Harmonic Multifractal Spectra for Multiple SLEs

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RANDOM SHAPES

Random Curves, Surfaces

& Transport

IPAM

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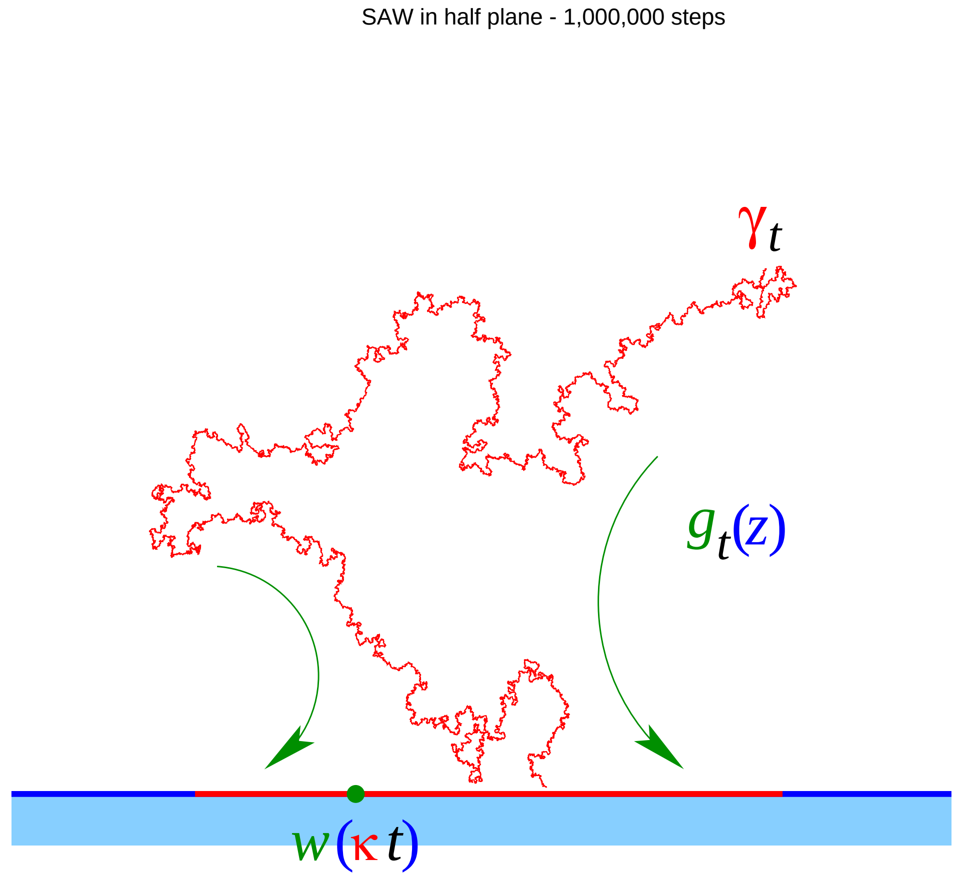
Stochastic Löwner Evolution (SLE$_\kappa$) (SCHRAMM)



$$\partial_t g_t(z) = \frac{2}{g_t(z) - w(\kappa t)}$$

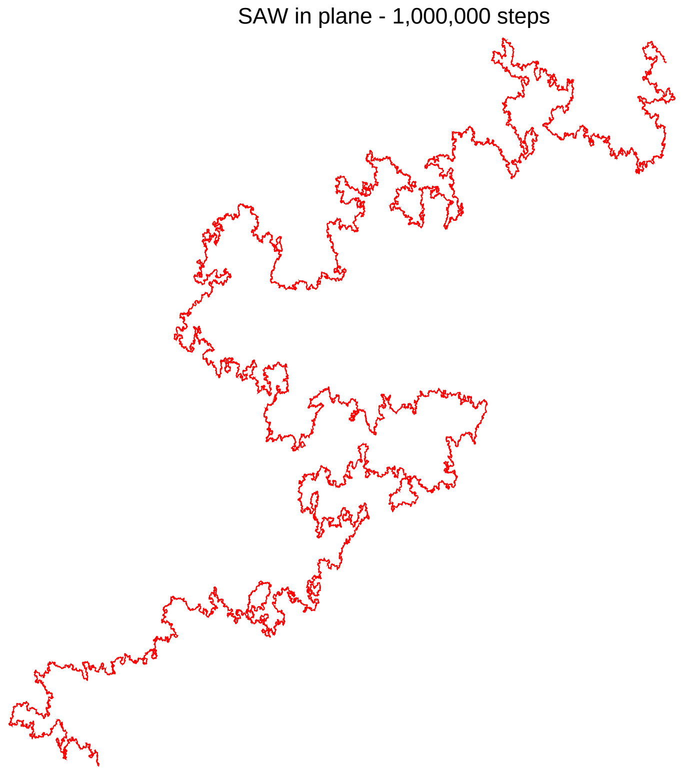
A Stochastic Löwner Evolution?

A Macroscopic SLE



$$\partial_t g_t(z) = \frac{2}{g_t(z) - w(\kappa t)}$$

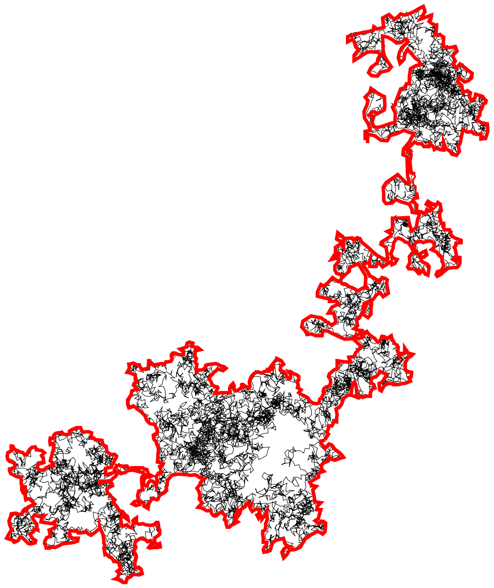
Self-Avoiding Walk & $\mathrm{SLE}_{\kappa=8/3}$



(Courtesy of T. Kennedy)

B. Nienhuis (1982): $D = \frac{4}{3}$, $\langle r^2 \rangle \propto N^{3/2} a^2$

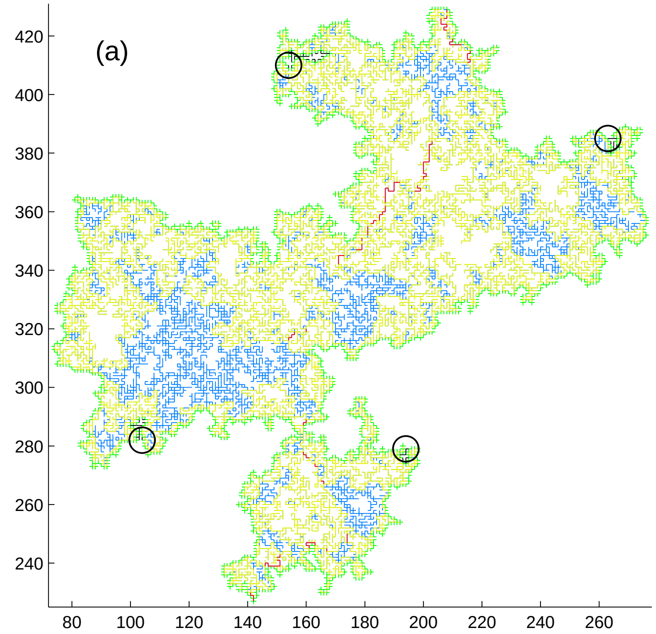
Planar Brownian Frontier



Mandelbrot conjecture (1982): Hausdorff dimension $D = \frac{4}{3}$.
In the plane, the Brownian frontier is the scaling limit of a self-avoiding walk.

B. D., 1998 (quantum gravity); G. F. Lawler, O. Schramm & W. Werner, 2000 (SLE). [Percolation External Perimeter: *M. Aizenman, B. D. & A. Aharony, 1999; S. Smirnov; LSW, 2001; V. Beffara, 2002.*]

Percolation Cluster Hull & Frontier

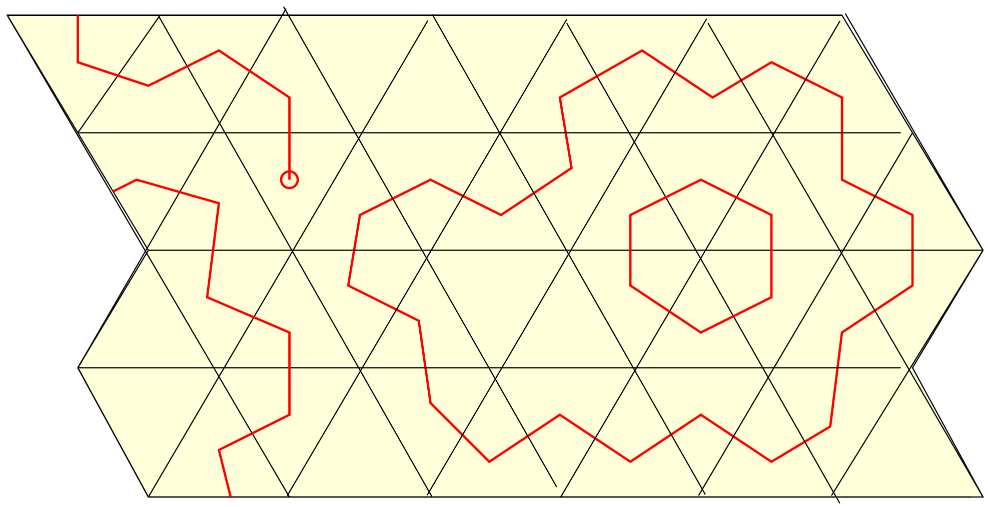


(Courtesy of J. Asikainen et al., 2003)

Hull: $D_{\text{Hull}} = \frac{7}{4}$ *(M. Rosso & B. Sapoval, 1984; B. D. & H. Saleur, 1987; S. Smirnov, 2001; V. Beffara, 2002; F. Camia & C. M. Newman, 2003, 2005); External Perimeter:* $D_{\text{EP}} = \frac{4}{3}$ *(M. Aizenman, B. D. & A. Aharony, 1999; LSW, 2001; Beffara, 2002;)* [DUALITY].

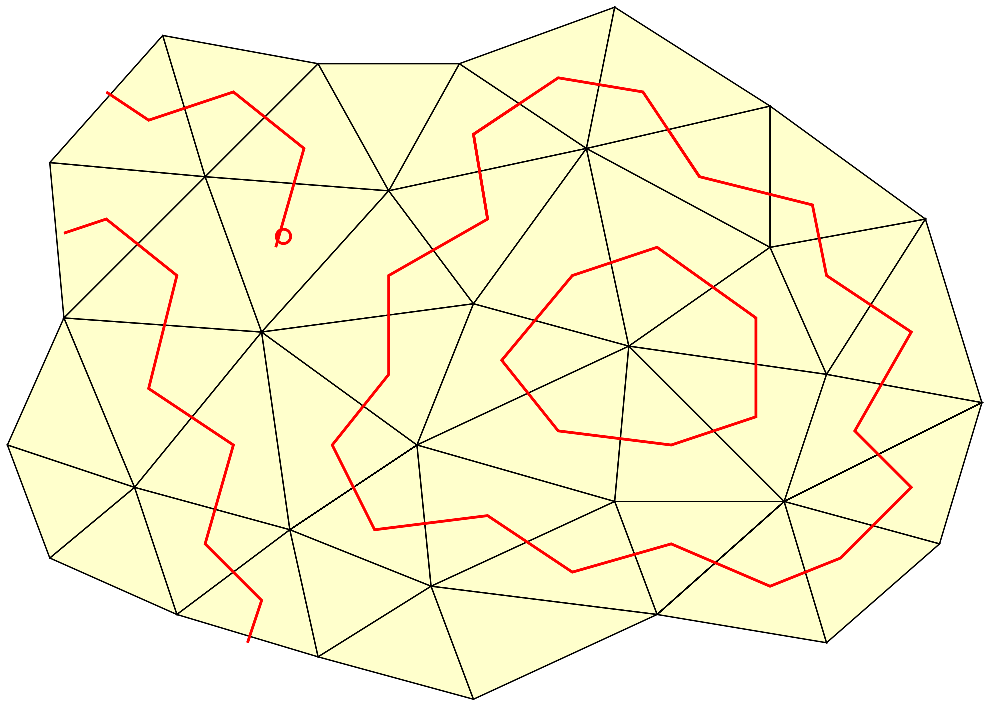
2D Quantum Gravity

Statistical Mechanics on a Regular Lattice



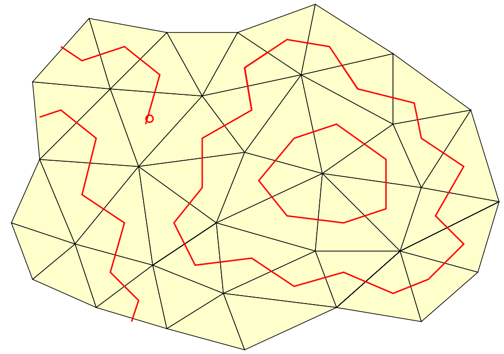
Random lines on the (dual of) a regular triangular lattice.

Statistical Mechanics on a Random Lattice



Statistical model on a random planar triangular lattice.

Partition Function on a Random Lattice



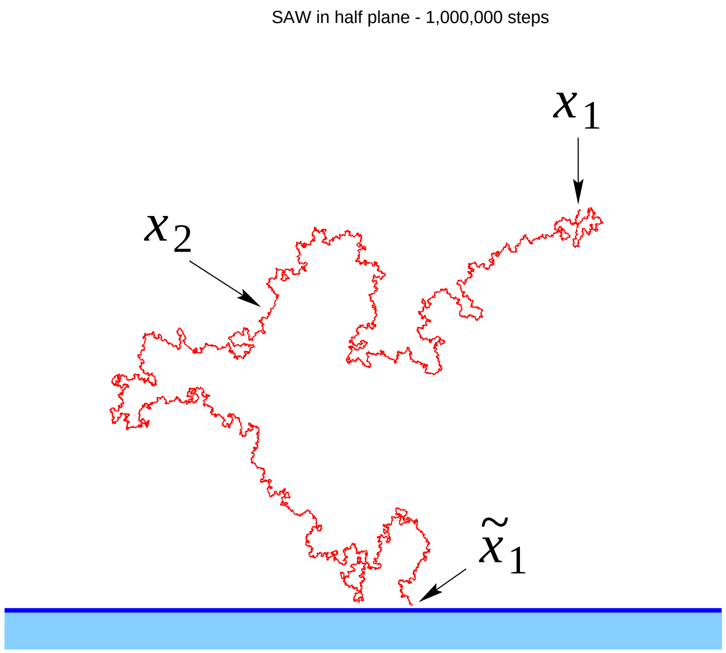
Statistical model $\mathcal{M}$ on random lattice G.

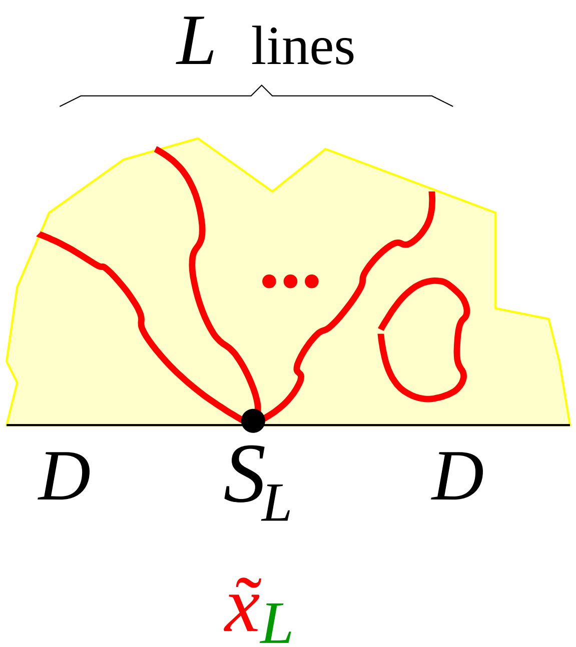
$$Z(\beta) = \sum_{\text{planar } G} e^{-\beta|G|} \, \mathcal{Z}_G$$

$\mathcal{Z}_G$: partition function of the statistical model $\mathcal{M}$ on G.

DOUBLE CRITICAL POINT of $\mathcal{M}$ & G at β_c

Conformal Weights of a Random Path in $\mathbb{C}$ or $\mathbb{H}$





Critical Behavior

Probabilities or partition functions in $\mathbb{C}$ or $\mathbb{H}$

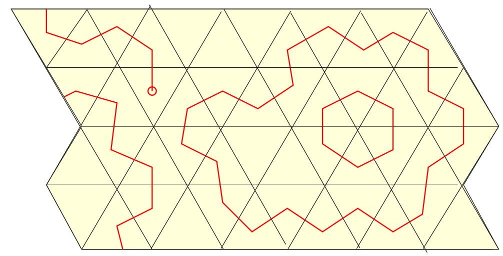
$$\mathcal{Z} \propto \left(\frac{r}{R}\right)^{2x}, \quad \tilde{\mathcal{Z}} \propto \left(\frac{r}{R}\right)^{\tilde{x}}$$

Partition functions in QG

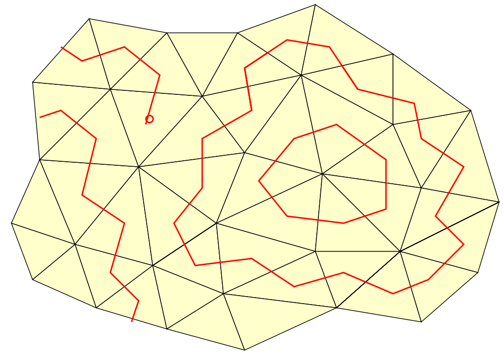
$$Z \propto \langle |G| \rangle^{-\Delta}, \quad \tilde{Z} \propto \langle |\partial G| \rangle^{-\tilde{\Delta}}$$

Typical lattice area $\langle |G| \rangle$, boundary length $\langle |\partial G| \rangle$

KPZ *Knizhnik, Polyakov, Zamolodchikov (88)*



A “conformal operator” O (e.g. creating a line extremity) has conformal weight $x = U(\Delta)$ in $\mathbb{C}$ (or $\tilde{x} = U(\tilde{\Delta})$ in $\mathbb{H}$)



where Δ (or $\tilde{\Delta}$) is the corresponding conformal weight in quantum gravity (or boundary Q. G.)

KPZ & SLE

Conformal dimensions Δ in QG and x in $\mathbb{C}$

$$x = U(\Delta) = \frac{1}{4}\Delta(\kappa\Delta + 4 - \kappa)$$

Inverse KPZ map

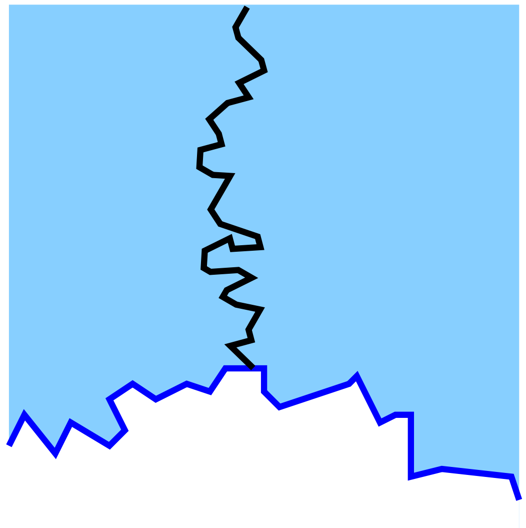
$$\Delta = U^{-1}(x) = \frac{1}{2\kappa}\left(\sqrt{16\kappa x + (\kappa - 4)^2} + \kappa - 4\right)$$

Duality $\kappa \to 16/\kappa$

$$x = U_{\kappa}^{-1}(x) \times U_{16/\kappa}^{-1}(x)$$

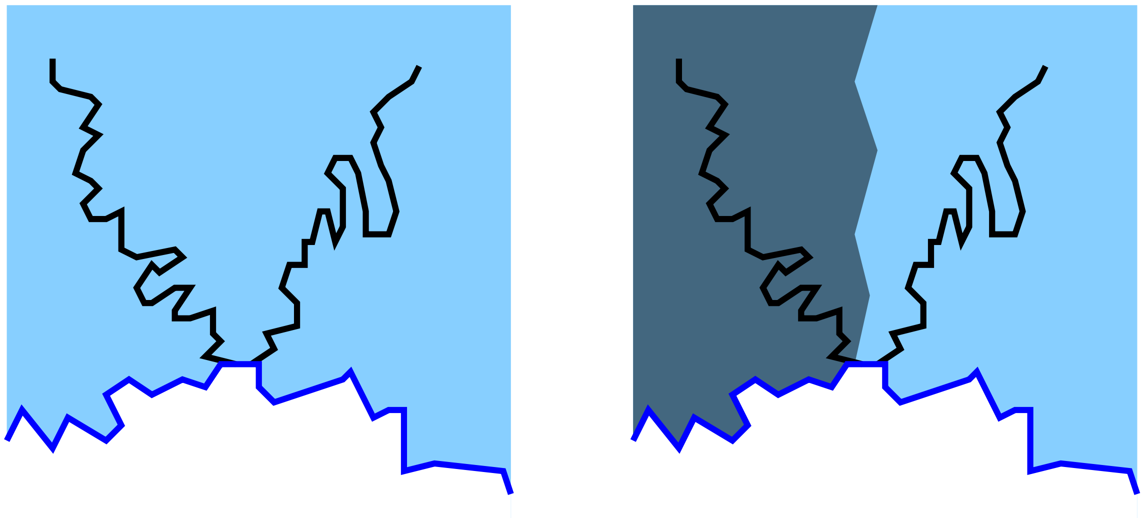
QG Rules

SLE & Boundary QG

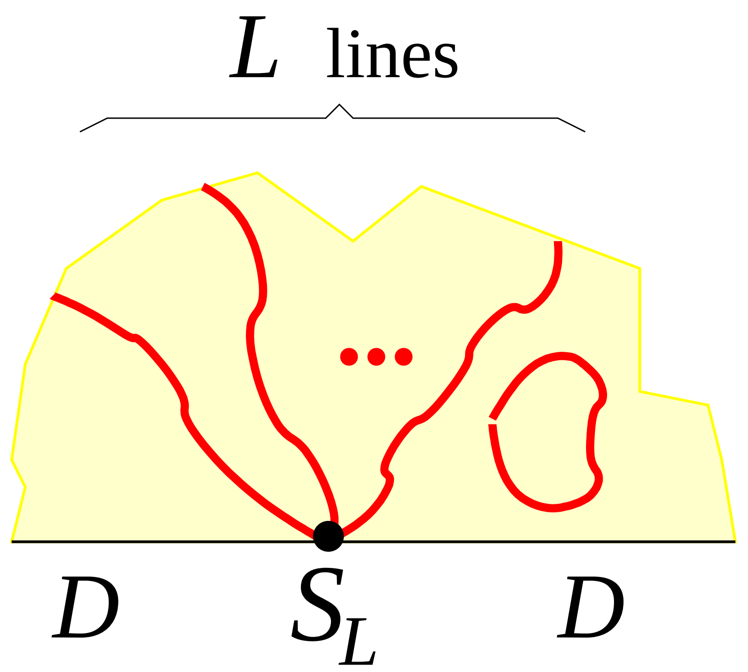


$$\tilde{\Delta}_1 = U_\kappa^{-1}(\tilde{x}_1) = \frac{2}{\kappa}$$

Boundary Quantum Gravity is Additive



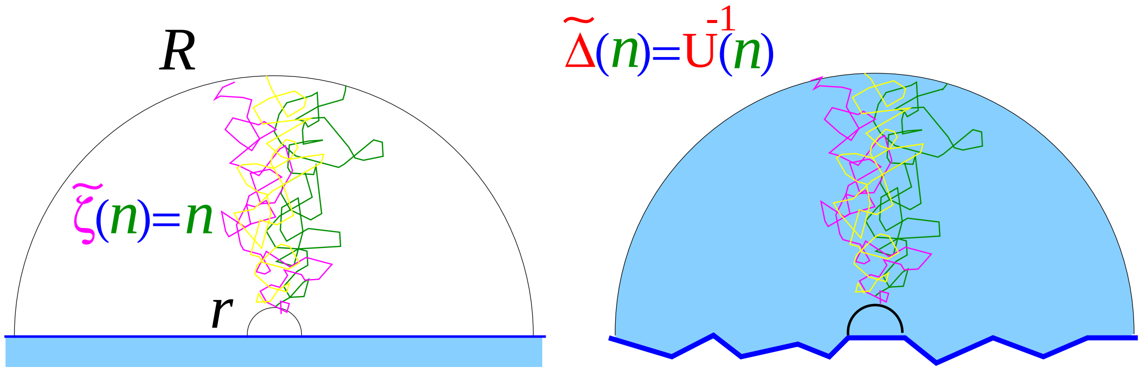
$$U_{\kappa}^{-1}(\tilde{x}_2) = \tilde{\Delta}_2 = 2\tilde{\Delta}_1 = 2U_{\kappa}^{-1}(\tilde{x}_1) = \frac{4}{\kappa}$$



An L-star vertex at the Dirichlet boundary, with conformal weight $\tilde{\Delta}_L$ in QG & $\tilde{x}_L = U_\kappa(\tilde{\Delta}_L)$ in $\mathbb{H}$

$$U_\kappa^{-1}(\tilde{x}_L) = \tilde{\Delta}_L = L\tilde{\Delta}_1 = LU_\kappa^{-1}(\tilde{x}_1) = \frac{2L}{\kappa}$$

Brownian Packet in QG

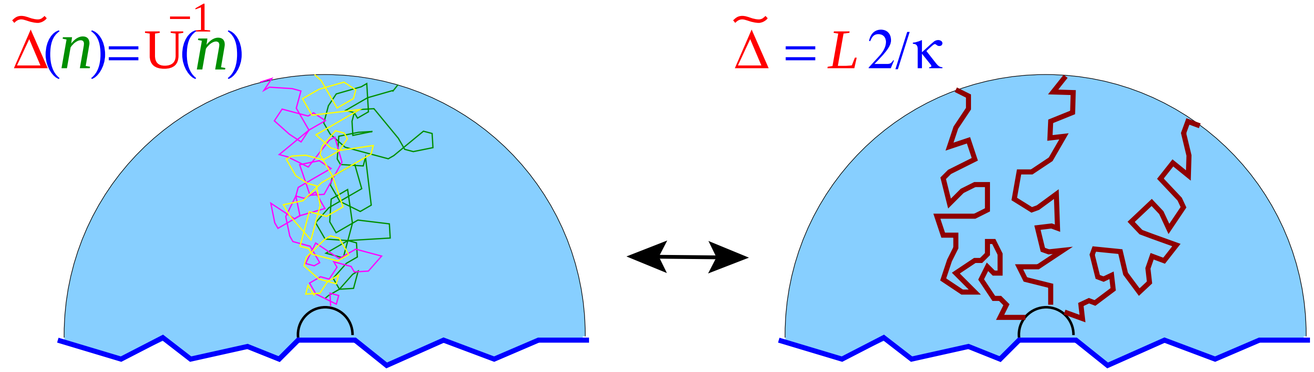


Left: Dirichlet boundary conditions for a packet of *n* independent Brownian paths in $\mathbb{H}$; *right:* its conformal weight $\tilde{\Delta}$ in boundary QG

$$\tilde{\Delta}(n) = U_{\kappa}^{-1}(n) = \frac{1}{2\kappa}\left(\sqrt{16\kappa n + (\kappa-4)^2} + \kappa - 4\right)$$

The Brownian paths, independent in a fixed metric, are strongly coupled by the metric fluctuations in quantum gravity.

SLE Transmutation



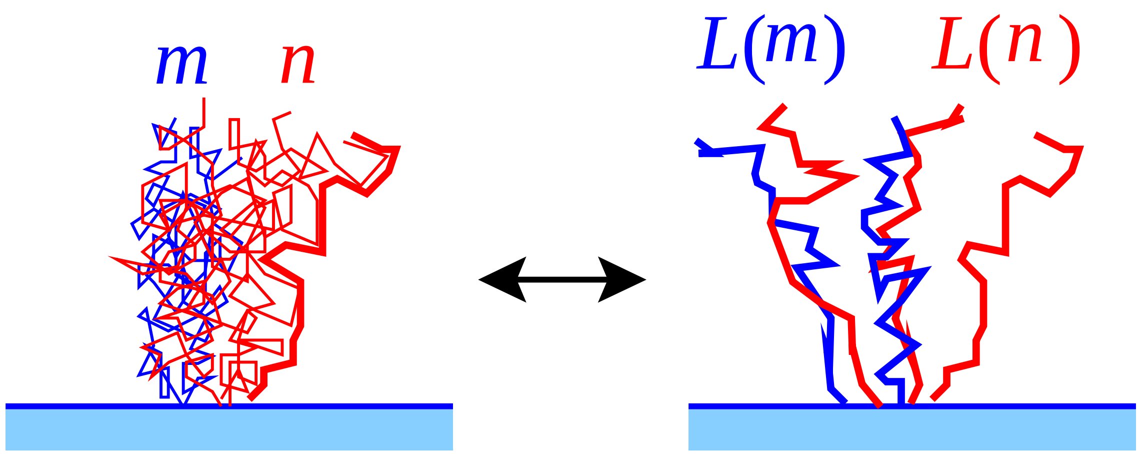
n independent Brownian paths $\Longleftrightarrow$ L mutually-avoiding

SLE paths:

$$L = \frac{U_\kappa^{-1}(n)}{U_\kappa^{-1}(\tilde{x}_1)} = \frac{\kappa}{2} U_\kappa^{-1}(n)$$

from ADDITIVITY OF BOUNDARY QG

Brownian Hiding Exponents and SLE(8/3) *(W. Werner, 2003)*



$$U^{-1}_{\kappa=8/3}(n) = \frac{2}{\kappa} L(n) = \frac{3}{4} L(n)$$

Hiding Exponents

Combining conformal dimensions $\tilde{\Delta}$ in boundary QG and $\tilde{x}$ in $\mathbb{H}$

$$\tilde{x}_{m,n} = U\left[\frac{3}{4} + U^{-1}\left[m + U\left(U^{-1}(n) - \frac{3}{4}\right)\right]\right]$$

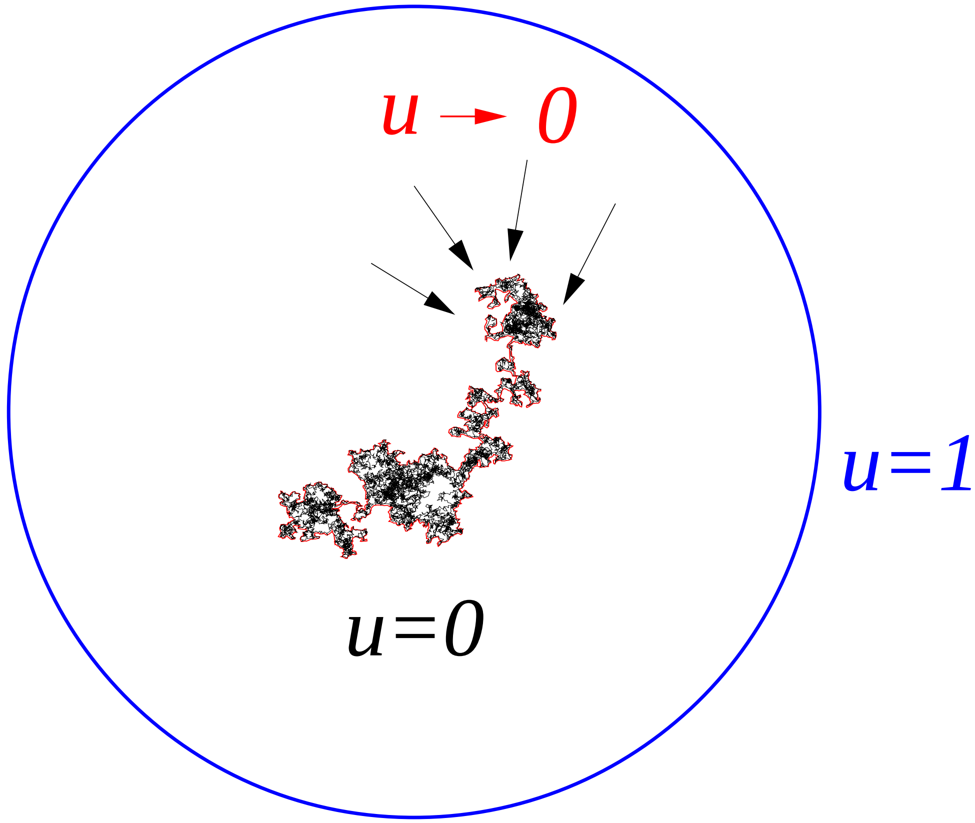
$$\tilde{x}_{m,n} = m + n + \frac{1}{4}\sqrt{24m + \left(\sqrt{1+24n} - 3\right)^2} - \frac{1}{4}\left(\sqrt{1+24n} - 3\right)$$

Harmonic Measure

&

Multifractality

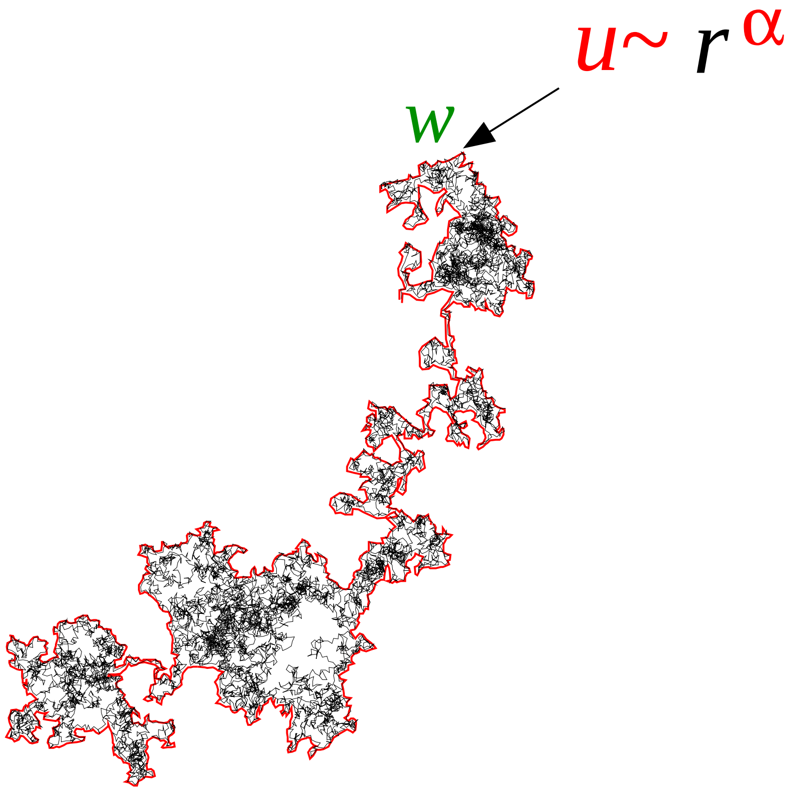
Potential Distribution



Laplace Eq.: $\Delta u = 0$

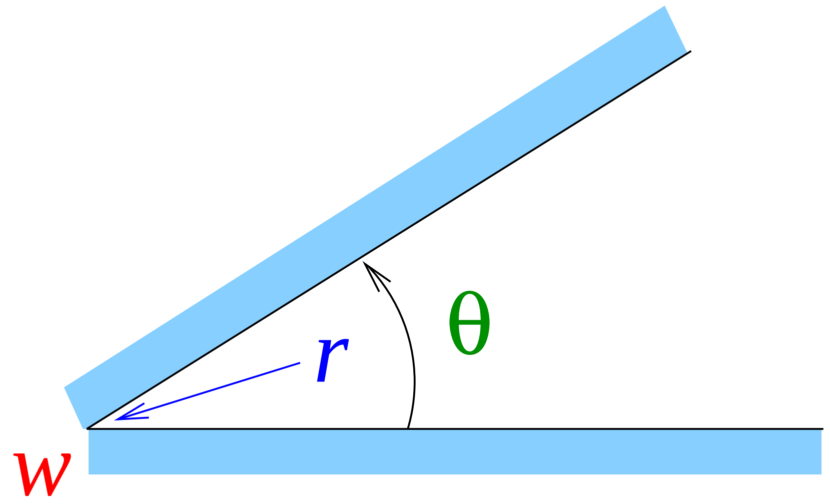
Multifractality

(T. C. Halsey & al., 1986)



$$w \in \mathcal{F}_{\alpha} : \quad \dim \mathcal{F}_{\alpha} = f(\alpha)$$

Electrostatic Angles

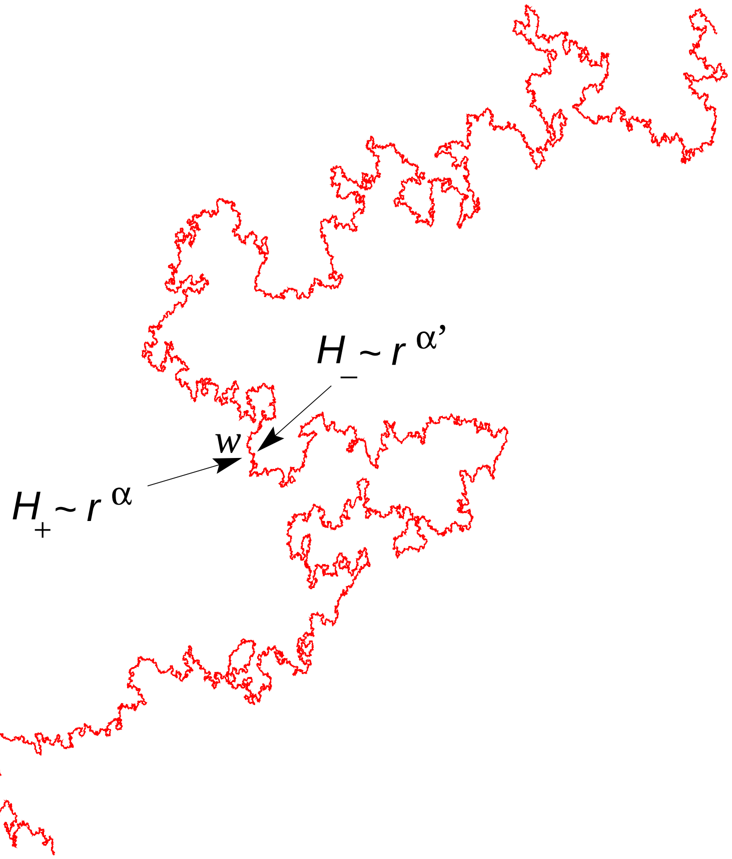


$$u(w,r) \sim r^{\alpha}, \quad \alpha = \pi/\theta$$

$$\theta \in (0,\ 2\pi],\quad \alpha \in \left[\frac{1}{2},\ +\infty\right)$$

Spikes, Fjords

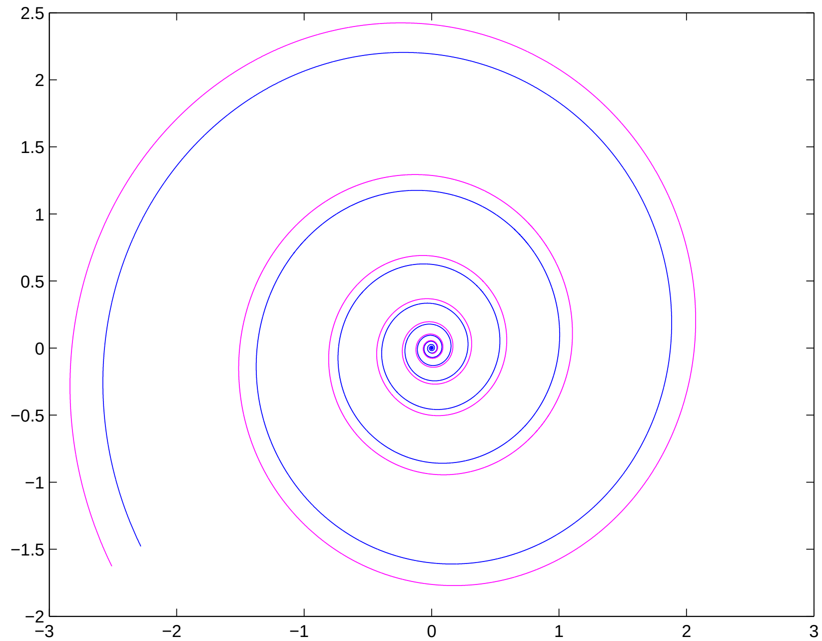
Double-Sided Potential



$$w \in \mathcal{F}_{\alpha,\alpha'}: \quad \dim \mathcal{F}_{\alpha,\alpha'} = f(\alpha, \alpha')$$

Equipotentials

Logarithmic Spirals



A point w on the frontier with a double logarithmic spiral.

Winding angle: $\varphi(w, r) = \lambda \ln r$

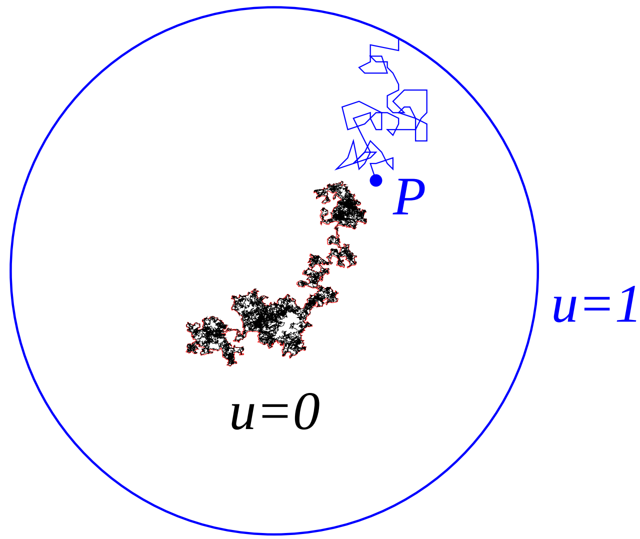
Mixed Multifractal Spectrum

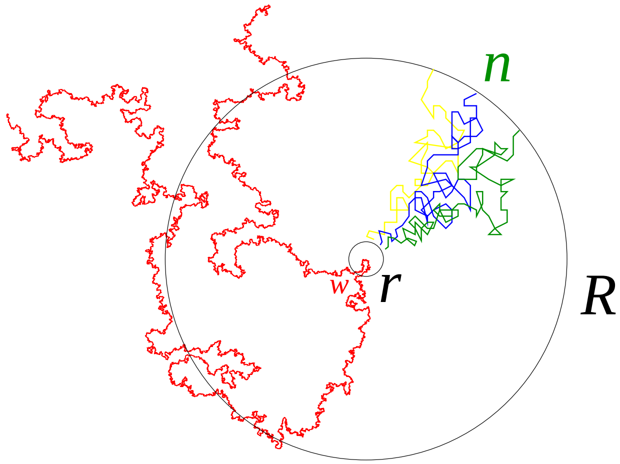
(I. Binder, 1996)

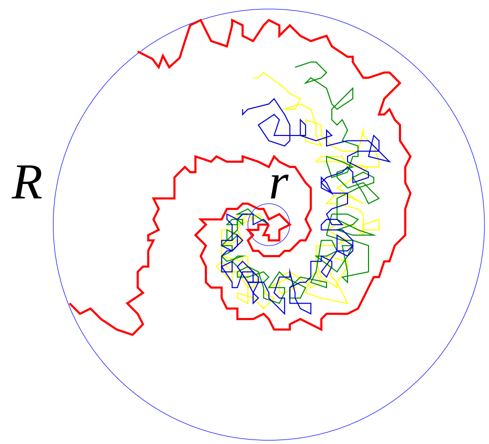
$$w \in \mathcal{F}_{\alpha,\lambda} \iff \left\{ \begin{array}{l} u(w,r) \sim r^{\alpha} \\ \varphi(w,r) \sim \lambda \ln r \end{array} \right\}$$

$$\dim \mathcal{F}_{\alpha,\lambda} = f(\alpha,\lambda)$$

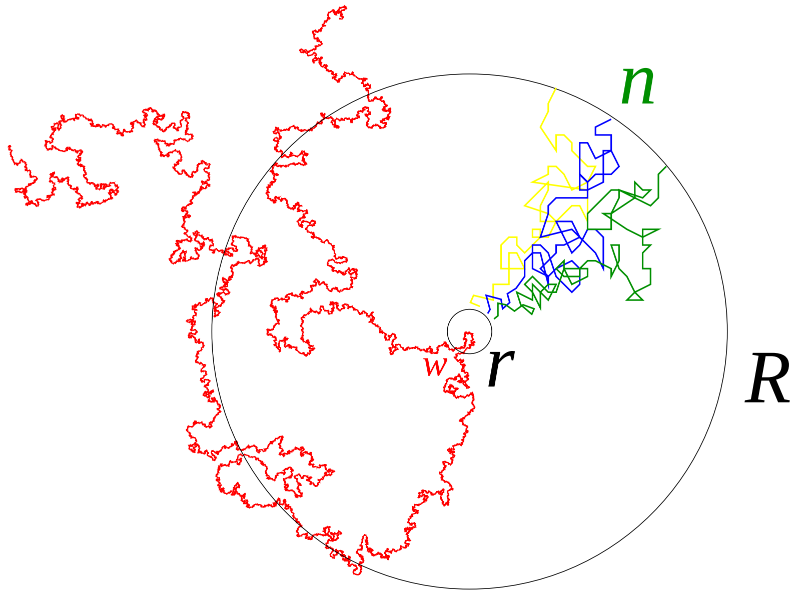
Potential & Brownian Paths







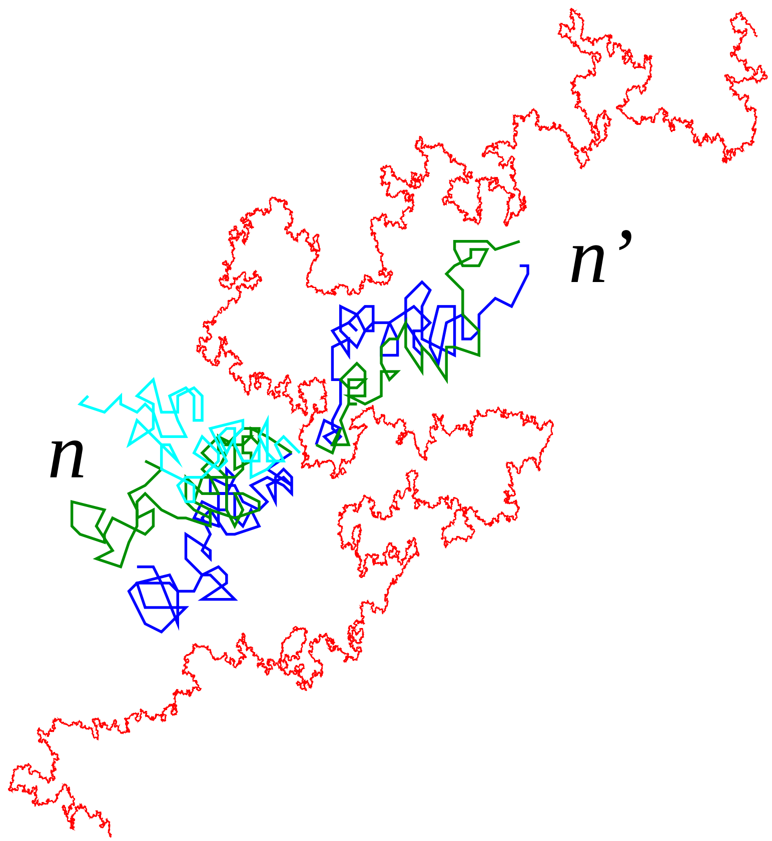
Harmonic Measure Exponents



$$\sum_{w} H^{n}(w,r) \approx (r/R)^{2x(n)-2}$$

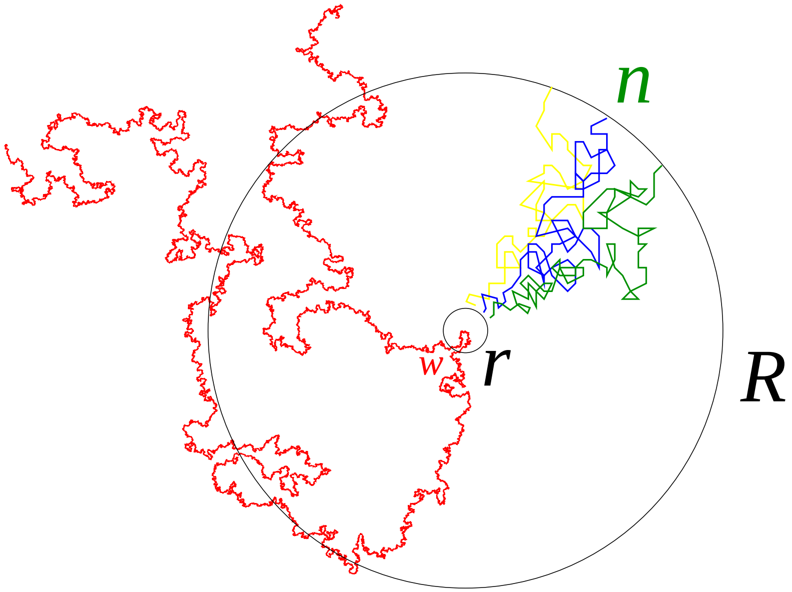
$H(w,r)$: harmonic measure in ball $B(w,r)$

Generalization



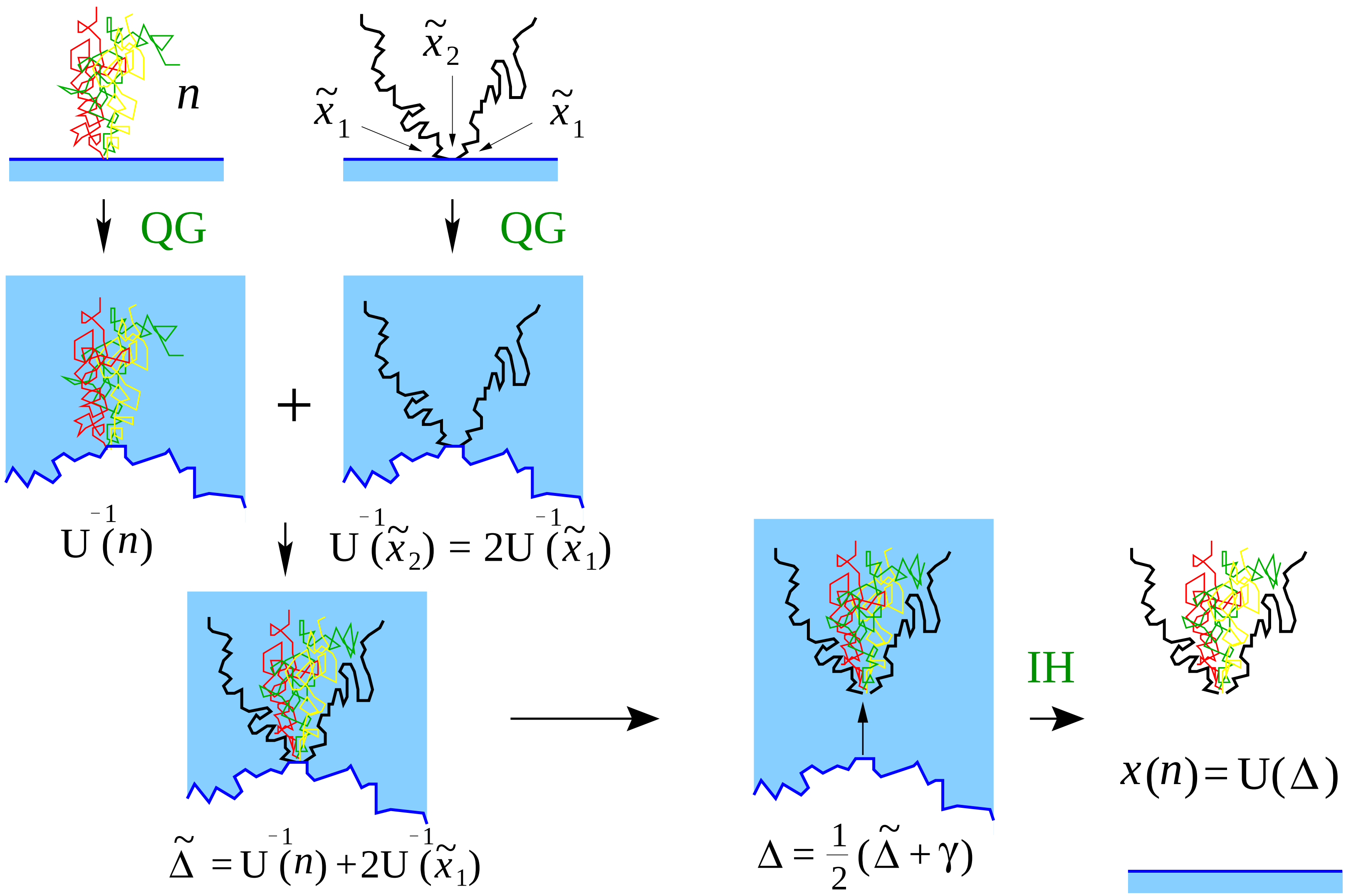
Double harmonic measure moments.

Multifractal Exponents & QG



$$x(n) = U\left(\frac{1}{2}U^{-1}(n) + \frac{1}{2}\right)$$

Quantum Gravity Construction



Quantum Gravity Construction

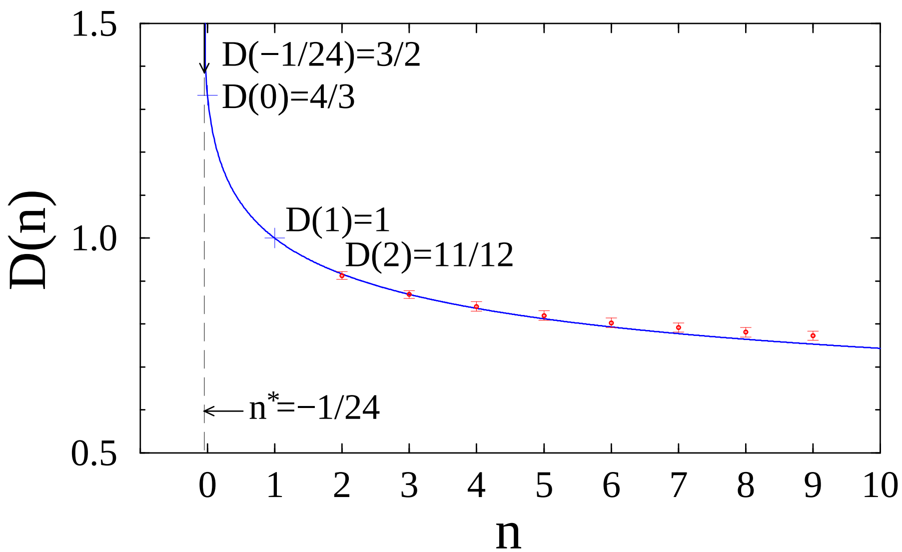
- Boundary

$$\begin{aligned}\tilde{\Delta} &= U^{-1}(n) + 2U^{-1}(\tilde{x}_1) \\ &= U^{-1}(n) + \frac{4}{\kappa}\end{aligned}$$

- Bulk

$$\begin{aligned}\Delta &= \frac{1}{2}\left(\tilde{\Delta} + 1 - \frac{4}{\kappa}\right) \\ &= \frac{1}{2}U^{-1}(n) + \frac{1}{2}\end{aligned}$$

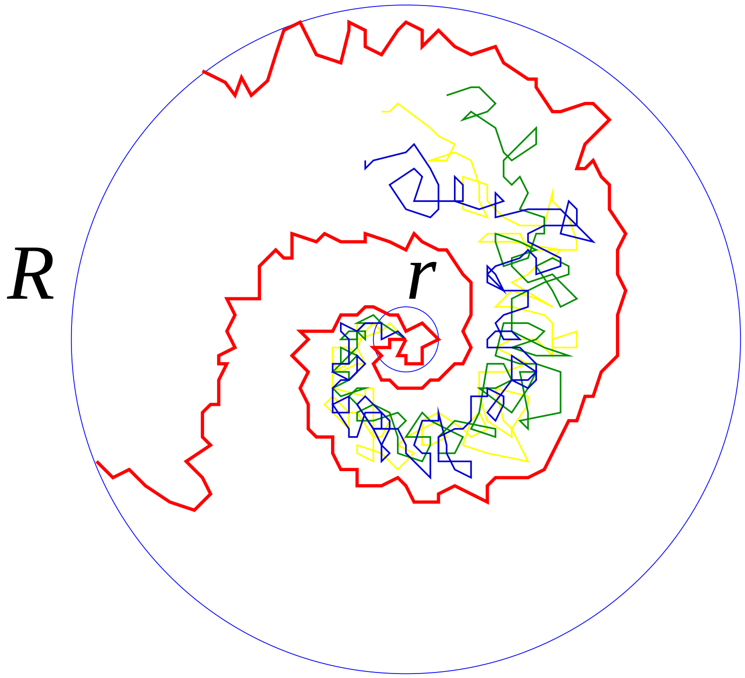
General Dimensions $D(n) = 2\,[x(n) - 1]/(n - 1)$



$$D(n) = \frac{1}{2} + \frac{25}{24}\left(\sqrt{\frac{24n+1}{25}} - 1\right)^{-1}$$

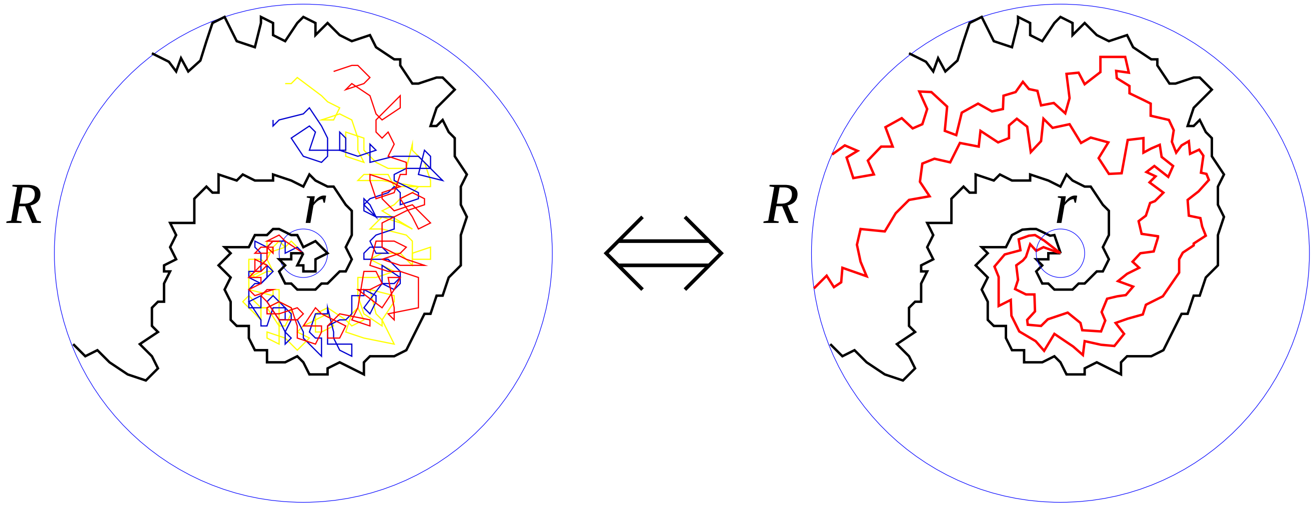
for percolation, self-avoiding or random walk (D., 1999);
data for percolation (red dots) (P. Meakin et al., 1988).

Rotation Exponents $x(n, p)$



A packet of n independent random walks winding with & avoiding the two frontier paths.

SLE Transmutation

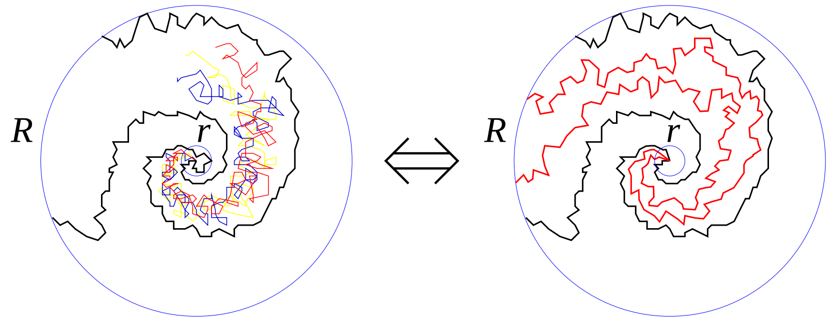


Total Number of Simple Paths: $\# = 2 + L$

n independent Brownian paths $\Longleftrightarrow$ L mutually-avoiding SLE paths:

$L = \frac{U_\kappa^{-1}(n)}{U_\kappa^{-1}(\tilde{x}_1)} = \frac{\kappa}{2} U_\kappa^{-1}(n)$, from ADDITIVITY OF BOUNDARY QG

Equivalent Path #: $2 + \frac{U^{-1}(n)}{U^{-1}(\tilde{x}_1)}$



$$x(n,p) = x(n) - \frac{1}{8}\frac{p^2}{2x(n)+b-2}$$

$$b = \frac{1}{2\kappa}\left(2+\frac{\kappa}{2}\right)^2$$

Multifractal Spectra

Double Legendre Transform

Define $\tau(n,p) = 2x(n,p) - 2$:

$$\alpha = \frac{\partial \tau}{\partial n}(n,p), \quad \lambda = \frac{\partial \tau}{\partial p}(n,p)$$

$$f(\alpha,\lambda) = \alpha n + \lambda p - \tau(n,p)$$

$$n = \frac{\partial f}{\partial \alpha}(\alpha,\lambda), \quad p = \frac{\partial f}{\partial \lambda}(\alpha,\lambda)$$

Multifractal Scaling Law

By Double Legendre Transform of $x(n, p)$

$$f(\alpha, \lambda) = (1+\lambda^2) f\left(\frac{\alpha}{1+\lambda^2}\right) - b\lambda^2$$

$$b = \frac{1}{2\kappa}\left(2+\frac{\kappa}{2}\right)^2$$

Multifractal Spectra

Legendre Transform & Scaling Law

$$f(\alpha) = \alpha + b - \frac{b\,\alpha^2}{2\alpha - 1}, \qquad b = \frac{1}{2\kappa}\left(2 + \frac{\kappa}{2}\right)^2$$

$$\begin{aligned} f(\alpha, \lambda) &= (1+\lambda^2)\, f\left(\frac{\alpha}{1+\lambda^2}\right) - b\lambda^2 \\ &= \alpha + b - \frac{b\,\alpha^2}{2\alpha - 1 - \lambda^2} \end{aligned}$$

B. D., 1998 ; I. Binder & B. D., 2002, 2007

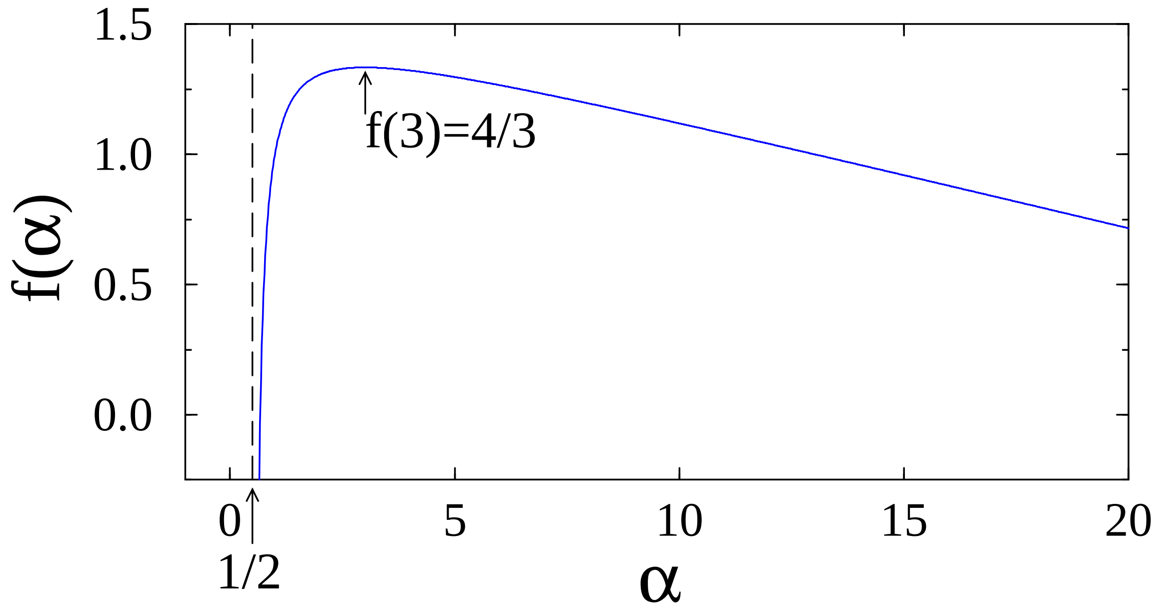
(Beliaev & Smirnov, 2005 ; Rushkin et al., 2006)

Multiple Multifractal Spectra

$$f\left(\alpha,\alpha';\lambda\right) = b-\frac{\kappa}{8}\left(\frac{1}{1+\lambda^2}-\frac{1}{2\alpha}-\frac{1}{2\alpha'}\right)^{-1} - \frac{b-2}{2}\left(\alpha+\alpha'\right)$$

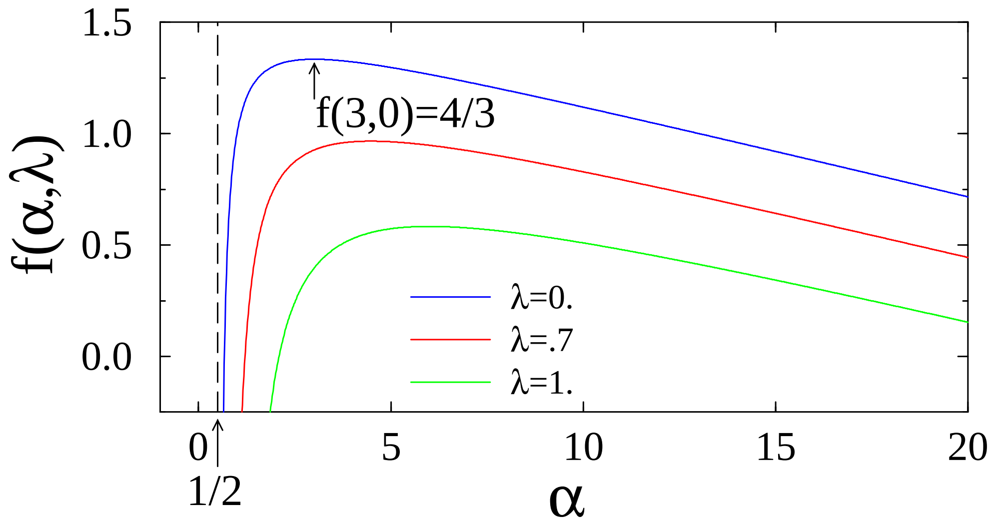
$$f\left(\alpha_{i=1,\dots,L};\lambda\right) = b-L^2\frac{\kappa}{32}\left(\frac{1}{1+\lambda^2}-\sum_{i=1}^{L}\frac{1}{2\alpha_i}\right)^{-1} - \frac{b-2}{2}\sum_{i=1}^{L}\alpha_i, \quad b=\frac{1}{2\kappa}\left(2+\frac{\kappa}{2}\right)^2$$

Brownian Multifractal Spectrum



$$f(\alpha) = \alpha + b - \frac{b\alpha^2}{2\alpha - 1}, \quad b = \frac{25}{12}$$

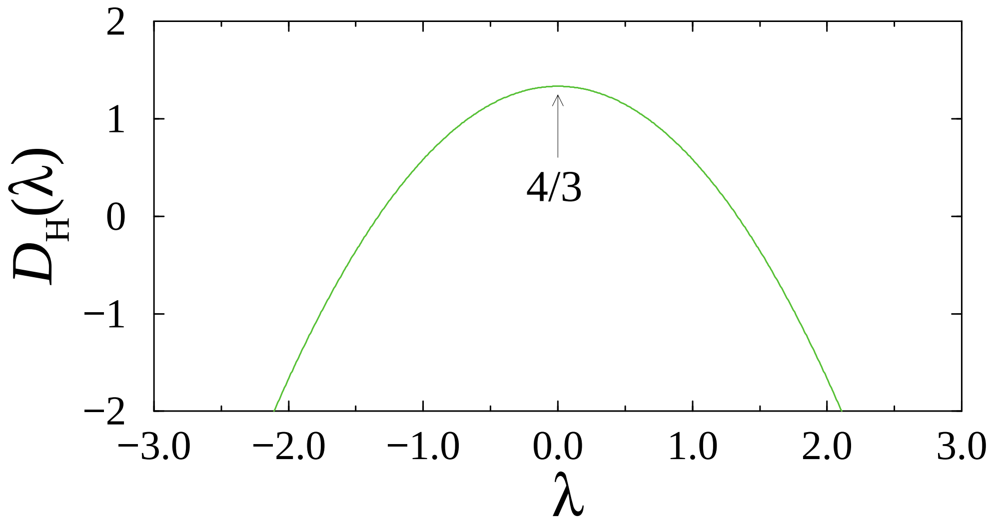
Brownian Mixed Spectrum $f(\alpha, \lambda)$



$f(\alpha, \lambda)$ for the Brownian frontier, percolation and SAW, and various spiralling rates λ.

The maximum $f(\alpha = 3, \lambda = 0) = 4/3$ is the Hausdorff dimension of the frontier.

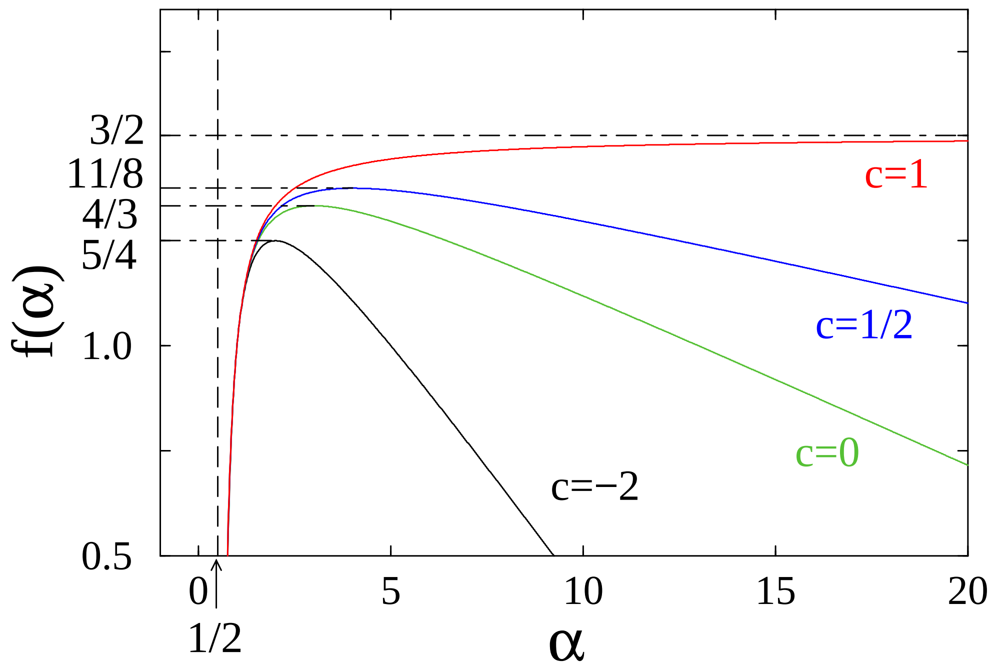
Brownian Rotation Dimensions



$$D(\lambda) = \sup_{\alpha} f(\alpha, \lambda) = \frac{4}{3} - \frac{3}{4}\lambda^2$$

Dimension $D(\lambda)$ of λ -spiral points on the Brownian frontier.

Multifractal Spectra $f(\alpha)$



Loop-erased RW ($c = -2$, SLE_2); *Brownian & percolation frontiers, and SAW's* ($c = 0$, $\mathrm{SLE}_{8/3}$); *Ising clusters* ($c = \frac{1}{2}$, SLE_3); $Q = 4$ *Potts clusters* ($c = 1$, SLE_4).

Duality in Percolation

Hull & External Perimeter Dimensions

$$D_{\mathrm{Hull}} = \frac{7}{4} \qquad \geqslant \frac{3}{2}$$

$$\left(D_{\mathrm{Hull}} - 1\right)\left(D_{\mathrm{EP}} - 1\right) = \frac{1}{4}$$

$$D_{\mathrm{EP}} = \frac{4}{3} \qquad \leqslant \frac{3}{2}$$

SLE Duality

$$D_{\mathrm{EP}}(\kappa) = D_{\mathrm{H}}(\kappa),\ \kappa \leqslant 4$$

$$D_{\mathrm{EP}}(\kappa) = D_{\mathrm{H}}(\kappa' = 16/\kappa), \quad \kappa \geqslant 4$$

$$\frac{1}{4} = [D_{\mathrm{EP}}(\kappa) - 1][D_{\mathrm{H}}(\kappa) - 1]$$

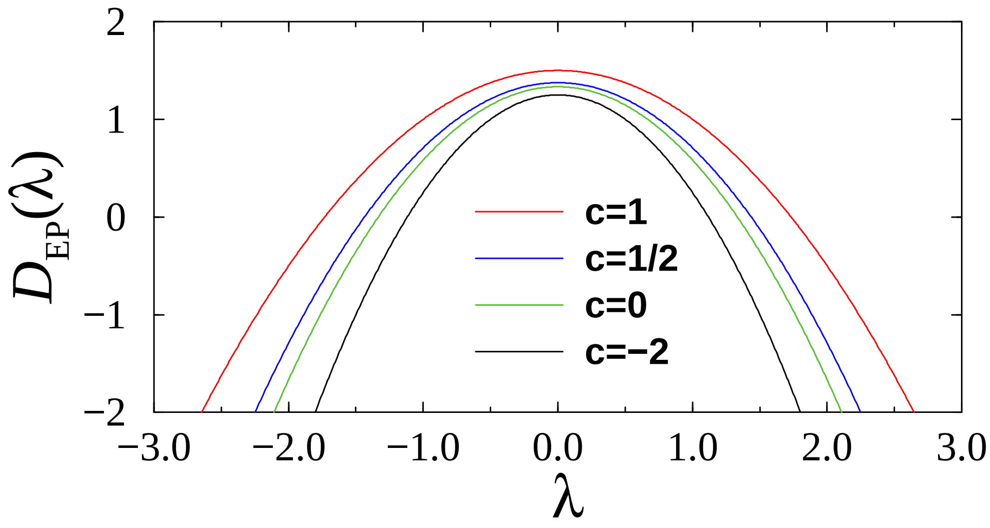
Duality: the external perimeter of $\mathrm{SLE}_{\kappa \geqslant 4}$ is the simple path of $\mathrm{SLE}_{[(16/\kappa) \leqslant 4]}$

Rotation Dimensions

$$D(\lambda) = \sup_{\alpha} f(\alpha, \lambda)$$

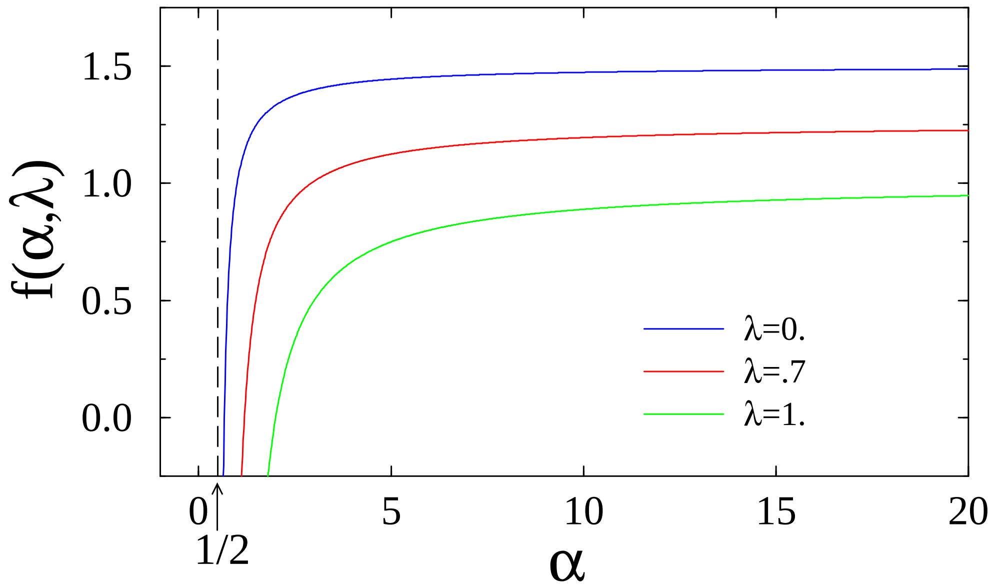
$$= D_{\mathrm{EP}} - (b - D_{\mathrm{EP}})\,\lambda^2$$

$$b \geqslant 2, \quad D_{\mathrm{EP}} \leqslant \frac{3}{2}$$



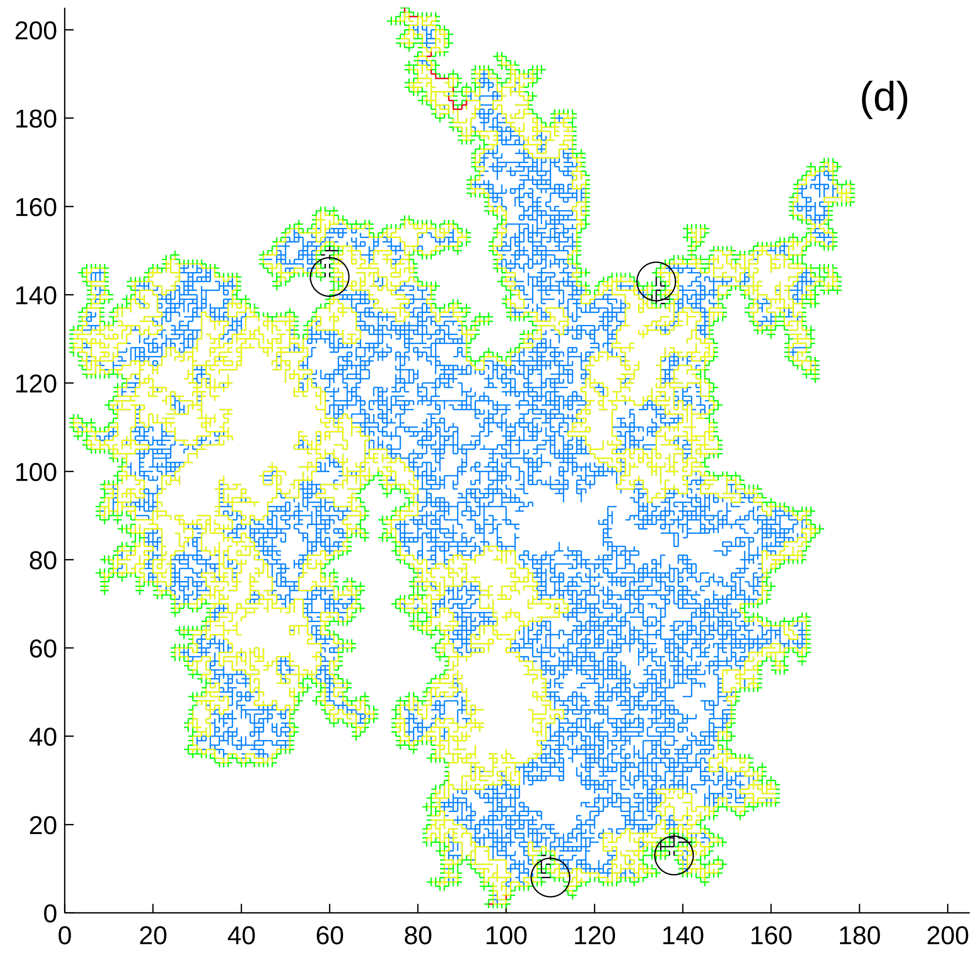
Dimensions $D(\lambda)$ of the external frontier as a function of rotation rate λ: loop-erased RW ($c=-2; \mathrm{SLE}_2$); Brownian & percolation frontiers, and SAW's ($c=0; \mathrm{SLE}_{8/3}$); Ising clusters ($c=\frac{1}{2}; \mathrm{SLE}_3$); $Q=4$ Potts clusters ($c=1; \mathrm{SLE}_4$) (or "Ultimate Norway").

The Pivotal $c = 1$ Case



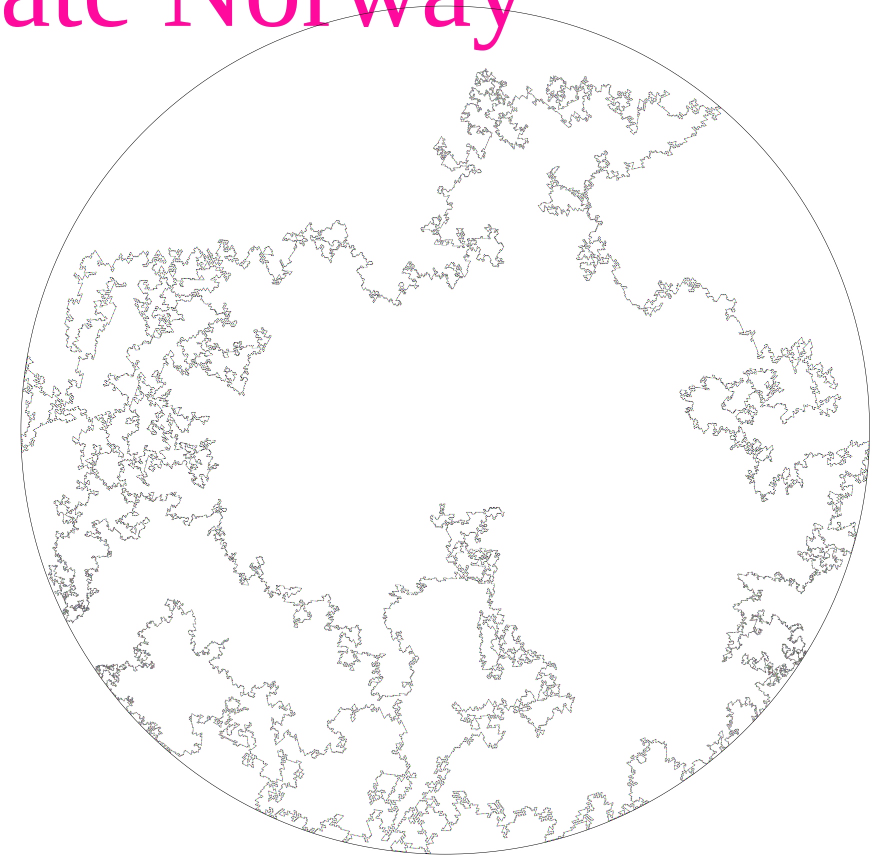
Left-sided mixed spectra $f(\alpha, \lambda)$ *for the "Ultimate Norway" ($c = 1$).*

Potts FK Cluster $(Q = 4)$



Cluster; *Hull*; *External Perimeter*.

Ultimate Norway



The "Ultimate Norway", i.e. the frontier of a $Q = 4$ *Potts cluster or* $\mathrm{SLE}_{\kappa=4}$, *the self-dual conformally invariant random curve* $(c = 1)$ *with maximal Hausdorff dimension* $D = 3/2$ *(courtesy of D. Wilson).*