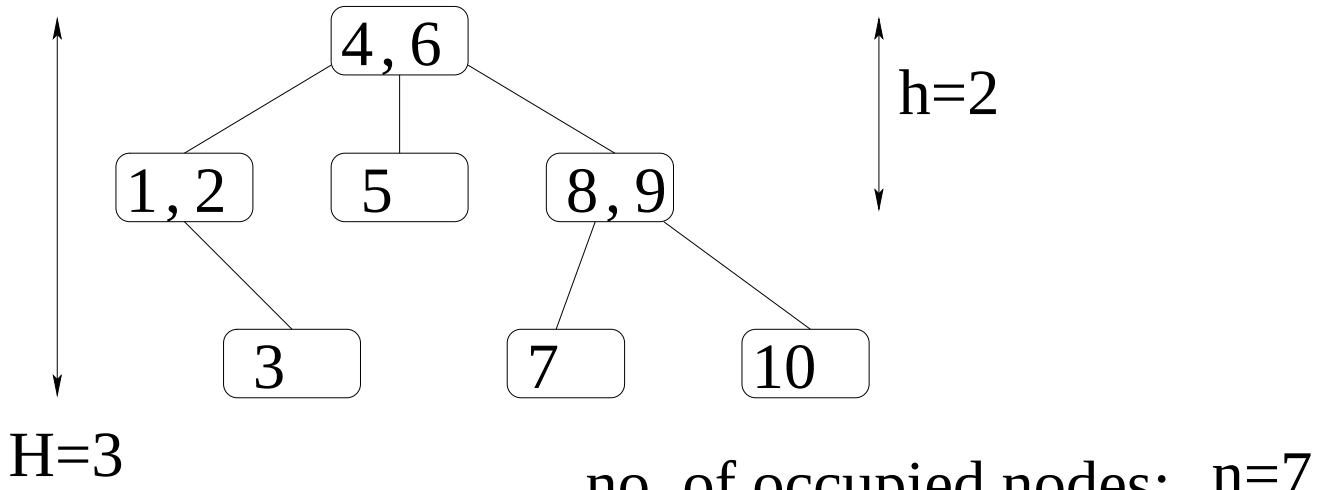
Generalization to m-ary Search Trees

$m = 2 \rightarrow$ Binary Tree

Random Sequence: $\{6, 4, 5, 8, 9, 1, 2, 10, 3, 7\}$

Each node can contain atmost $(m - 1)$ elements.



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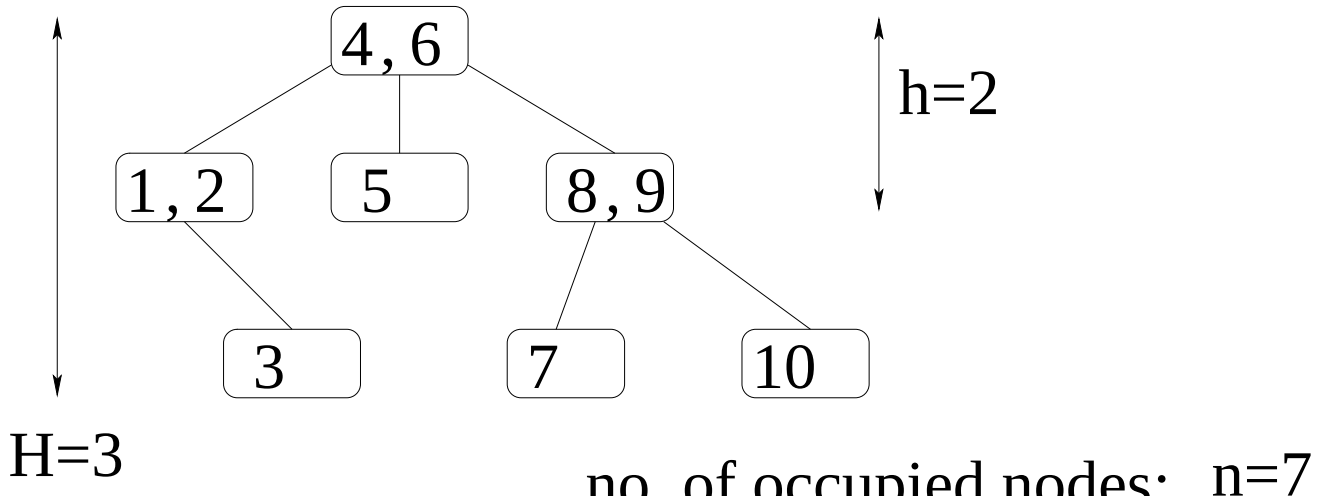

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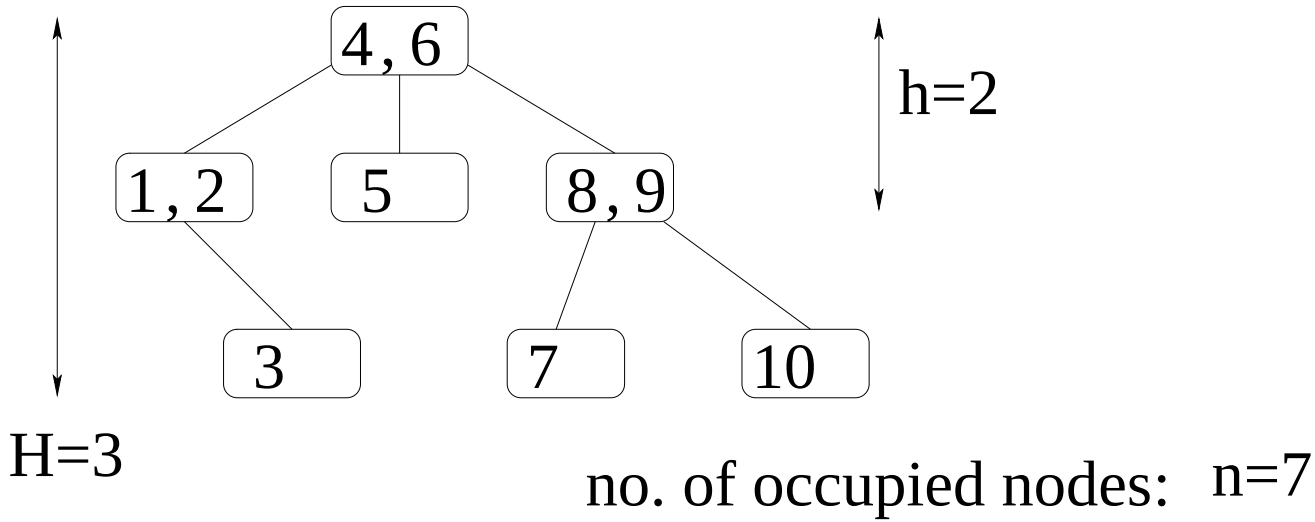

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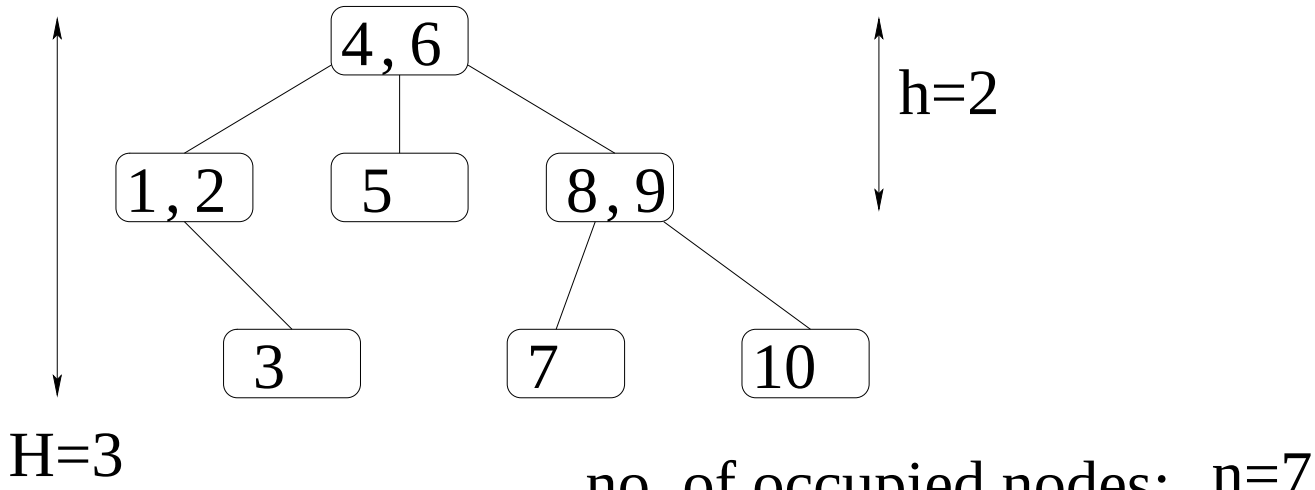

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No. of NON-EMPTY nodes: $n = 7 \rightarrow$ No. of nodes required to store the data

Random m-ary Search Tree Model: *RmST*

$N = 10$ data elements: $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
Each permutation $\longrightarrow$ an m-ary tree.

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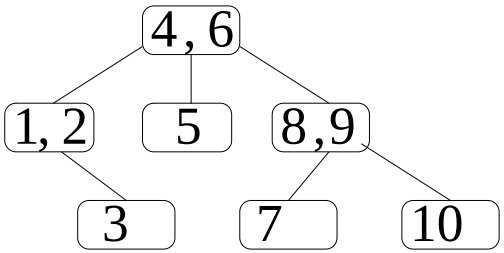

H=3, h=2, n=7

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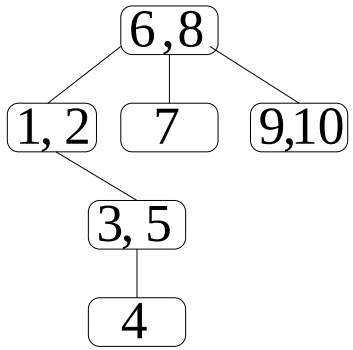

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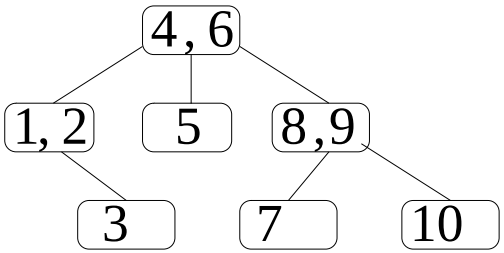

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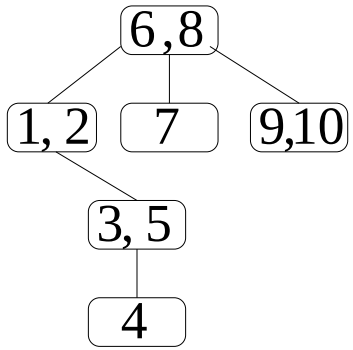

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Q: Statistics of HEIGHT H_N, BALANCED HEIGHT h_N and the no. of NON-EMPTY NODES n_N for RANDOM data of size N?

Asymptotic Results for RmST: for large data size N

(1) Height H_N:

- $\langle H_N \rangle \approx a_m \log(N) + b_m \log(\log(N))$ (??) $+ \ldots$

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The correction terms $\rightarrow$ conjectured by Hattori and Ochiai (simulations, 2001).

Other results by Robson (2001-), Reed (2001-), Drmota (2001-).

Asymptotic Results for RmST: for large data size N...continued

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A striking PHASE TRANSITION occurs for the Variance: $\nu_N = \langle \left(n_N - \langle n_N \rangle \right)^2 \rangle$.

$$\begin{aligned} \nu_N &\sim N && \text{for } m \leq 26 \\ &\sim N^{2\theta(m)} && \text{for } m > 26 \end{aligned}$$

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Q: Why 26? What is the mechanism of this Phase Transition and how generic is it? Can one calculate $\theta(m)$ exactly ?

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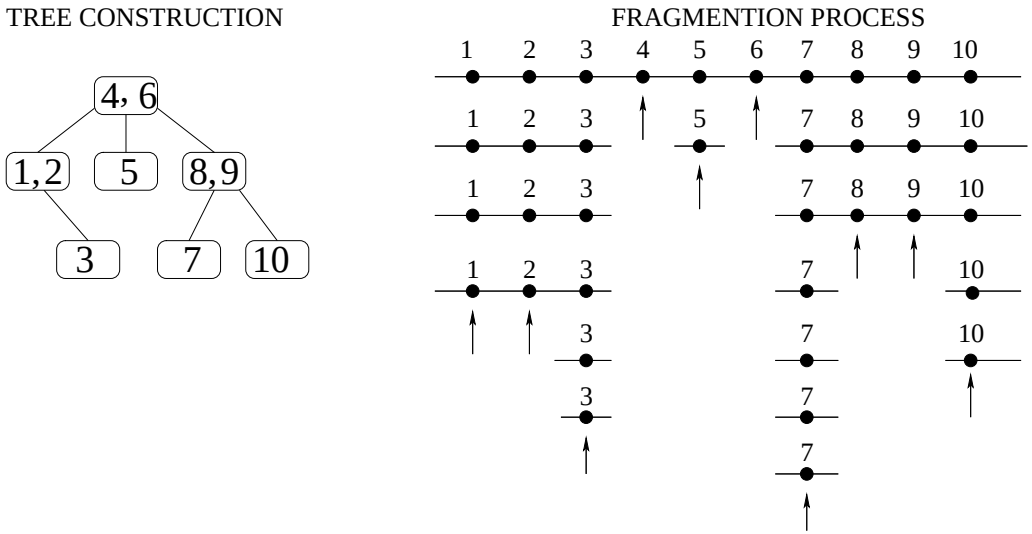
- Various other generalizations: Vector Data

The Mapping to a Fragmentation Process

Construction of the Tree $\rightarrow$ Dynamical Fragmention Process: Split an interval into m pieces with the break points chosen randomly. An interval can split iff it contains atleast one point.

Ex: Consider the data: $\{6, 4, 5, 8, 9, 1, 2, 10, 3, 7\}$ on a $(m = 3)$-ary tree



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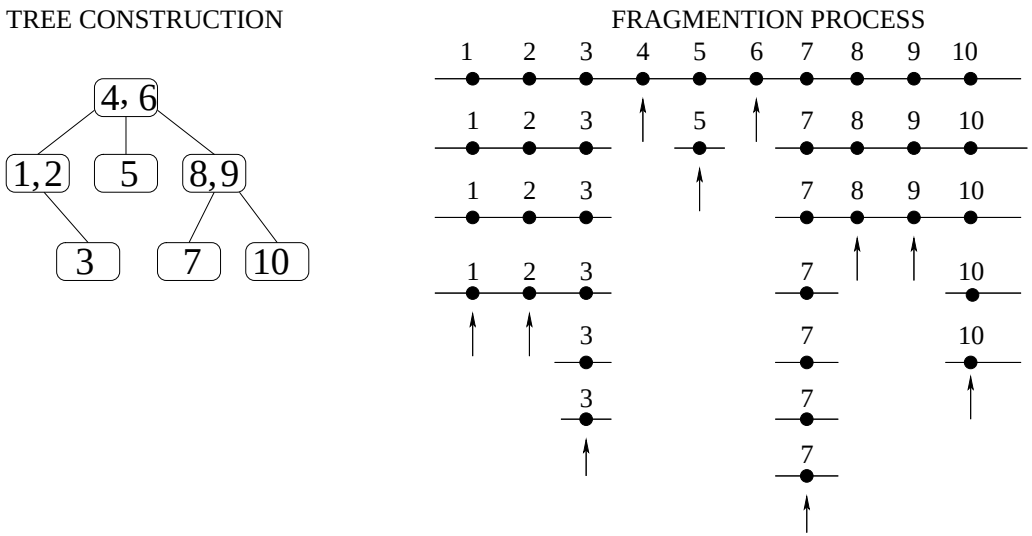
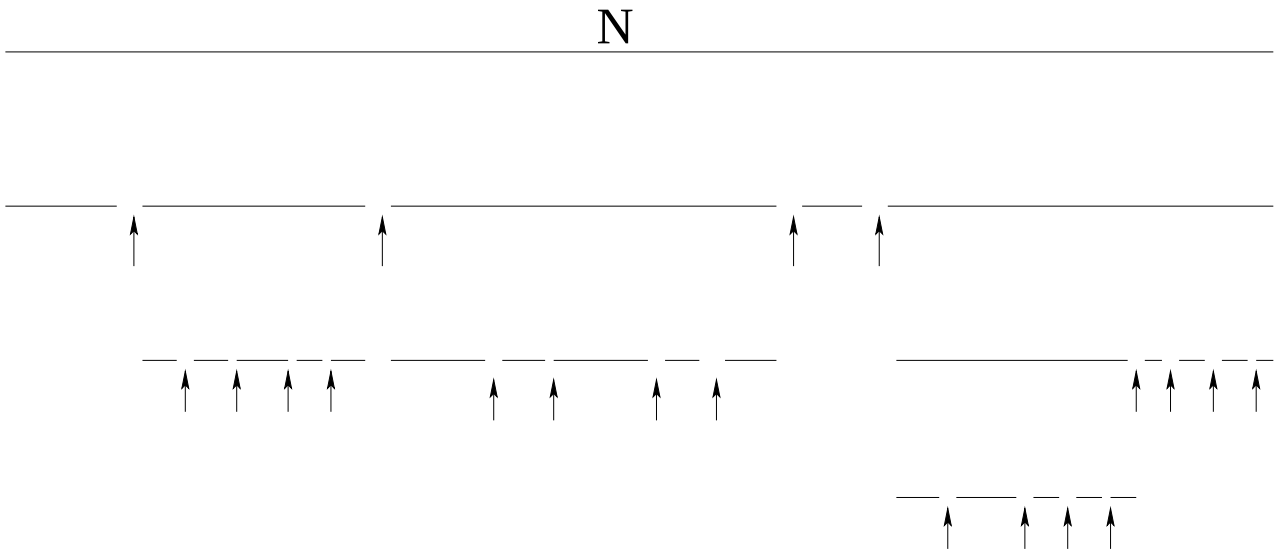


NOTE:

No. of NONEMPTY nodes n=7= No. of SPLITTING EVENTS

Fragmentation Process: Continuum Limit



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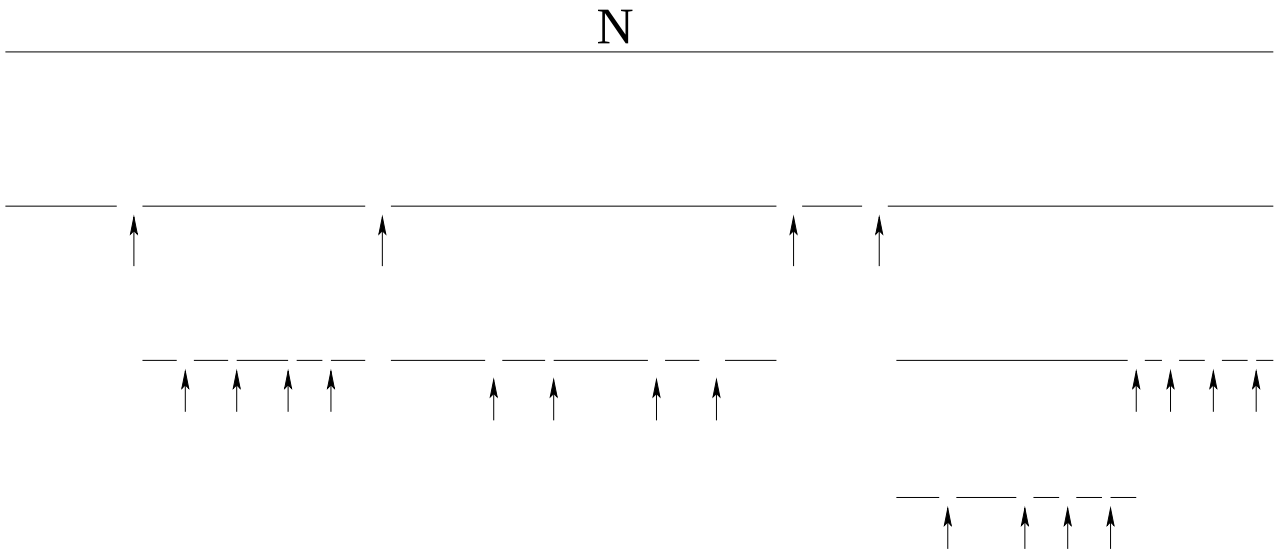

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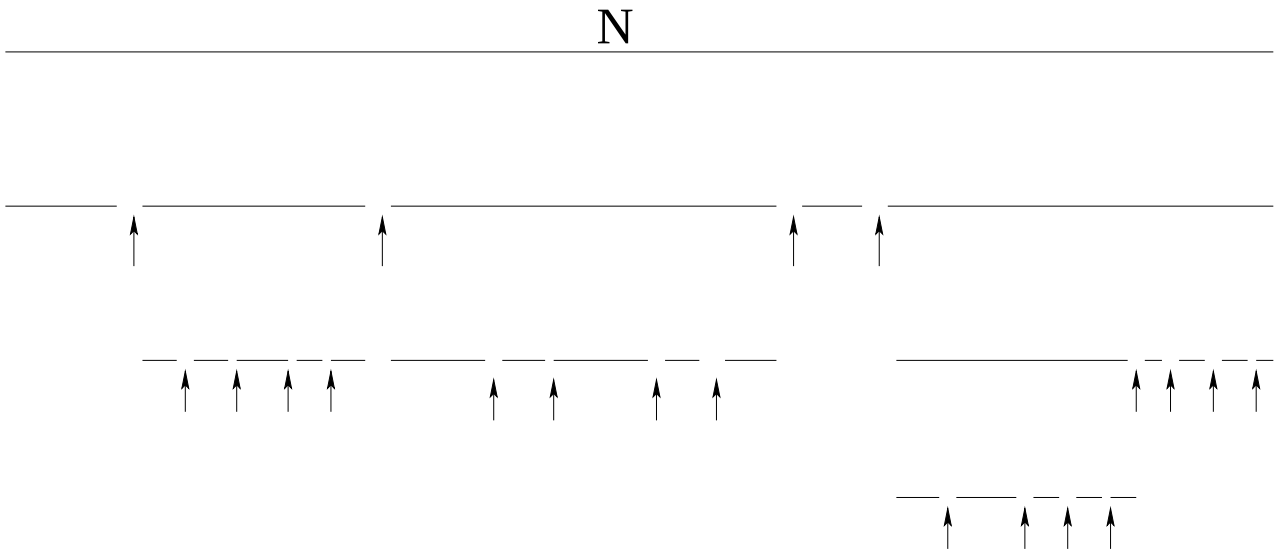

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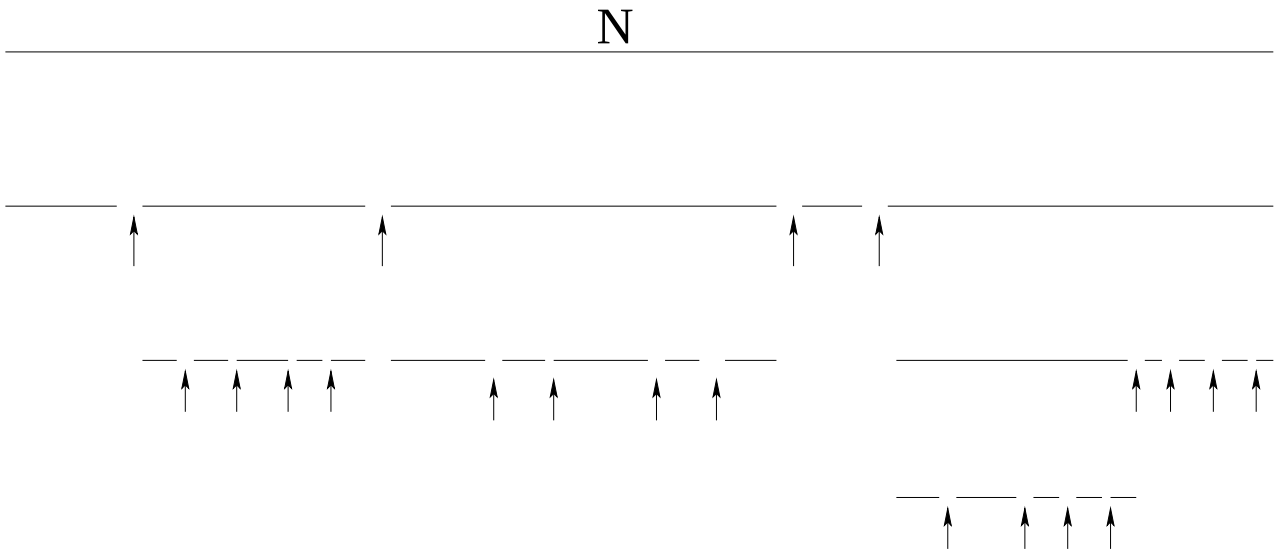

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4. Repeat the process till all pieces have length < 1 and then STOP.

DICTIONARY Between the Search Tree and the Fragmentation Process:

m-ary SEARCH TREE $\equiv$ FRAGMENTATION PROCESS

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Height H_N:

- Prob$[H_N < n]$= Prob[$l_1 < 1$, $l_2 < 1$, ... after n steps]

Balanced Height h_N:

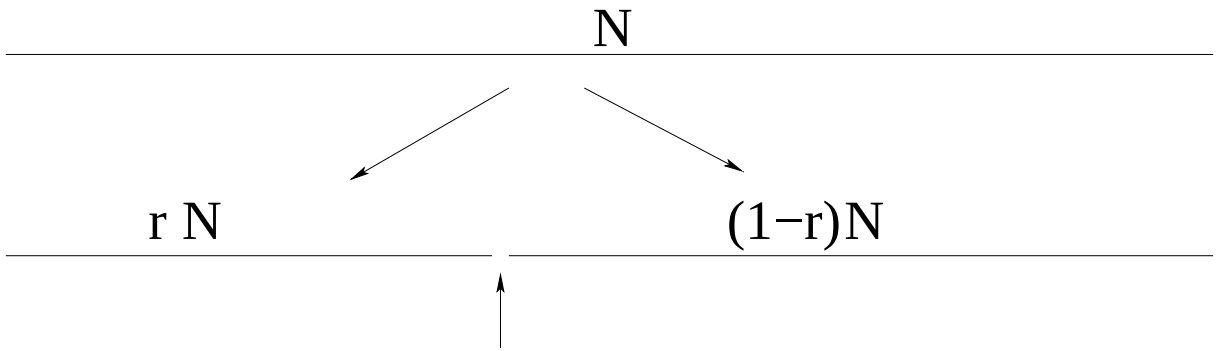
- Prob$[h_N > n]$=Prob[$l_1 > 1$, $l_2 > 1$, ... after n steps]

Number of Nonempty Nodes n_N $(m > 2)$:

- Prob$[n_N = n]$= Prob[there are n SPILLITING EVENTS till the end of the Fragmentation process].

Analysis of HEIGHT H_N

P(n,N)= Prob[$H_N < n$]= Prob[$l_1 < 1$, $l_2 < 1$, ... after n steps starting with initial length N]



Recursion: $P(n,N) = \int_0^1 P\,(n-1, rN)\, P\,(n-1, (1-r)N)\, dr$ starting with $P(n,1) = \theta(n-1)$.

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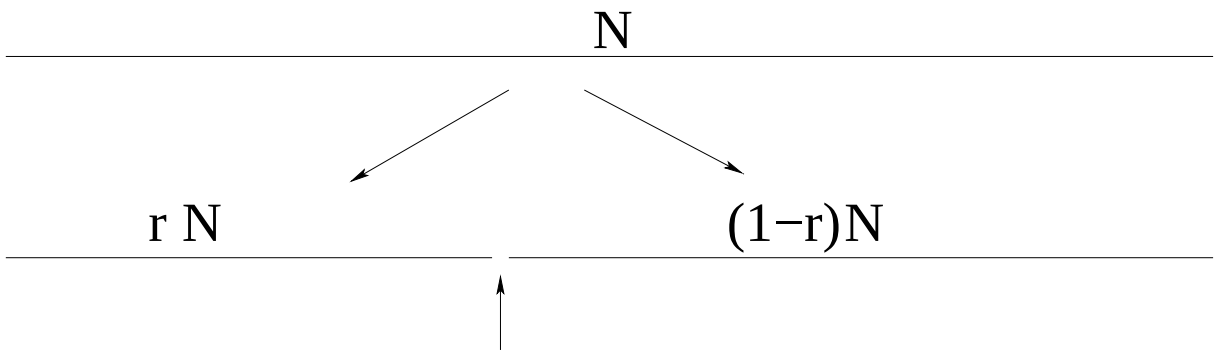

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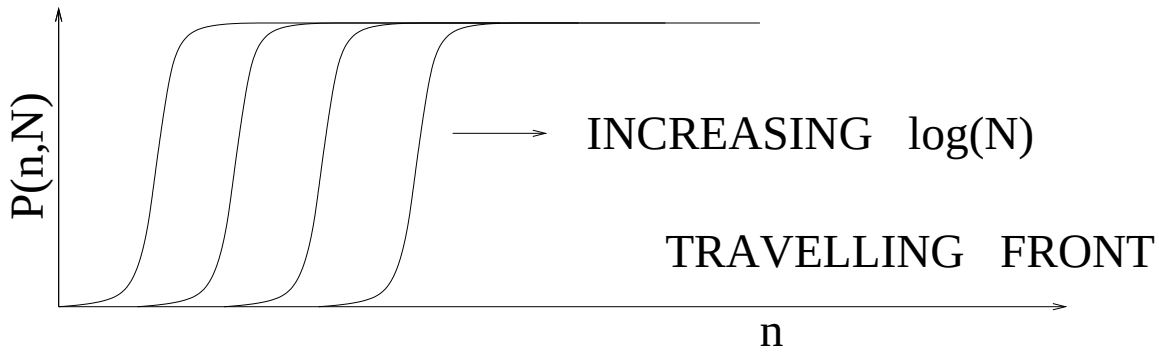

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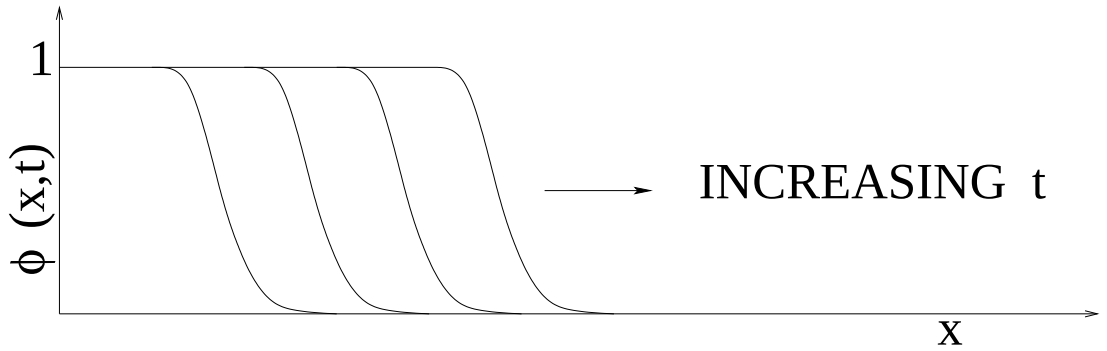

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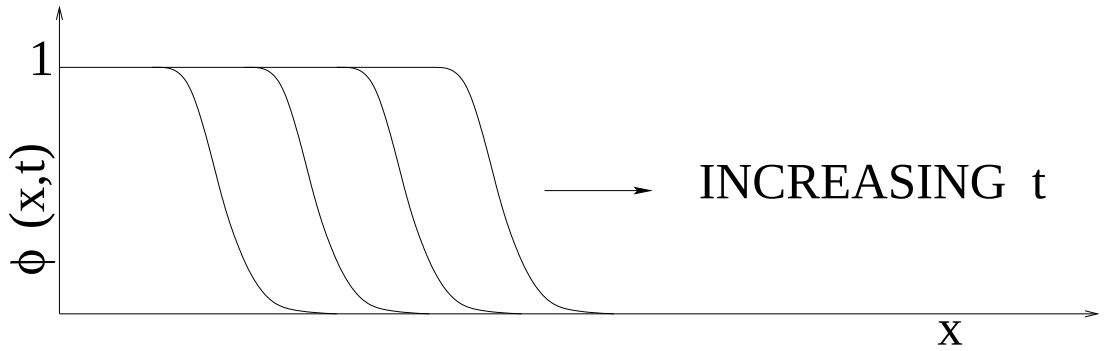
$\phi(x) = 1 \rightarrow$ STABLE Fixed point. $\phi(x) = 0 \rightarrow$ UNSTABLE Fixed point.



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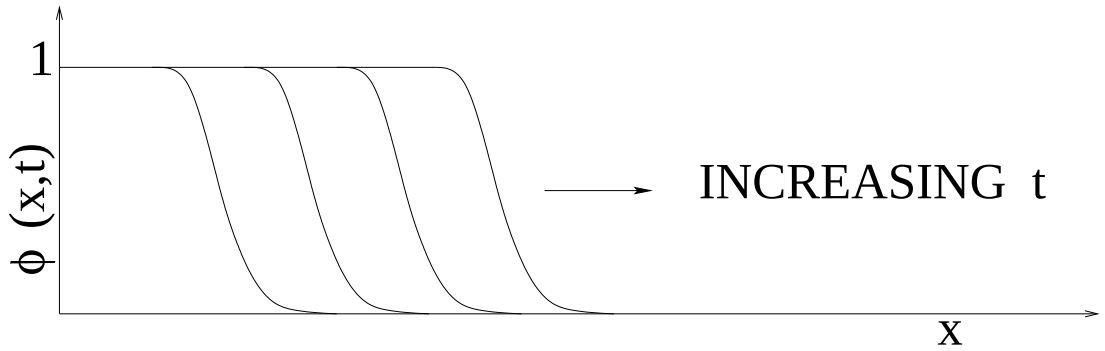

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Q: How to determine the Front Velocity v?

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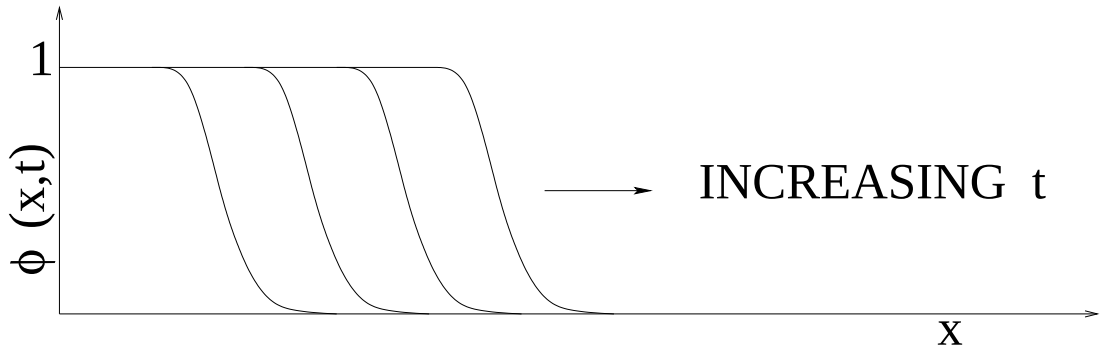

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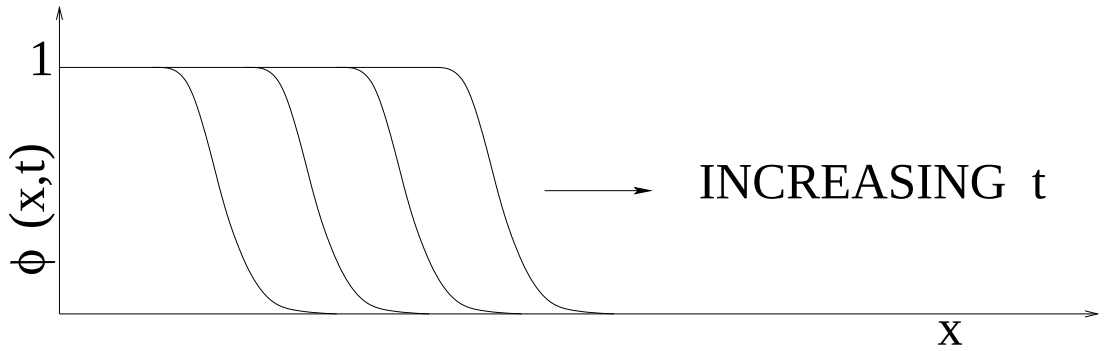

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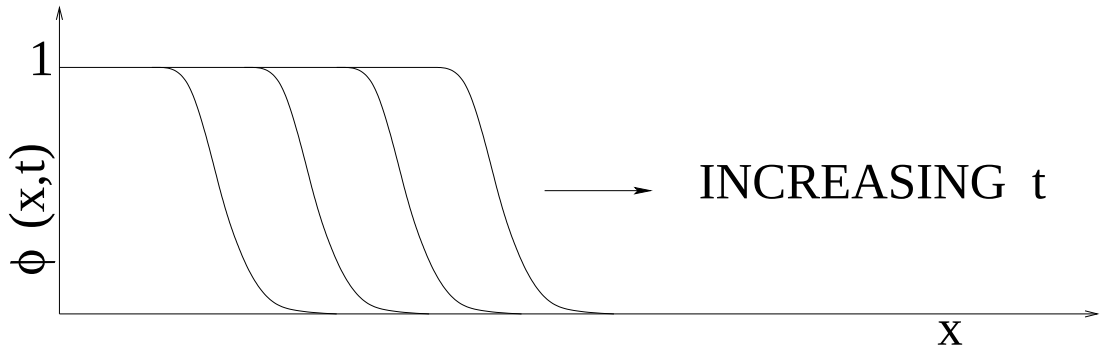

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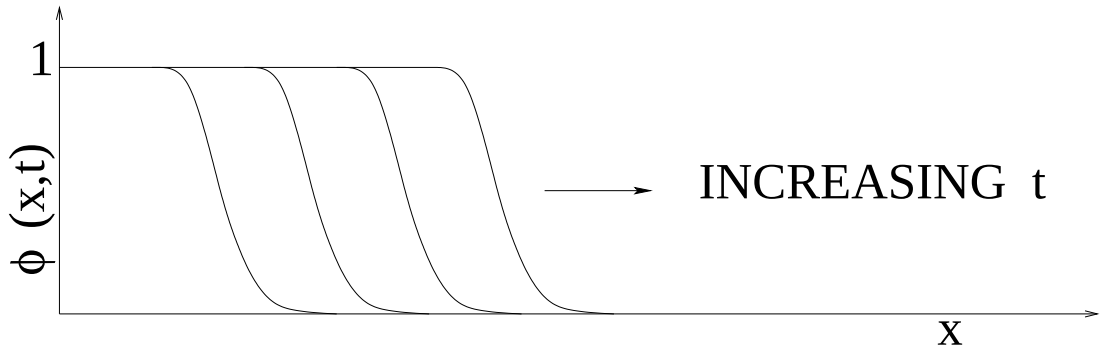

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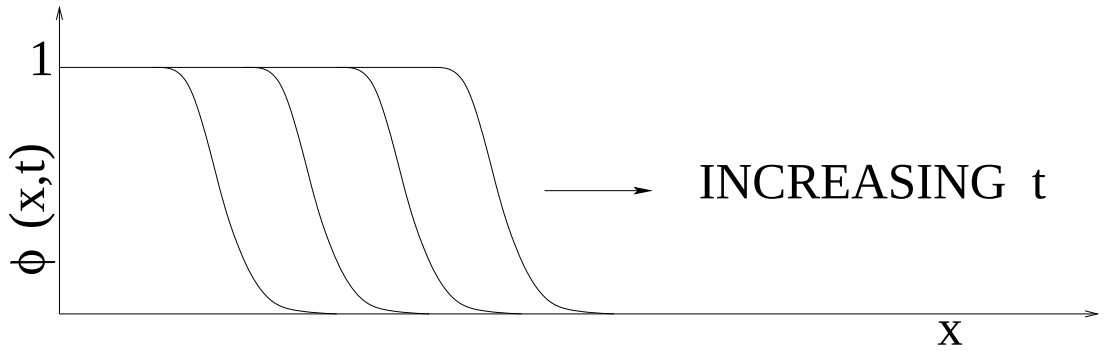

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(Bramson, Brunet & Derrida, van Saarloos, ...)

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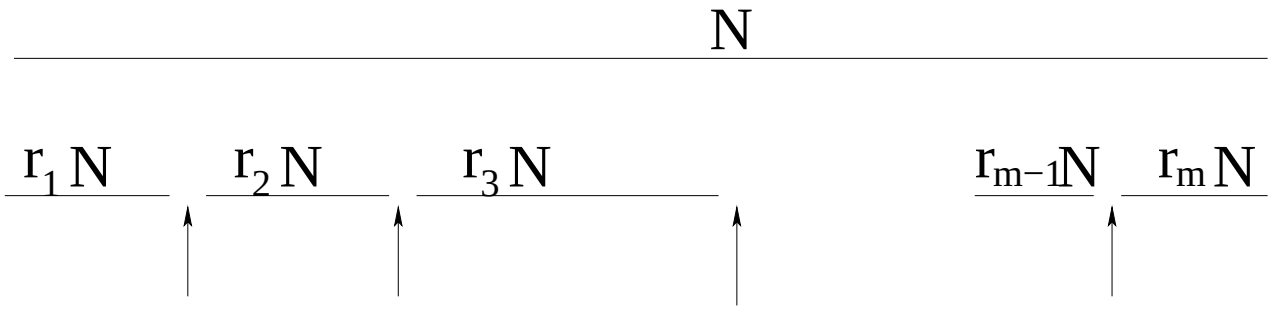
Same strategy holds for the Balanced Height h_N.

No of Non-Empty Nodes:



$$r_1 + r_2 + r_3 + \ldots\ldots + r_m = 1$$

No. of Non-empty nodes $n(N)$ in the tree $\equiv$ Total no. of Splitting Events in the fragmentation process till the end, starting with the initial length N

Recursion:

$$n(N) \equiv n(r_1 N) + n(r_2 N) + n(r_3 N) + \cdots + n(r_m N) + 1; \qquad \sum_i^n r_i = 1$$

The marginal distribution of any fragment: $\eta_m(r) = (m-1)(1-r)^{m-2}$

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For $m > m_c = 26.0461...$,$\theta(m) = \lambda(m)$. (Dean and S.M., 2002).

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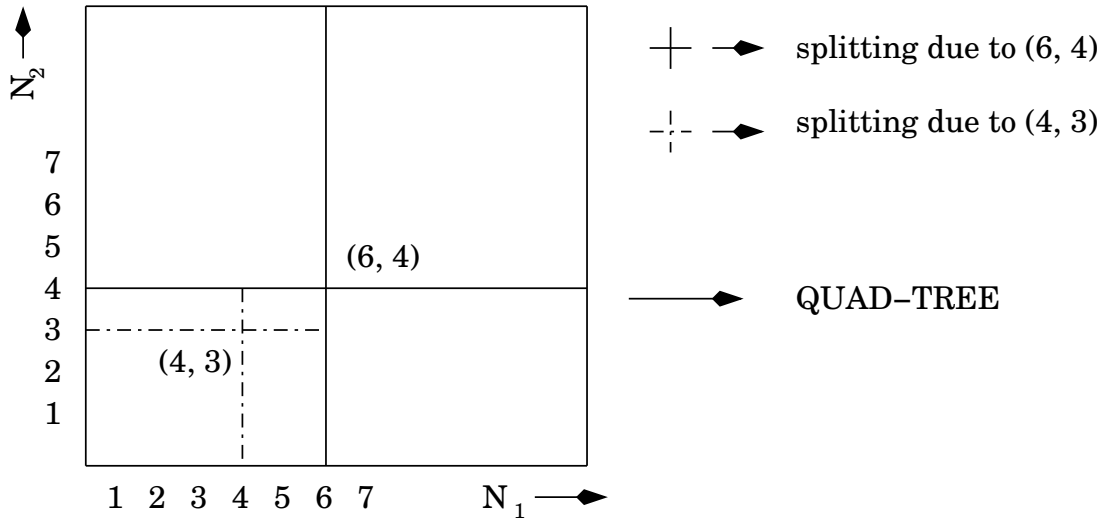

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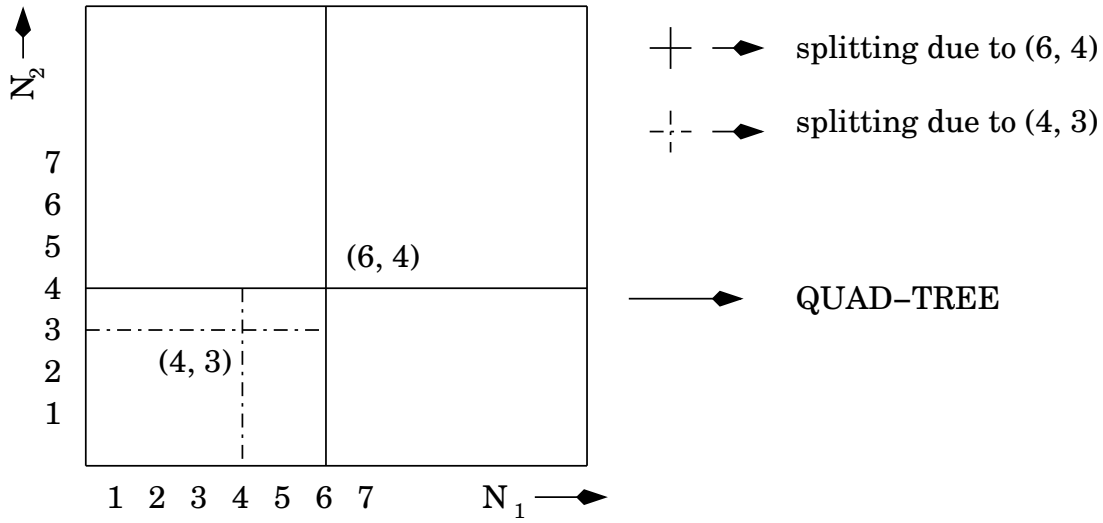

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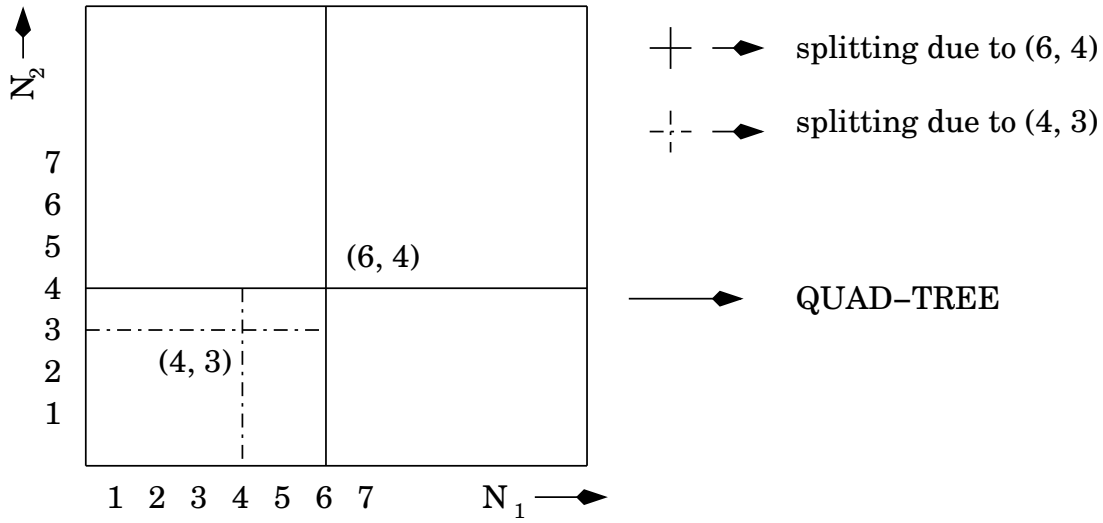

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Is there a PHASE TRANSITION in the variance of n_N?

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No. of Non-empty Nodes n_N : $\langle n_N \rangle \approx \frac{2}{D} V$ where $V = N^D$.

Variance ν_N has a Phase Transition:

$$\nu_N \sim V \qquad \text{for } D \leq D_c$$
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Exact Results for Vector Data of N D-tuples for Large N:

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$\theta(D) = 2\cos(\frac{2\pi}{D}) - 1 \rightarrow$ increases continuously with D for $D > D_c$

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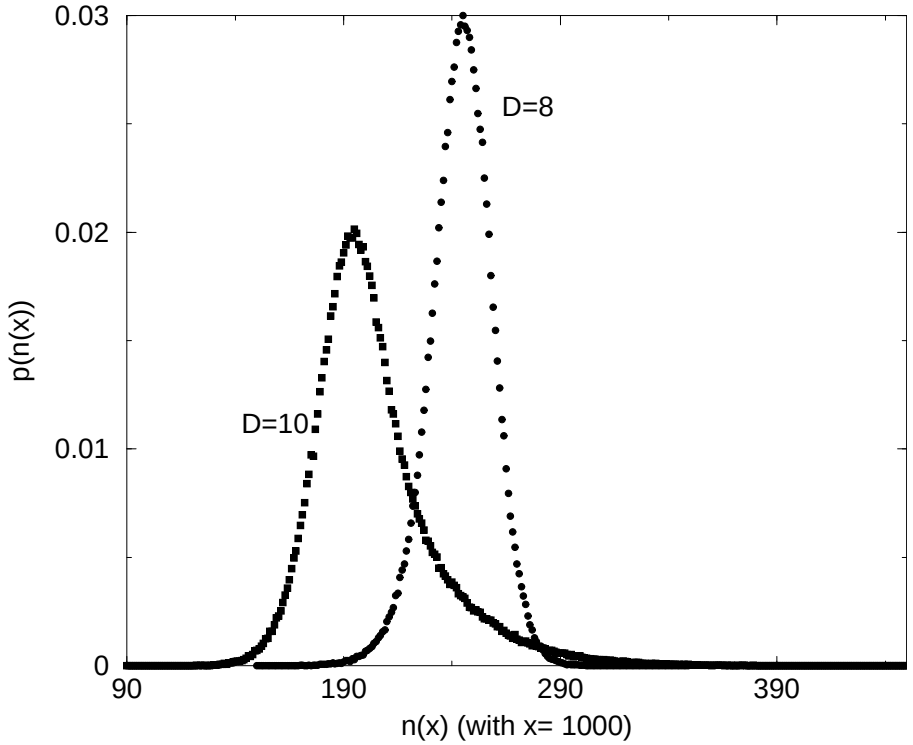

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Perspectives: Lots of beautiful open problems in Sorting and Search that may be possible to handle by using statistical physics techniques.

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