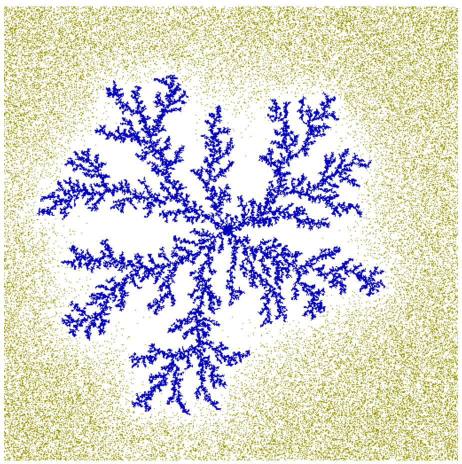
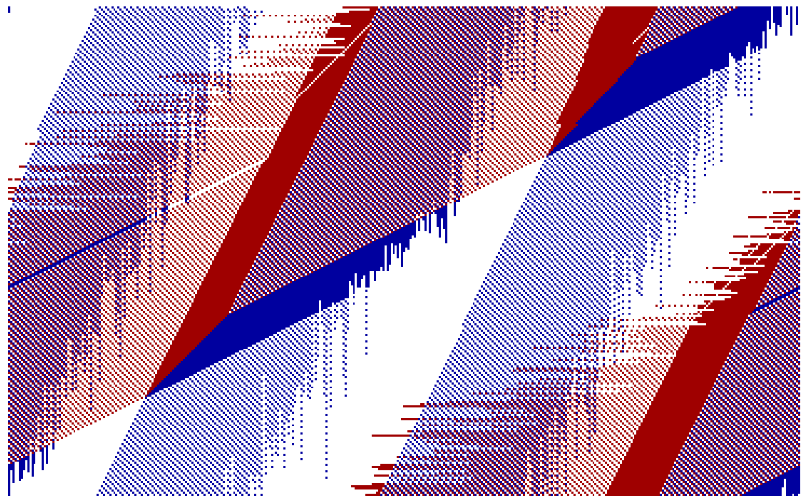
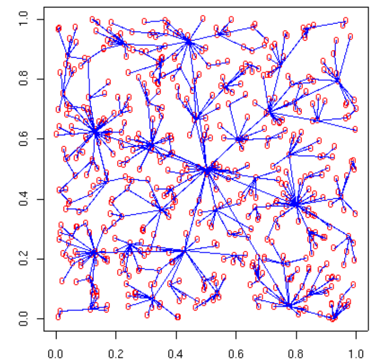
# “Jamming, counting and changing phase”

## (Statistical physics, computer simulation and probability theory)








Raissa D’Souza *UC Davis*

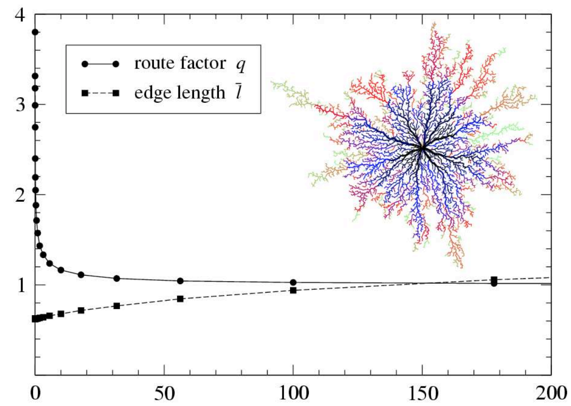
*Dept of Mechanical and Aeronautical Eng.*

*Center for Computational Science and Eng.*

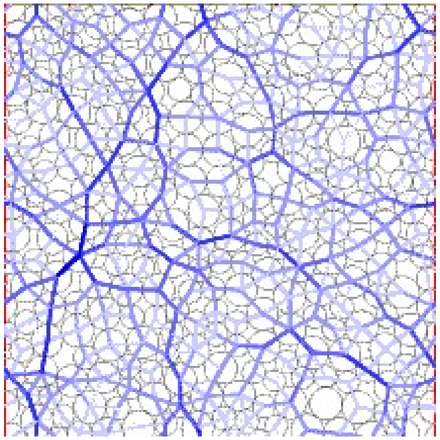
and *Santa Fe Institute*

# Transport phenomena

Flocking behavior




Collection/distribution networks



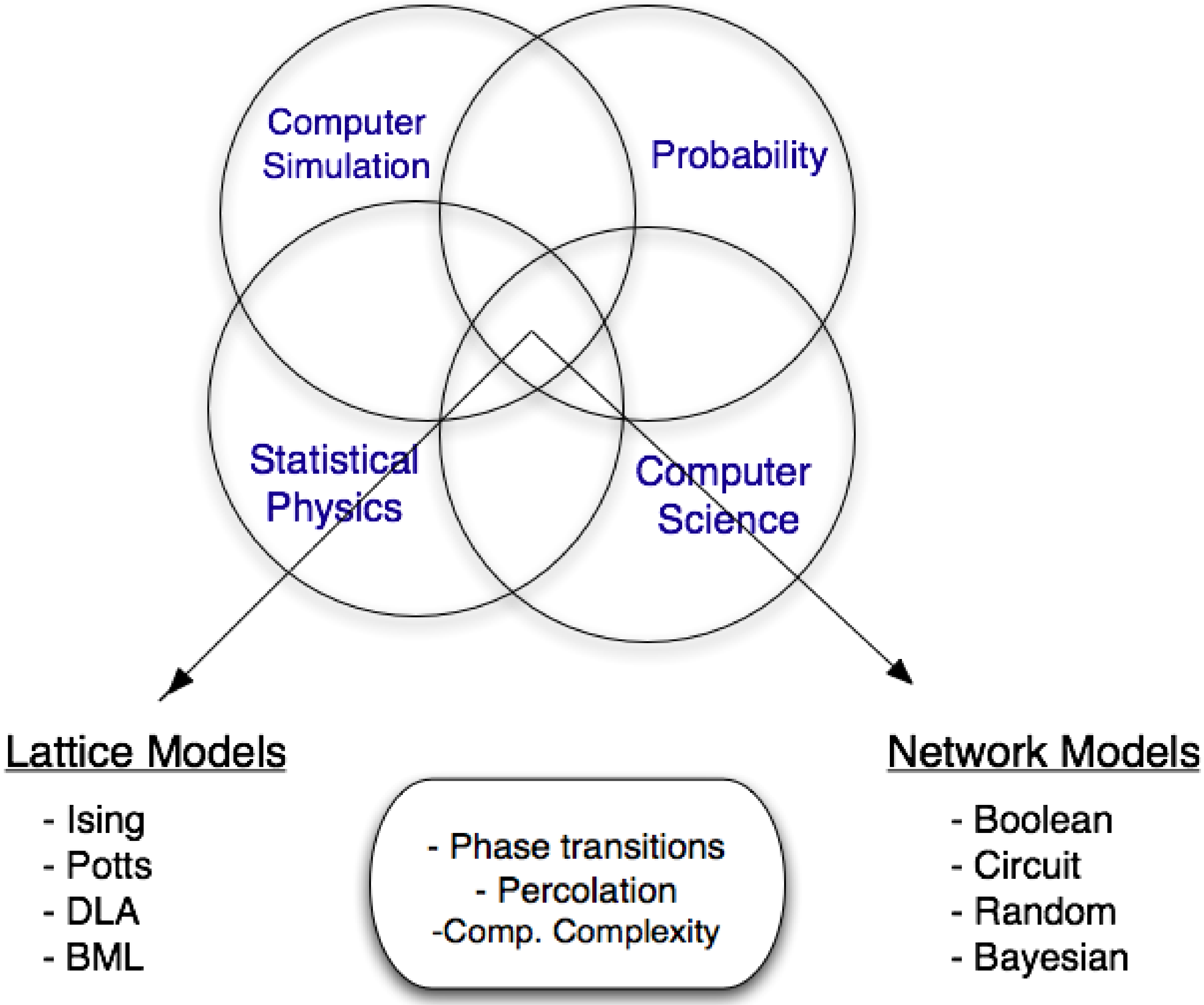
Granular materials

- Traffic flow on highways, congestion of packets on the Internet, flow of nutrients through the body, flocking behavior of birds in flight, formation of river networks, etc.

- Self-organized behaviors

- Abrupt onset of jamming

- → Need simple models of jamming and flow. (What makes a system jam?)

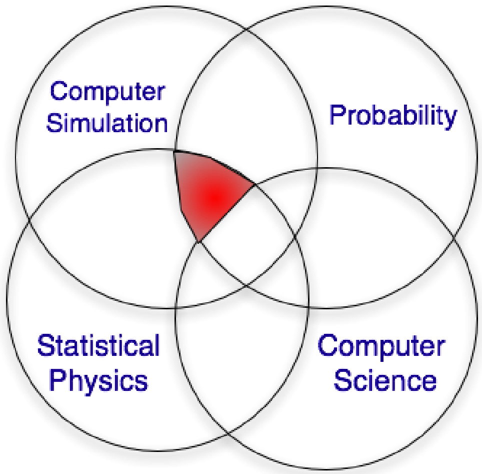

Computer Simulation
Probability
Statistical Physics
Computer Science
Lattice Models
- Ising
- Potts
- DLA
- BML
- Phase transitions
- Percolation
-Comp. Complexity
Network Models
- Boolean
- Circuit
- Random
- Bayesian

Statistical Physics grounded in the study of lattice models.

- Phase transitions (equilibrium – Rigorous treatment )
  - Ising
  - Potts
  - Percolation

- Kinetic critical phenomena (typically non-linear, dynamical, typically accessed through Simulation )
  - DLA (+ variants iDLA, rDLA, roto-router)
  - BML (jamming and flow)

These models are highly influential
— general paradigms foundations.




## Stat. Phys., simulation and probability

- Intimately related through these shared lattice models.
- Rigorous results typically extremely difficult to obtain.
- Progress through simulation: “experimental mathematics”.
  - Numerical data
  - Visualization
  - Intuition
- Rigorous results may contradict results from simulation! (Bootstrap percolation)

# Statistical physics, simulation and probability
# Common Ground

- Length and time scales:
  - Simulation (small)
  - Mathematics (asymptotic results)
  - Physics (somewhere in-between)

- Role of synchrony:
  - in pattern formation
  - probabilistic vs deterministic (games, roto-router)

- Lattice artifacts/boundary constraints

# Statistical Physics, Simulation and Probability: Interplay and Caveats:

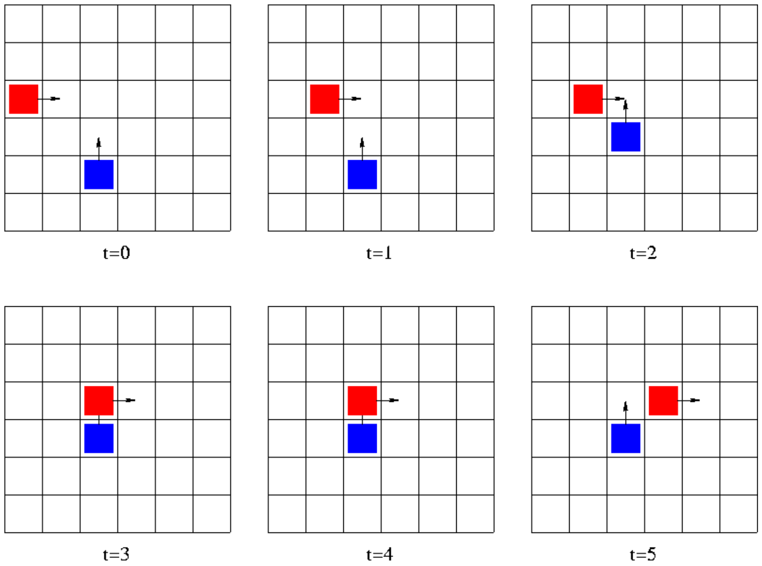
(Review paper in *Complexity* **12** (2) 3039, 2006:
"BML revisited: statistical physics, computer simulation and probability")

| Model Discussed | Significance |
|---|---|
| BML | Dynamical jamming transition and traffic flow. |
| Honey-comb dimer model | Importance of aspect ratio. |
| Spatial Prisoner's Dilemma | Role of synchrony. |
| Random-turn games | Randomizing a deterministic system. |
| Bootstrap Percolation | Importance of lenthscales and timescales. |
| Rotor-router | To be discovered? |

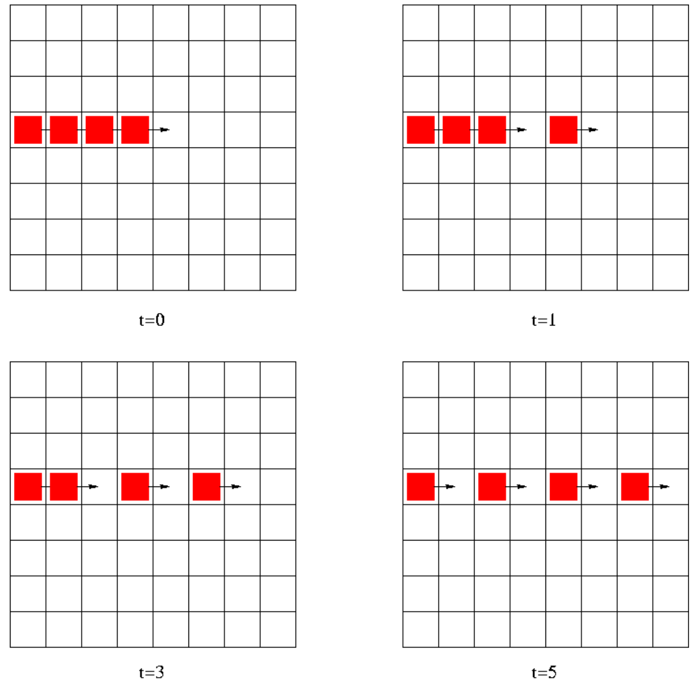
# The Biham-Middleton-Levine (BML) traffic model, 1992

- Self-organization/Phase transitions/kinetic pathways of granular flow

- Modeling vehicular and Internet traffic, especially congestion and traffic jams.



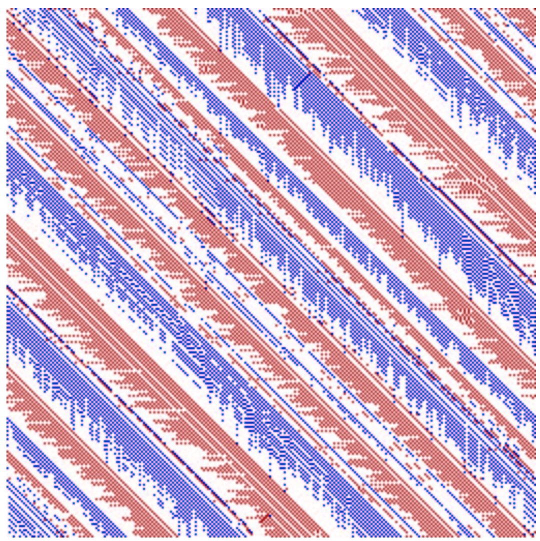

- Two species: east-bound (red), and north-bound (blue), living on a torus.



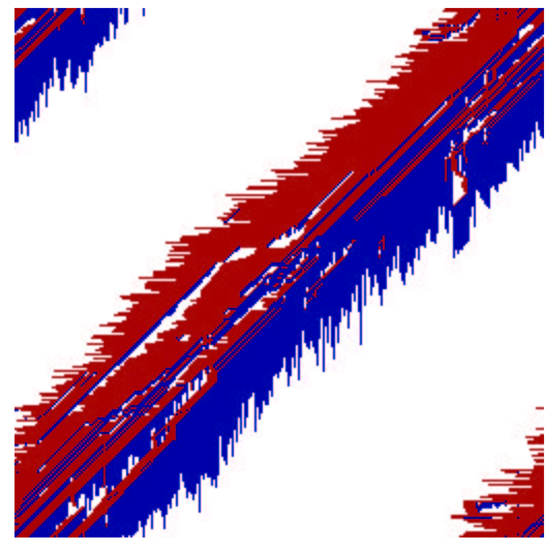

- Initially distributed at random with density $\rho$, on $L \times L$ lattice.
- Each species attempts to move on alternating time steps.
- $2L$ conservation laws, for an $L \times L$ lattice.
- Paradigm for self organization and phase transitions, as well as a foundation for modeling traffic flow.

# Standard understanding

- Evidence of a phase transition from all cars moving, $v = 1$, to all cars jammed, $v = 0$, as a function of initial density $\rho$.



$\rho < \rho_c, \quad v = 1$



$\rho > \rho_c, \quad v = 0$

# Standard understanding, cont.

- Almost all progress has been numerical. Only rigorous for $\rho \to 0$ and $\rho \to 1$. The latter very recent progress,

  $[$Angel, Holroyd, Martin, *Elec. Commun. in Probability*, 10(17) 2005.$]$
  Blocking paths and supercritical percolation theory, $\rho \to 1$.

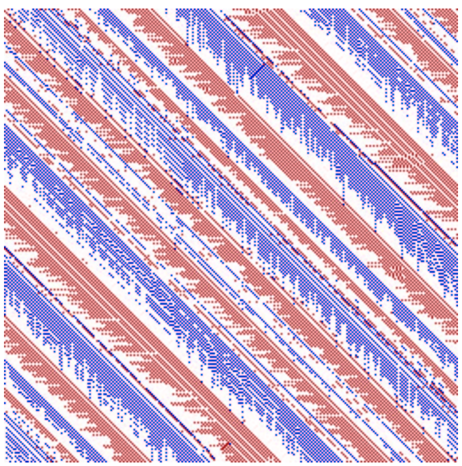
  $[$ Austin, Benjamini, *arXiv:math/0607759* $]$
  Show that $v = 1$ if less than $L/2$ cars, $\rho \to 0$.

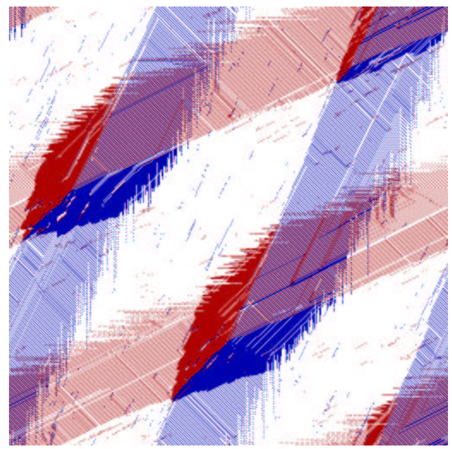
- BML serves as a theoretical underpinning to more complex traffic models. (Over 200 citations to BML'92 referring to the existence of a phase transition.)

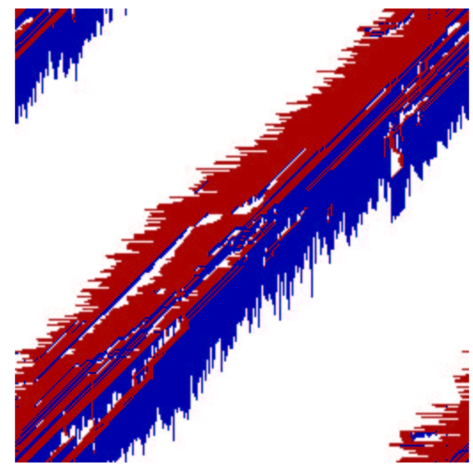
- Extensions: motion of charged beads (Hubler); flow of information in social organization (lateral and up the hierarchy) and jamming.

# What we find

(D'Souza, PRE 2005)







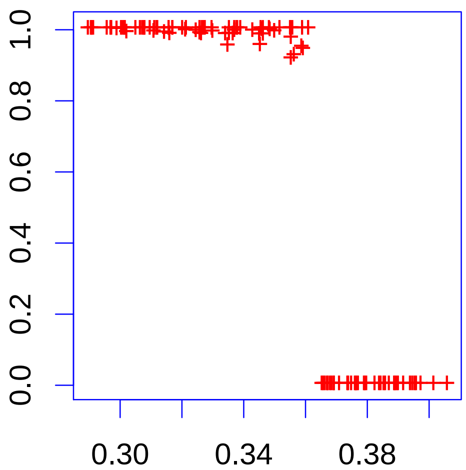
$\rho < \rho_c$

??

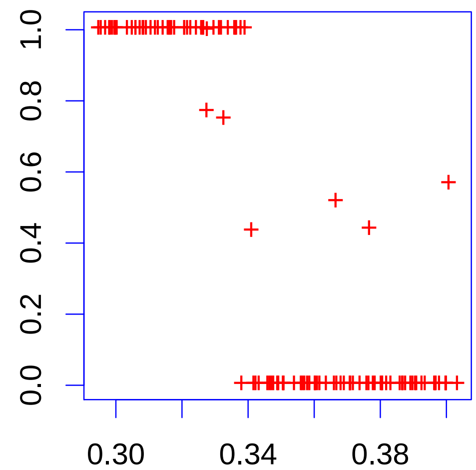
$\rho > \rho_c$

# Numerical results (Square Lattices)

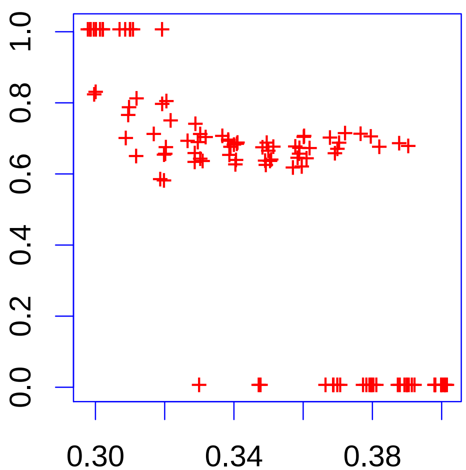
(L=64)



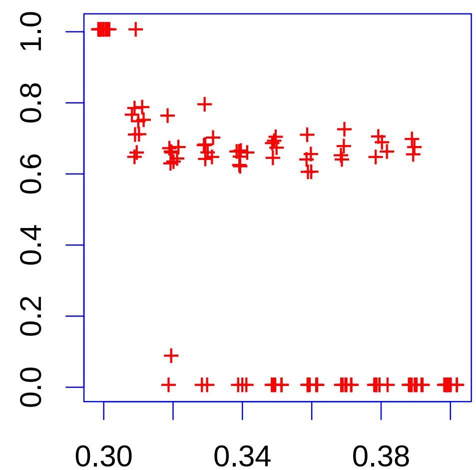

(L=128)




(L=256)

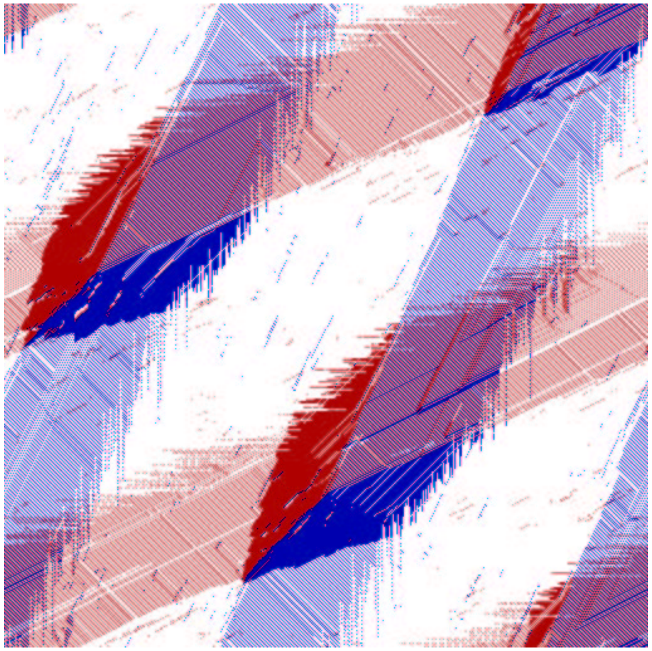



(L=512)




A quantized intermediate state, with $v \sim 2/3$??

# Typical intermediate configuration, on $L \times L$ lattice



- Macroscopic structure stable (maybe metastable??); but persists for $t > 10^8$.

- Microscopic fluctuations, so not exactly periodic.

- Disordered chains of particles at random positions.

## Aside: Honey-comb dimer model, Kenyon and Wilson 2004.

- Variant of bond percolation.
- Connected paths on a honey-comb lattice.
- At the critical point, partition function depends on shape of domain.
- Found other number theoretic dependencies.

# What this means for BML

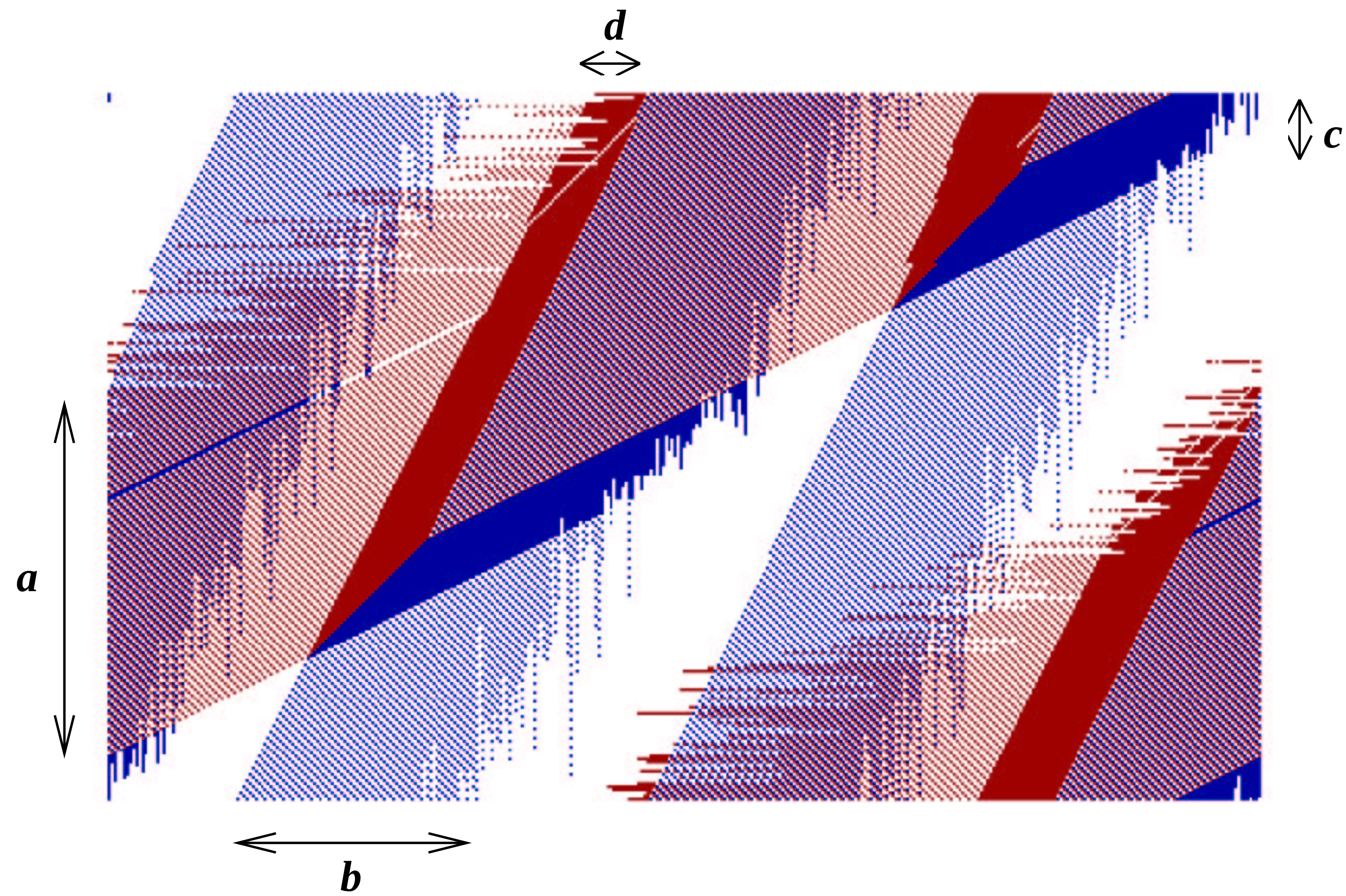
- From square to $(L \times L')$-lattices where aspect ratio is relatively prime.

- Effectively simulates a larger system. (Correlations have larger length scales – increased time between subsequent interactions of two particles).

- Choose, $L$, $L'$, two successive Fibonacci numbers (“the most relatively prime”, in a computational complexity sense).

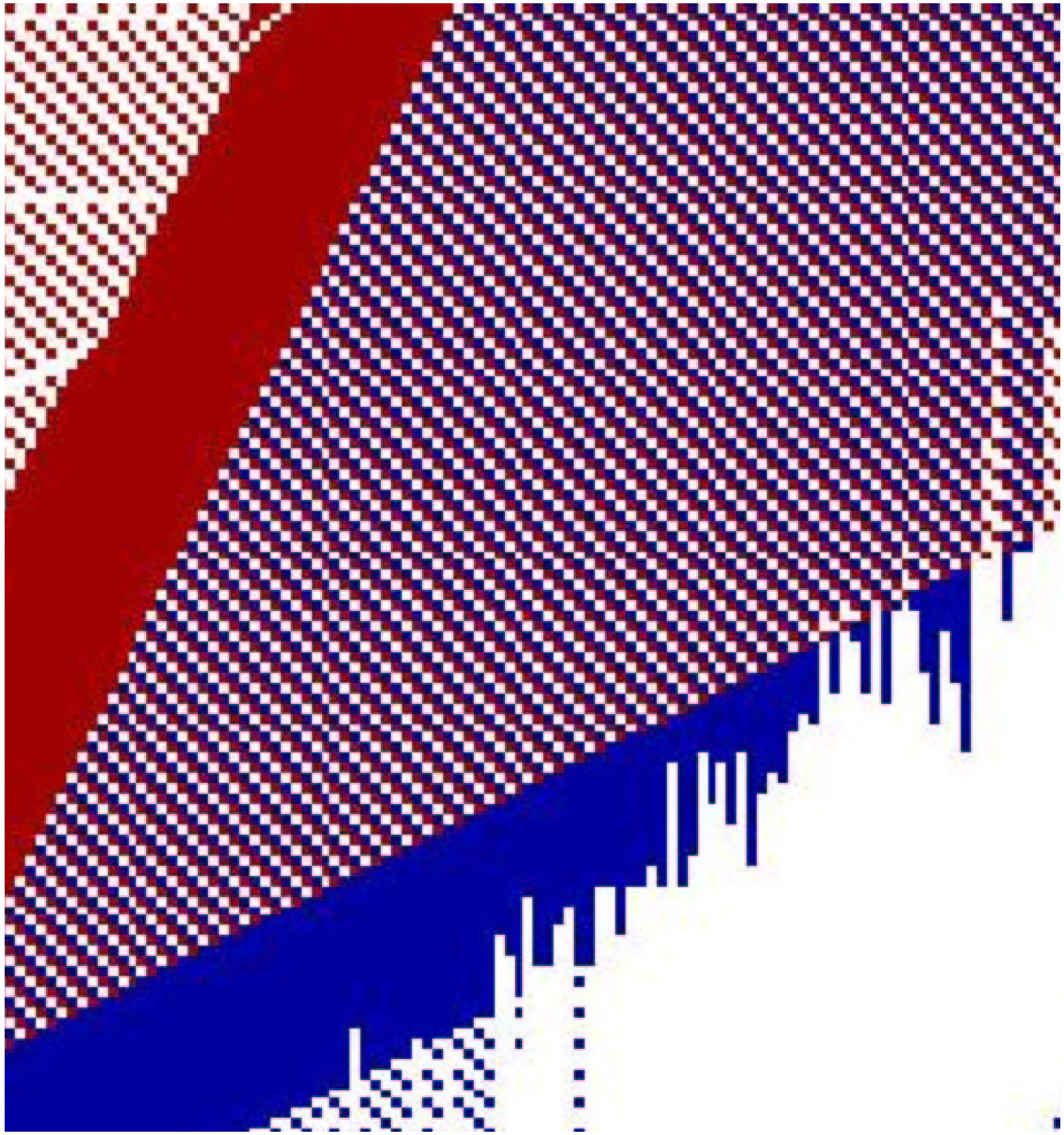
- A crisp regular geometric structure emerges.

# Understanding the geometric structure

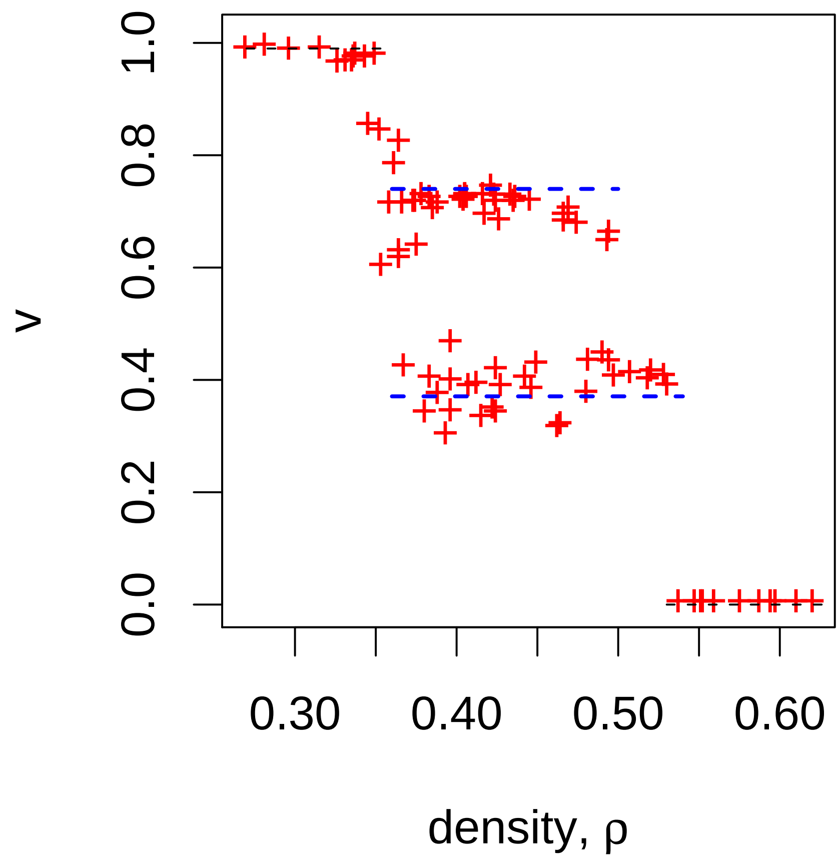



These structures are PERIODIC!

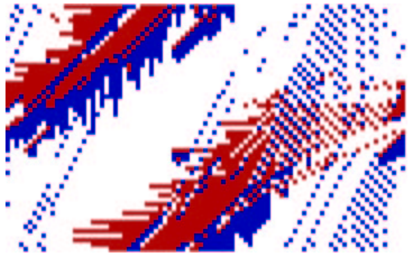
Kinetic pathway: a large jam involving almost all particles forms, then sheds.

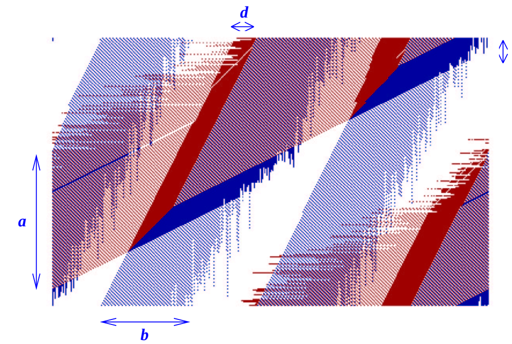


# Numerical results ("Fibonacci" lattices)




- For $0.35 < \rho < 0.5$, only the intermediate states accessible!
- Each realization settles into periodic limit cycle.
- First occurrence of v=0, has $\rho = 0.53$.






Kinetic pathway (Local):

- Red bands with slope $1/2$ and blue with slope $2$.
- Interface slope, $s \approx 1$.
- Bands have $\rho_b = 1/3$: red and blues on non-interacting lattices.

Discreteness constraint (Global):

- Number of bands ($\omega_b$ and $\omega_r$) must be integers.

# Interplay between Global and Local

Conservation laws: width of the bands.

- $a = (3/2)\rho L'$ $\qquad$ $b = (3/2)\rho L/\omega_b$.

Geometric constraints (Global)

$$\begin{aligned} k + k' &= L && (1) \\ \frac{1}{2}k + sk' &= \omega_r L' && (2) \\ m + m' &= L && (3) \\ 2m + sm' &= \omega_b L' && (4) \\ k' &= m' && (5) \end{aligned}$$

# Winding number:

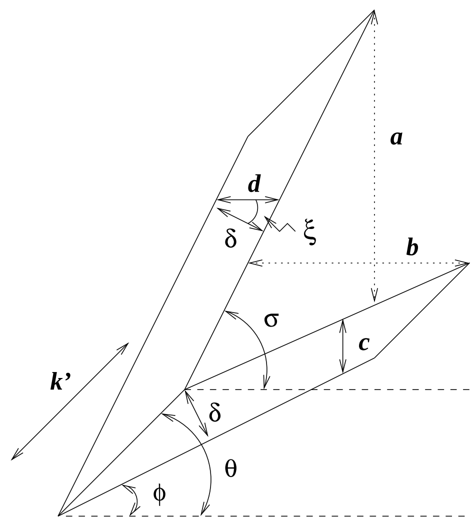
Predicting the number of red bands, $\omega_r$, blue bands $\omega_b$:

$$\omega_b = \frac{2L}{L'} - \frac{(2-s)}{(2s-1)}\left(2\omega_r - \frac{L}{L'}\right)$$

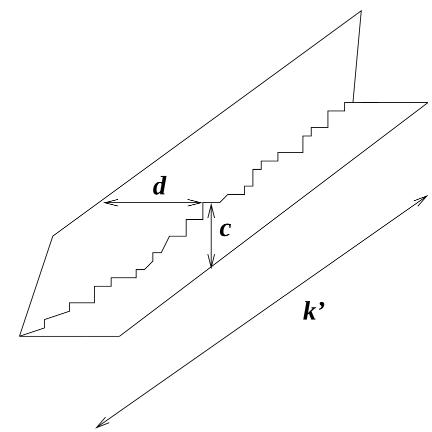
Depends on $L/L'$, and interface slope, $s$.

| $L/L'$ | $s$ | $\omega_r$ | $\omega_b$ |
|---|---|---|---|
| 1 | 1 | 1/2 | 2 |
| 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 4 |
| 2 | 1 | 2 | 2 |
| (1+$\sqrt{5}$)/2 | 1.17 | 1 | 3 |
| (1+$\sqrt{5}$)/2 | 7/10 | 1 | 2 |

# Asymptotic velocity




Type I jam




Type II jam

$$d = \delta/\cos(\xi) < d_{\max}$$

$$J \approx n_j\delta\,(\mathrm{l}_b + \mathrm{l}_r)$$

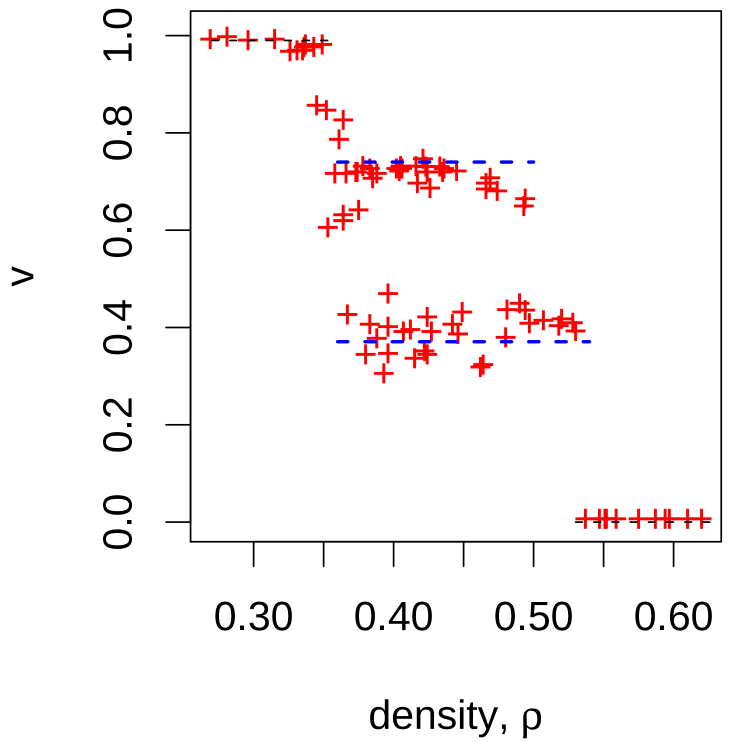
$$v = 1 - J/N = 0.7430$$

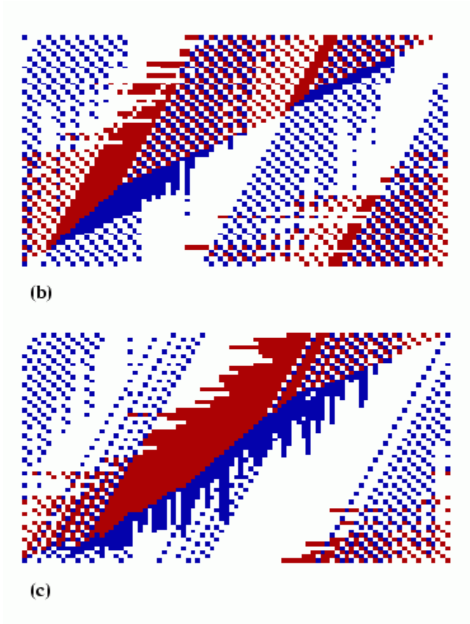
$$d = d_{\max} = (3/2)\rho L/\omega_b$$

$$J \approx k'(s \cdot d_{\max} + c_{\max})$$

$$v = 1 - J/N = 0.3707.$$

## Numerical results (Rectangular Fibonacci lattice: $L = 89$, $L' = 55$)



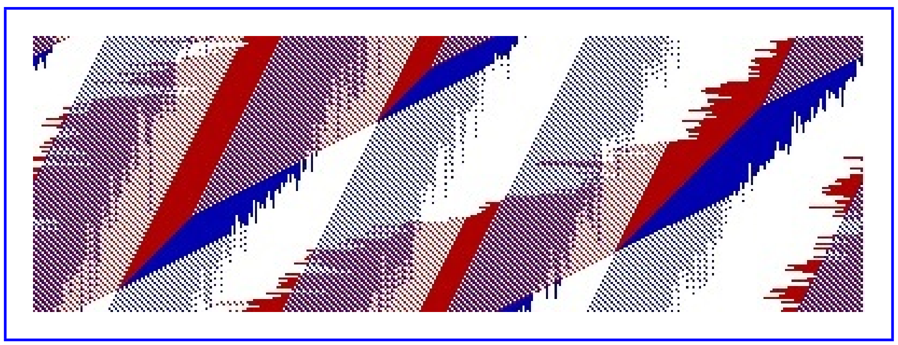

As predicted we see two types of winding number configurations. (And note, first occurrence of v=0, has $\rho = 0.53$.)

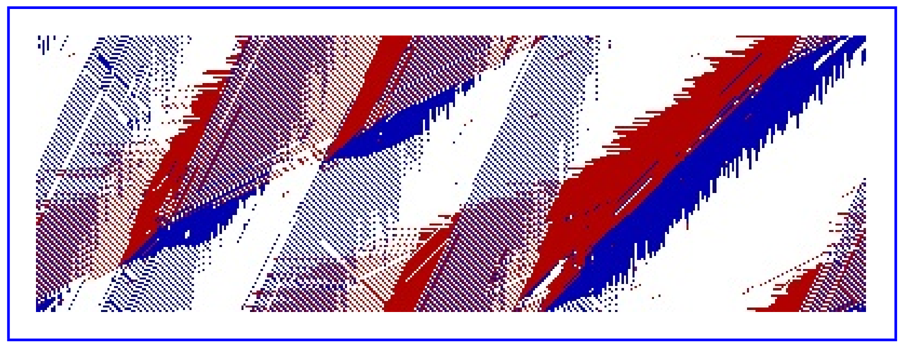
# BML Behavior?

- Evidence that $\rho_c$ decreases with increasing $L$ (but not all evidence consistent)

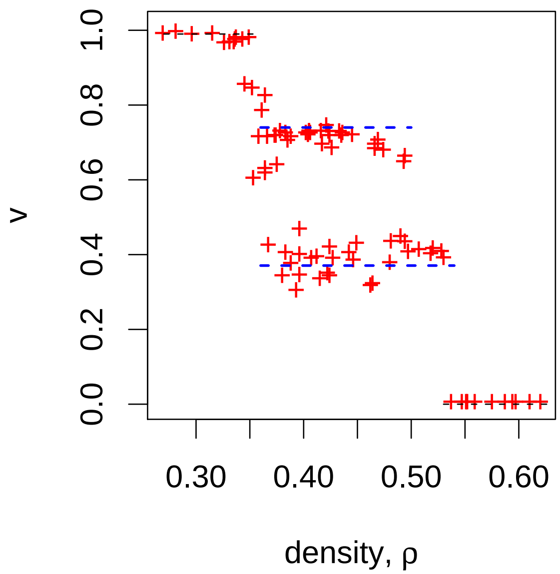
- See intermediate states for all aspect ratios, but for exact periodicity do we need relatively prime?

  Behavior for $L \times (3L+1)$ different from $L \times 3L$

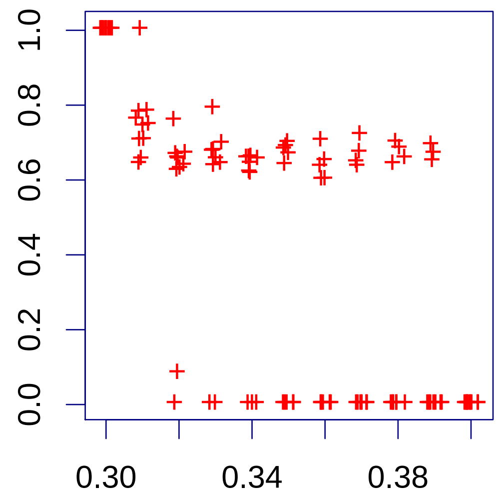




# Robustness of Intermediate states?




$L/L' \approx (1+\sqrt{5})2$




$L/L' = 1$

- Bifurcation points rather than phase transition.

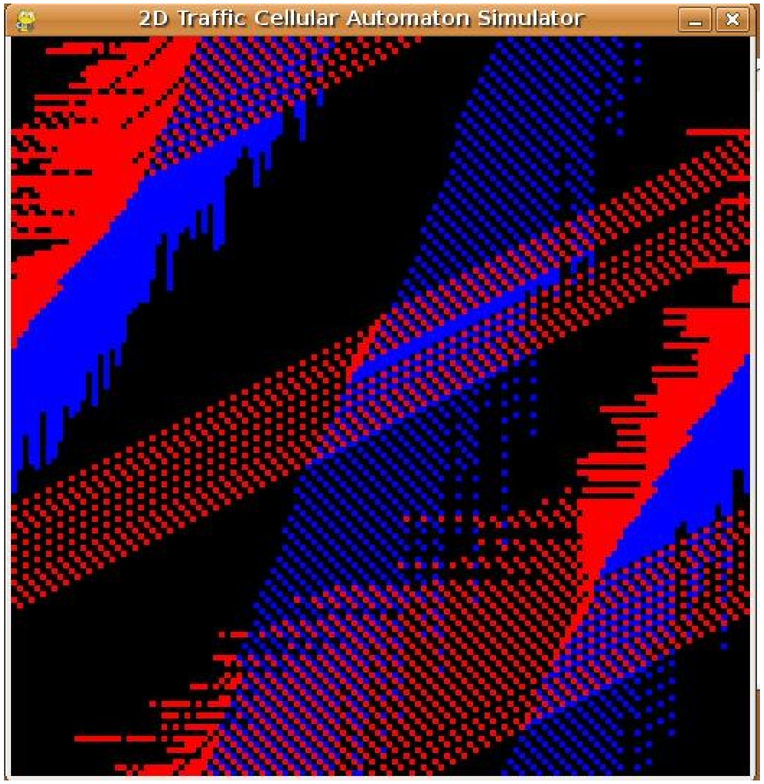
- How to characterize basins of attraction?
  - On Fibonacci lattices, for $0.35 < \rho < 0.5$, only the intermediate states accessible!
  - Even for square aspect ratios, intermediate states significant fraction of phase space.

- BML is not strictly monotonic!

- Are the periodic intermediate states pathologies/artifact of Fibonnaci?

## Latest results
## (Nicholas Linesch, UC Davis applied math undergrad)

Periodic states occur even on square aspect ratios!
(More than metastable, converged to limit cycle)



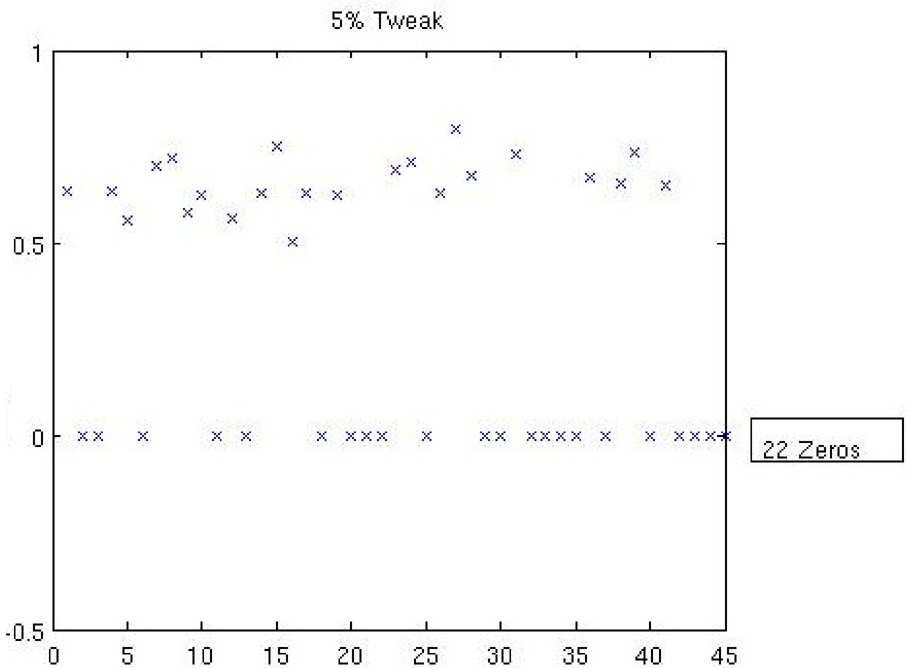

- $L = 256$, $\rho = 0.34$
- $\tau = 25344 \approx N$.

# Nicholas's results in more detail

- Generated 45 random initial conditions:

  28 converged to the $v \sim 2/3$ state
  14 converged to $v = 0$
  2 converged a periodic limit cycle, with $v \sim 1/2$

- Stability: (8 configs flip with 5% tweak):




Can we quantify what makes a system jam?

## Role of Synchrony (global clocks/global coordination)
## Looking for a “better” model

Full synchrony up to now, but
Two variants:

1. Poisson updating for car motion
   – In “red-phase” each red-car flips a coin to advance.

2. “Traffic lights” initialized in random phase, but flip simultaneously
   – Previously all lights initialized in east-phase.

# Results

- No longer get self-organization/patterns in free-flow!
  – Do not achieve $v = 0$ state.

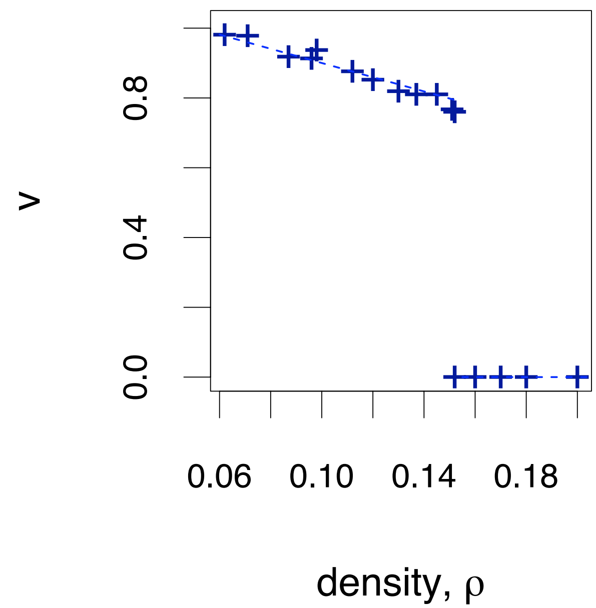
- No longer get the intermediate states discussed above.

- Synchrony previously been observed as crucial mechanism for self-org: Spatial Prisoner's Dilemma

  – Nowak and May, *Nature* 1992
  – Huberman and Glance, *PNAS* 1993

- Synchrony and games (random-turn games)

# Results of Asynch, cont




- Poisson updating: $v \sim (1 - \rho^{1/2})$ below transition

- Still get a phase transition
  - $\rho_c \approx 0.15$ (Note in "standard" BML, $\rho_c \approx 0.3$)
  - Conjecture: Should be more amenable to analysis

- Relation to infinite system?? Also shows evidence that $\rho_c$ decreases with increasing $L$. So does $\rho \to 0$ as $L \to \infty$?

# Implications from BML

For traffic flow/congestion:

- Substantial throughput ($v \sim 2/3$) in regime where previously believed $v = 0$. (On all lattices, not just special L,L' choices)

- Onset of jamming can be delayed based on geometry.

For simulation (i.e., Why had no one seen this before?):

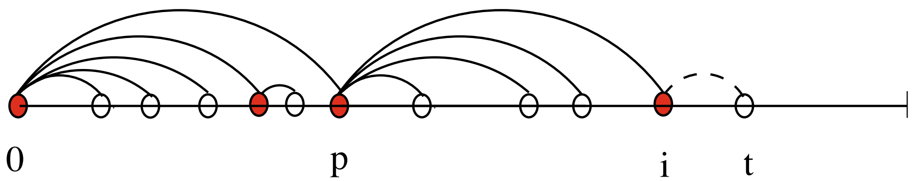
- Need to replicate the highly influential models.

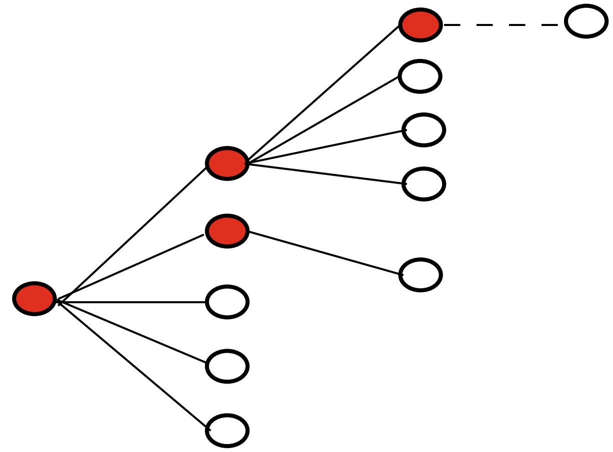
- Visualization helps, but ingrained traditions may have prevented their discovery (discard configurations, dismiss unexpected).

- Importance of lattice aspect ratio.

- In which regimes are synchrony and global coordination reasonable assumptions? (Appear to be essential components for certain types of self-organization.)

# From lattice to network



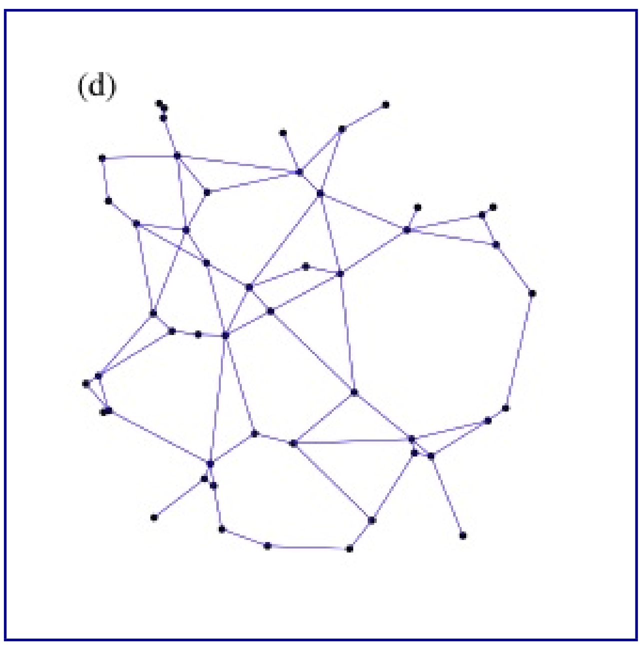
Can we use such protocols to delay congestion on networks?

Optimization in network growth

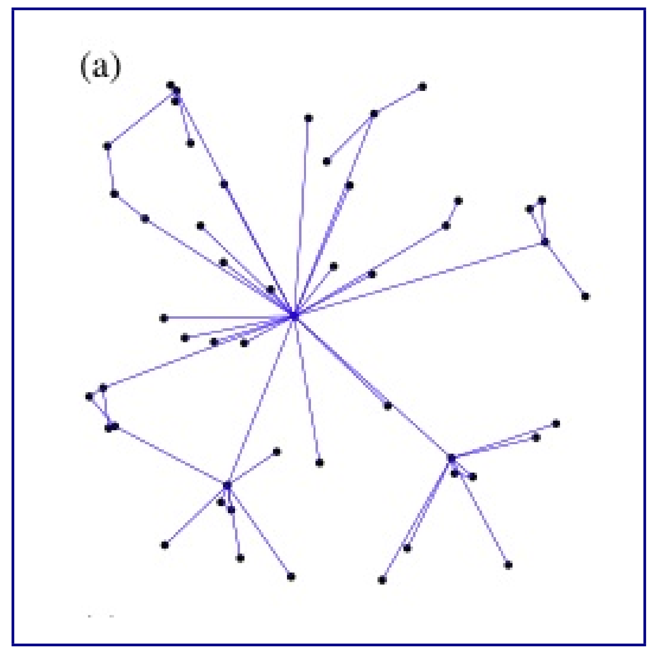
“Power of two choices” in randomized algorithms

# Optimization in network growth

- Gastner and Newman (arXiv:cond-mat/0407680, Eur. Phys. J. B (2006)).
- Cost associated with 1) length of all edges AND
- 2) Either (a) *Euclidean distance* or (b) *hop distance*
- Minimize linear combination of 1) and 2):



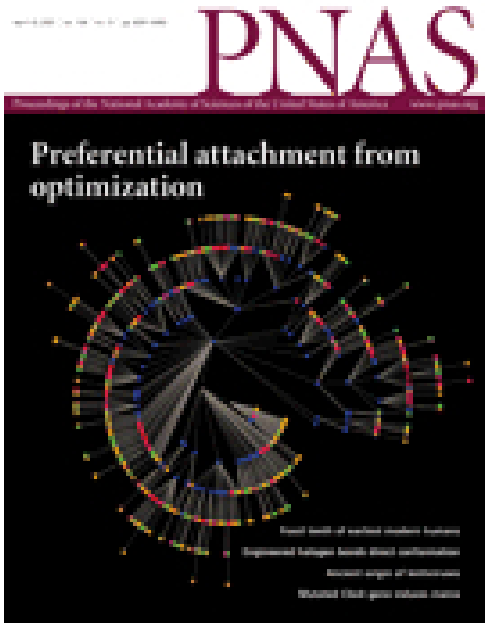

(a) Road-like




(b) Hub-and-spokes

## Optimization can give rise to Preferential Attachment in Networks

(R.D, Borgs, Chayes, Berger, Kleinberg, PNAS 2007)



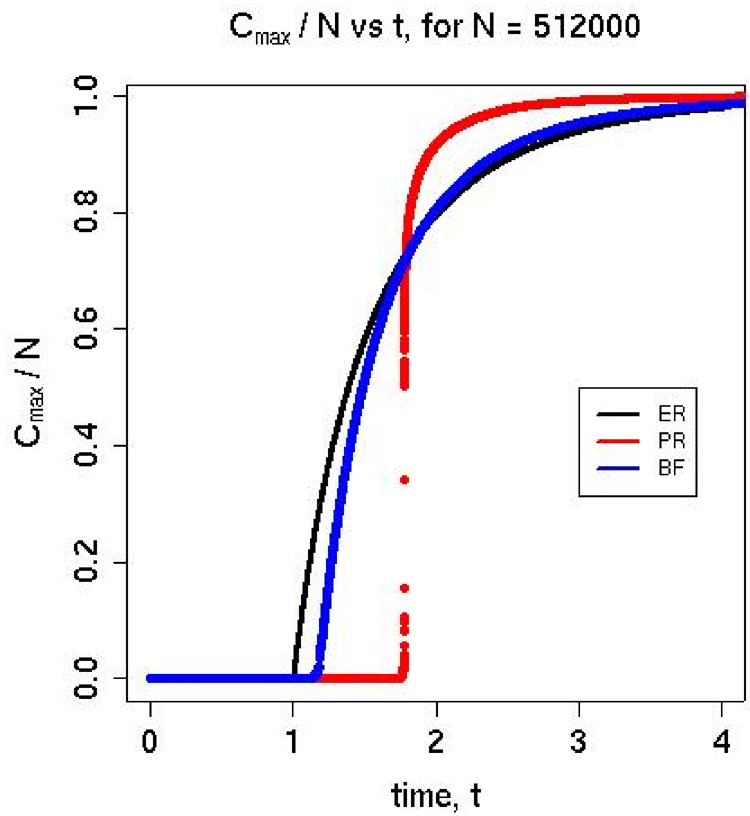

Hear this talk

IPAM the week of May 7th.

# The power of two choices and $G(n,p)$
# (The Erdös-Rényi Random graph)

Achlioptas, Spencer, RD

Altering the nature of phase transitions (not in *transport* but in *connectivity*)



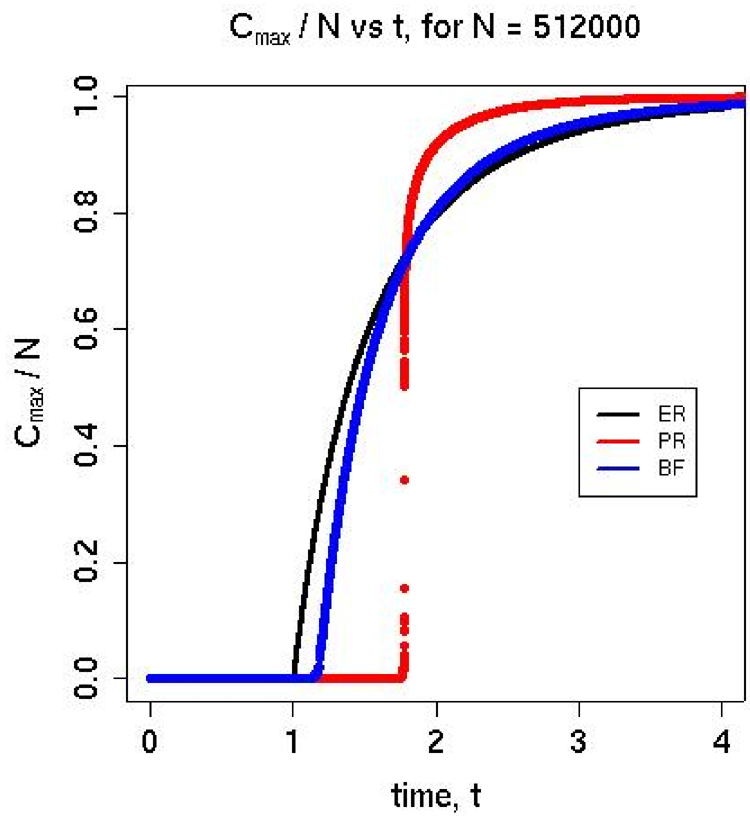
# Bounded size versus unbounded size

- At each round, now given two edges and get to chose which one to accept.

- "Bounded size" , accept edge if one of the components of size $S \leq$ B.
  – Can enhance or delay onset
  – Will always produce a smooth transition (rigorous results exist: Bohman-Frieze, Wormwald-Spencer).

- Unbounded size , choose edge that minimizes the resulting sum or product of the size of the merged component.
  – Enhance or delay onset
  – Can produce a discontinuous transition!

# Discontinuous transition




- $\Delta t \equiv$ time required to go from sublinear to linear in size.
- Show numerically: $\Delta t \propto N^{-2/3} \rightarrow 0$ as $N \rightarrow \infty$.

# Choice and Network Growth

P.Krapivsky, RD, C.Moore

- Growth of random network with *choice*
  At each discrete time step given $k$ candidate edges.
  $k = 1$ random, $k > 1$ choice.

- Minimize depth: depth distribution
  $k = 1$ Poisson
  $k = 2$ traveling wave solution.
  *Small choice makes big difference*

- Maximize degree: degree distribution
  For all small $k$, exponential decay.
  For $k \gg 1$, power law for $d < k$, exp decay above.
  *Need much choice to see change.*

# Conclusions: Jamming and phase transitions

- BML model: simple model with self-organization and phase transition/bifurcation, but the picture far more complex than believed for many years.

- We can easily delay the onset of full jamming by altering boundary conditions/extreme sensitivity to boundary conditions.

- Connections with other phase transitions?